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A characterization of the Radon transform and its dual on Euclidean space

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A. HERTLE [3] gave a characterization of the Radon transform by investigating its behaviour under rotations, dilations and translations and making some restrictions on its range. In this paper we present an alternative characterization of the Radon transform R, for which we do not need any restriction on its range. We also consider the boomerang transform B [5], which is essentially the dual of the Radon transform.

By definition

$$R(f)_{(\omega,p)} = \int_{\langle x,\omega\rangle = p} f(x)\,\mathrm{d}x, \quad B(h)_{(x)} = \int_{S_x^{n-1}} h(\langle \omega, x\rangle \omega)\,\mathrm{d}\omega,$$

where $\omega\in S^{n-1}$, $0\leqq p\in\mathbf{R}$, $\mathrm{d}x$ and $\mathrm{d}\omega$ is the surface measure, h and f are good enough functions on $\mathbf{R}^n$ (for example continuous functions with compact support) and

$$S_x^{n-1} = \{\omega\in S^{n-1}\colon \langle\omega, x\rangle \geqq 0\}.$$

One can easily verify (see [4]), that the boomerang transform has the following properties:

(1) $B(a_1 f_1 + a_2 f_2) = a_1 B(f_1) + a_2 B(f_2)$

(2) $B(f\circ U) = B(f)\circ U$

(3) $B(f(\delta x))_{(y)} = B(f)_{(\delta y)}$

(4) $B(f)_{(y+z)} = B\left(f\left(x\dfrac{\langle x, x+z\rangle}{\langle x, x\rangle}\right)\right)_{(y)}$

(5) If $f_k\to f$ locally uniformly on $\mathbf{R}^n$ then $B(f_k)_{(x)}\to B(f)_{(x)}$ for any $x\in\mathbf{R}^n$,

where $a_1, a_2\in\mathbf{R}, f, f_1, f_2, f_k$ are functions on $\mathbf{R}^n$, U is an orthogonal transformation, $0<\delta\in\mathbf{R}$ and $x, y\in\mathbf{R}^n$.

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Theorem 1. *If a nonzero function-transformation β has properties* (1)—(5) *then* $\beta=\frac{\beta(1)}{B(1)}B$ *on* $C(\mathbf{R}^n\setminus\{0\})$.

Proof. Let $f_i(x)=|x|^i$. The following simple equations prove, that $\beta(f_i)=$ $=f_i\cdot c_i'$, where the constants c_i' depends only on $i\in\mathbf{N}$.

$$\beta(f_i)_{(\delta x)}=\beta(f_i(\delta y))_{(x)}=\delta^i\beta(f_i)_{(x)}$$
$$\beta(f_i)_{(Ux)}=\beta(f_i(Uy))_{(x)}=\beta(f_i)_{(x)}.$$

This implies that if i is even

$$c_i'(t+|x|)^i=\beta(f_i)_{(x+(tx/|x|))}=\beta(f_i(y+te_y\langle e_y,e_x\rangle))_{(x)}=$$
$$=\sum_{j=0}^{i}\binom{i}{j}t^{i-j}\beta(|y|^j\langle e_x,e_y\rangle^{i-j})_{(x)},$$

where $e_x=\frac{x}{|x|}$ and $t\in\mathbf{R}$. Since this is an equality of two polynomials, we have obtained for even i that

$$(*)\qquad c_j'f_j(x)=\beta(|y|^j\langle e_x,e_y\rangle^{i-j})_{(x)}.$$

Now this equation and the condition (4) imply

$$\beta(\langle e,e_y\rangle^i)_{(e_0)}=\beta(\langle e,e_y\rangle^i)_{(e+e_0)}=c_i',$$

where e, e_0 are unit vectors. Let $c_i=B(f_i)/f_i$. The integration of our last equation over $S_{e_0}^{n-1}$ gives

$$c_i'B(1)=\int_{S_{e_0}^{n-1}}\beta(\langle e,e_y\rangle^i)_{(e_0)}\,de=\beta\Big(\int_{S_{e_0}^{n-1}}\langle e,e_y\rangle^i de\Big)_{(e_0)}=\beta(c_i)_{(e_0)}=c_i\beta(1).$$

This way, we have derived the following equation for any even i:

$$B(f_i)=\frac{\beta(1)}{B(1)}B(f_i).$$

Furthermore if i is even

$$\beta(f_i(x+y))=\beta(\langle x+y,x+y\rangle^{i+2})$$
$$=\sum_{k,l,m}\binom{i/2}{k,l,m}2^m|x|^{2l+m}\beta(|y|^{2k+m}\langle e_x,e_y\rangle^m),$$

where $k+l+m=i/2$ and $k,l,m\in\mathbf{N}$. Taking into account $(*)$ this gives rise to (by calculating $B(f_i(x+y))$)

$$\beta(f_i(x+y))=\frac{\beta(1)}{B(1)}B(f_i(x+y)).$$

Since the translates of radial polynomials are locally uniformly dense in $C(\mathbf{R}^n)$ this completes the proof.

Theorem 2. *If a nonzero function-transformation ϱ has the following properties then $\varrho = cR$ on $C_c(\mathbf{R}^n)$, the compactly supported continuous functions, for some $c \in \mathbf{R}$.*

(1′) $\varrho(a_1 f_1 + a_2 f_2) = a_1 \varrho(f_1) + a_2 \varrho(f_2)$

(2′) $\varrho(f \circ U) = \varrho(f) \circ U$

(3′) $\varrho(f(\delta x))_{(\omega, r)} = \delta^{1-n} \varrho(f)_{(\omega, \delta r)}$

(4′) $\varrho(f(x+y))_{(\omega, r)} = \varrho(f)_{(\omega, r + \langle \omega, y \rangle)}$

(5′) *If $f_k \to f$ locally uniformly on $\mathbf{R}^n$ then $\varrho(f_k) \to \varrho(f)$ (with common support).*

The notations are the same as in Theorem 1.

Proof. Let $\hat{f} = B(\varrho(f))$. Then we have

(a) $(a_1 f_1 + a_2 f_2)\hat{} = a_1 \hat{f}_1 + a_2 \hat{f}_2$

(*b*) $(f \circ U)\hat{} = \hat{f} \circ U$

(c) $(f(\delta x))\hat{}(y) = \delta^{1-n} \hat{f}(\delta y)$

(d) $(f(x+y))\hat{}(z) = \hat{f}(z+y)$

(e) If $f_k \to f$ locally uniformly on $\mathbf{R}^n$ then $\hat{f}_k \to \hat{f}$ (with common support).

It is well known [1], that if a transform satisfies (a), (d) and (e) then there is a function g on $\mathbf{R}^n$ such that

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y)\, g(x-y)\, \mathrm{d}y.$$

A short calculation with (b) and (c) gives that

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y) \frac{d}{|x-y|} \mathrm{d}y,$$

where d is a suitable constant. This implies by [2, p. 104] that $\hat{f}(x) = cB(R(f))$ for some constant c. Thus we have for any function f that

$$B(\varrho(f) - cR(f)) = 0.$$

From Theorem 1.3 of [4] we know the invertibility of the boomerang transform on the set of radial continuous functions. So if we prove that the continuity of f implies the continuity of $\varrho(f)$ then we shall obtain that ϱ is equal to cR on the continuous radial functions. Since the translates of radial continuous functions are locally uniformly dense in $C_c(\mathbf{R}^n)$ this will imply the theorem.

Thus to finish the proof we only have to prove the continuity of $\varrho(f)$ for any continuous radial function f. For any $\varepsilon>0$ we have

$$\varrho(f)_{(r+\varepsilon)}-\varrho(f)_{(r)}=\varrho\left(\left[\frac{r+\varepsilon}{r}\right]^{n-1} f\left(\frac{r+\varepsilon}{r}x\right)\right)_{(r)}-\varrho(f)_{(r)}=$$

$$=\varrho\left(\left[\frac{r+\varepsilon}{r}\right]^{n-1} f\left(\frac{r+\varepsilon}{r}x\right)-f(x)\right)_{(r)}$$

If ε tends to zero then

$$\left[\frac{r+\varepsilon}{r}\right]^{n-1} f\left(\frac{r+\varepsilon}{r}x\right)-f(x)\to 0$$

uniformly on $\operatorname{supp}(f)$. This just means that $\varrho(f)$ is a continuous function.

Theorem 1 (resp. Theorem 2) is valid for any other function space in which $C(\mathbf{R}^n\setminus\{0\})$ (resp. $C_c(\mathbf{R}^n)$) is dense.

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