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Fourier series with positive coefficients and generalized Lipschitz classes

J. NÉMETH*)

1. L. LEINDLER ([3]—[6]) investigated the relations between function classes defined by the rate of strong approximation of functions by Fourier series and the classes determined by the modulus of continuity of the functions.

Following G. G. LORENTZ [7] and R. P. BOAS [2] we shall prove theorems giving coefficient-conditions assuring that a function belong to function classes defined in terms of modulus of continuity by L. Leindler.

Combining these results and those of L. Leindler mentioned above we can get coefficient-conditions for a function to belong to a function class defined by the rate of strong approximation by Fourier series.

2. Before formulating our results we give a couple of definitions, notations and theorems.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx) \tag{1}$$

be its Fourier series. Denote by $s_n=s_n(x)=s_n(f;x)$ the n-th partial sum of (1). For any positive β and p L. LEINDLER [3] defined the following strong means and function classes

$$h_n(f;\beta;p)=\left\|\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k-f|^p\right\}^{1/p}\right\|$$

$$H(\beta,p,\omega)=\{f\colon h_n(f;\beta;p)=O(\omega(1/n))\},$$

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where $\|\cdot\|$ denotes the usual maximum norm and $\omega(\delta)$ is a modulus of continuity having the following properties

$$\omega(0)=0, \quad \omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2) \quad \text{for any} \quad 0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi.$$

Furthermore we consider the following function classes

$$H^\omega = \{f: \|f(x+h)-f(x)\| = O(\omega(h))\}$$
$$(H^\omega)^* = \{f: \|f(x+h)+f(x-h)-2f(x)\| = O(\omega(h))\}.$$

If $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$ then H^{δ^α} are the known Lipschitz classes.

In [3] L. LEINDLER (see also in [6], p. 153) proved, among others, the following result.

$$\left.\begin{array}{lll} (2) & H(\beta, p, \delta^\alpha) \equiv H^{\delta^\alpha} & \text{for } 0<\alpha<1 \\ (3) & H(\beta, p, \delta) \equiv (H^\delta)^* & \text{for } \alpha=1 \end{array}\right\} \text{ if } \beta > \alpha p.$$

G. G. LORENTZ [7] proved in 1948 that *if* $\lambda_n \downarrow 0$ *and* λ_n *are the Fourier sine or cosine coefficients of* f *then* $f\in H^{\delta^\alpha}$ $(0<\alpha<1)$ *if and only if* $\lambda_n=O(n^{-1-\alpha})$. This result and some others were generalized by R. P. BOAS [2] in 1967 as follows: *Let* $\lambda_n\geqq 0$ *and let* λ_n *be the Fourier sine or cosine coefficients of* f. *Then* $f\in H^{\delta^\alpha}$ $(0<\alpha<1)$ *if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O(n^{-\alpha}), \tag{4}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n^{1-\alpha}). \tag{5}$$

Combining this result and the result of L. Leindler mentioned above we have that if $\lambda_n\geqq 0$ and λ_n are sine or cosine coefficients of f then the following three relations are equivalent:

$$f\in H(\beta, p, \delta^\alpha); \quad \sum_{k=n}^{\infty} \lambda_k = O(n^{-\alpha}); \quad \sum_{k=1}^{n} k\lambda_k = O(n^{1-\alpha}), \tag{6}$$

if $0<\alpha<1$ and $\beta>\alpha p$.

L. Leindler has extended results (2) and (3) from H^{δ^α} to H^ω at least for certain special but more general class of moduli of continuity than $\omega(\delta)=\delta^\alpha$.

Next we give the definition of this class of moduli of continuity (see [4], [5] and in [6], p. 154). Let $\omega_\alpha(\delta)$ denote the modulus of continuity having the following properties for $0\leqq\alpha\leqq 1$:

i) for any $\alpha'>\alpha$ there exists a natural number $\mu=\mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n(\geqq 1); \tag{7}$$

ii) for every natural number ν there exists a natural number $N(\nu)$ such that

(8) $$2^{\nu\alpha}\omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n > N(\nu).$$

L. LEINDLER in [4], [5] (see also [6], p. 154) proved the following relation generalizing results (2) and (3).

(9) $$H(\beta, p, \omega_\alpha) \equiv H^{\omega_\alpha} \quad \text{for} \quad 0 < \alpha < 1;$$
(10) $$H(\beta, p, \omega_1) \equiv (H^{\omega_1})^* \quad \text{for} \quad \alpha = 1$$
$$\text{if} \quad \beta > \alpha p.$$

It is clear thát in order to get coefficient conditions of type (6) for $f \in H(\beta, p, \omega_\alpha)$ instead of $H(\beta, p, \delta^\alpha)$ it is sufficient to generalize the mentioned Boas results to class H^{ω_α}. These results are formulated in the next paragraph.

3. Theorems

Throughout the rest of this paper $g(x)$, $f(x)$, $\varphi(x)$ will denote continuous 2π periodic functions; furthermore $g(x)$ and $f(x)$ always denote the sum of sine series and cosine series, respectively. And $\varphi(x)$ denotes the sum of either sine or cosine series while λ_n will denote the Fourier coefficients of $g(x)$, $f(x)$ or $\varphi(x)$.

Theorem 1. *Let* $\lambda_n \geqq 0$. *Then* $\varphi \in H^{\omega_\alpha}$ $(0<\alpha<1)$ *if and only if*

(11) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_\alpha(1/n)),$$

or equivalently

(12) $$\sum_{k=1}^{n} k\lambda_k = O(n\omega_\alpha(1/n)).$$

Theorem 2. *Let* $\lambda_n \geqq 0$. *Then*

(13a) $$g \in H^{\omega_1}$$

if and only if

(13b) $$\sum_{k=1}^{n} k\lambda_k = O(n\omega_1(1/n)).$$

Theorem 3. *Let* $\lambda_n \geqq 0$. *Then*

(14a) $$\varphi \in (H^{\omega_1})^*$$

if and only if

(14b) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)).$$

Theorem 4. *Let* $\lambda_n \geqq 0$. *Then*

(15a) $$f \in H^{\omega_1}$$

if and only if

(15b) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)) \quad and \quad \sum_{k=1}^{n} k\lambda_k \sin kx = O(n\omega_1(1/n)).$$

Theorem 5. *Let* $\lambda_n \geqq 0$. *Then*

(16a) $$f \in H^{\omega_0}$$

if and only if

(16b) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_0(1/n)).$$

Furthermore

(17) $$g \in H^{\omega_0}$$

implies

(18) $$\sum_{k=1}^{n} k\lambda_k = O(n\omega_0(1/n)),$$

and from

(19) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_0(1/n))$$

(20) $$g \in H^{\omega_0}$$

follows.

Remark 1. Combining relation (9) of L. Leindler and Theorem 1 we get generalizations of the relations under (6). Namely that if $\lambda_n \geqq 0$ then for $0<\alpha<1$ with $\beta>\alpha p$

$$\varphi \in H(\beta, p, \omega_\alpha)$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_\alpha(1/n)),$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n\omega_\alpha(1/n)).$$

Remark 2. Using Theorem 1 we can prove the following result.

If $h \in H^{\omega_\alpha}$ $(0<\alpha<1)$ and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

is the Fourier series of $h(x)$ and $a_k \geqq 0$, $b_k \geqq 0$ then

$$\|s_n(x)-h(x)\| = O(\omega_\alpha(1/n)). \tag{21}$$

It should be noted that for arbitrary a_k, b_k according to the well-known Lebesgue result only

$$\|s_n(x)-h(x)\| = O(\omega_\alpha(1/n))\cdot \log n$$

can be obtained.

The proof of this remark is very simple. Really, consider

$$\|s_n(x)-h(x)\| \leqq \|s_n(x)-\sigma_n(x)\| + \|\sigma_n(x)-h(x)\| = I + II.$$

Using (12) and the fact that

$$s_n(x)-\sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n}(ka_k \cos kx + kb_k \sin kx)$$

I can be estimated as follows

$$I \leqq \frac{1}{n+1}\sum_{k=1}^{n} k(a_k+b_k) = O(\omega_\alpha(1/n)).$$

Here we used that if $h \in H^{\omega_\alpha}$, then both its sine-part and cosine-part also belong to H^{ω_α}. And finally from (9)

$$II = O(\omega_\alpha(1/n)).$$

Thus we have (21).

4. Lemmas

Lemma 1. (Lemma 2.6 of [6], p. 39.) *For any nonnegative sequence $\{a_n\}$ the inequality*

$$\sum_{n=1}^{m} a_n \leqq Ka_m \quad (m = 1, 2, \ldots;\ K > 0) \tag{22}$$

holds if and only if there exist a positive number c and a natural number μ such that for any n

$$a_{n+1} \geqq ca_n \tag{23}$$

and

$$a_{n+\mu} \geqq 2a_n \tag{24}$$

are valid.

Lemma 2. *If* $\mu_k \geqq 0$ *and* $\delta > \beta > 0$ *then*

(25a) $$\sum_{k=1}^{n} k^\delta \mu_k = O\big(n^\delta \omega_{\delta-\beta}(1/n)\big)$$

is equivalent to

(25b) $$\sum_{k=n}^{\infty} \mu_k = O\big(\omega_{\delta-\beta}(1/n)\big).$$

Proof. First we suppose that (25b) is true. By using Abel-rearrangement ([1], p. 71) we have

(26) $$\sum_{k=1}^{n} \mu_k k^\delta \leqq \sum_{k=1}^{n} (k^\delta-(k-1)^\delta) \sum_{\nu=k}^{\infty} \mu_\nu + 2 \sum_{\nu=1}^{\infty} \mu_\nu = I + II.$$

It is obvious by (25b) and the definition of $\omega_{\delta-\beta}(t)$ that II does not exceed $K \cdot n^\delta \omega_{\delta-\beta}(1/n)$*).

And I can be estimated as follows

(27) $$I \leqq K_1 \sum_{k=1}^{n} k^{\delta-1} \omega_{\delta-\beta}(1/k) \leqq K_2 \sum_{m=1}^{[\log n]} (2^m)^{\delta-1} \cdot 2^m \omega_{\delta-\beta}(1/2^m) =$$

$$= K_2 \sum_{m=1}^{[\log n]} 2^{m\delta} \omega_{\delta-\beta}(1/2^m) \leqq K_3 n^\delta \omega_{\delta-\beta}(1/n),$$

where the last estimation can be obtained by using property (7) of $\omega_\alpha(\delta)$ and Lemma 1. So taking into account (26) and (27), (25a) really follows from (25b). Now we suppose (25a).

Again Abel-rearrangement gives that

(28) $$\sum_{k=m}^{n} \mu_k = \sum_{k=m}^{n} k^\delta \mu_k k^{-\delta} = \sum_{k=m}^{n} (k^{-\delta}-(k+1)^{-\delta}) \sum_{l=1}^{k} l^\delta \mu_l + n^{-\delta} \sum_{k=1}^{n} k^\delta \mu_k.$$

Making n tend to infinity from (25a) and (28) we get

(29) $$\sum_{k=m}^{\infty} \mu_k = \sum_{k=m}^{\infty} (k^{-\delta}-(k+1)^{-\delta}) \sum_{l=1}^{k} l^\delta \mu_l = I_1.$$

I_1 can be estimated by (25a) as follows

(30) $$I_1 \leqq K_1 \sum_{k=m}^{\infty} k^{-1} \omega_{\delta-\beta}(1/k) \leqq K_2 \sum_{n=[\log m]}^{\infty} \omega_{\delta-\beta}(1/2^n) \leqq K_3 \omega_{\delta-\beta}(1/m).$$

The last step can be derived from property (8) of $\omega_\alpha(\delta)$. From (29) and (30) we have (25b). Thus Lemma 2 is completely proved.

*) $K, K_1, K_2, \ldots$ will denote positive absolute constants.

Lemma 3. *If* $\mu_k \geqq 0$, $\sum_{k=1}^{\infty} \mu_k < \infty$ *and* $0 < \beta \leqq 1$, *then*

$$\sum_{k=1}^{\infty} \mu_k (1-\cos kx) = O(\omega_\beta(x)) \tag{31a}$$

is equivalent to

$$\sum_{k=n}^{\infty} \mu_k = O(\omega_\beta(1/n)). \tag{31b}$$

Proof. Supposing first (31a), we have that

$$\sum_{k=1}^{[1/x]} k^2 \mu_k \frac{1-\cos kx}{k^2 x^2} = O(x^{-2} \omega_\beta(x)) \tag{32}$$

holds for any positive x.

Since $K \geqq t^{-2}(1-\cos t)\downarrow$ on $(0, 1)$, from (32) it follows that

$$\sum_{k=1}^{[1/x]} k^2 \mu_k = O(x^{-2} \omega_\beta(x)). \tag{33}$$

Putting $x = 1/n$ we get

$$\sum_{k=1}^{n} k^2 \mu_k = O(n^2 \omega_\beta(1/n)) \tag{34}$$

which, by using Lemma 2, is equivalent to

$$\sum_{k=n}^{\infty} \mu_k = O(\omega_\beta(1/n)) \tag{35}$$

which proves (31a)⇒(31b).

Now we suppose that (31b) is valid. But (31b), by using Lemma 2, is equivalent to

$$\sum_{k=1}^{n} k^2 \mu_k = O(n^2 \omega_\beta(1/n)). \tag{36}$$

Using (31b) and (36) we have that

$$\sum_{k=1}^{\infty} \mu_k (1-\cos kx) \leqq \sum_{k=1}^{[1/x]} \mu_k (1-\cos kx) + 2 \sum_{k=[1/x]}^{\infty} \mu_k =$$

$$= x^2 \sum_{k=1}^{[1/x]} k^2 \mu_k \frac{1-\cos kx}{k^2 x^2} + 2 \sum_{k=[1/x]}^{\infty} \mu_k = O(\omega_\beta(x))$$

which gives that (31b) really implies (31a). Thus Lemma 3 is completed.

5. Proofs of the theorems

Proof of Theorem 1. We detail the proof just for cosine, the line of the proof for sine is very similar.

Suppose that $f\in H^{\omega_\alpha}$ $(0<\alpha<1)$. Then since by the Paley's theorem from the continuity of $f(x)$ it follows that $\sum_{k=1}^{\infty}\lambda_k<\infty$, we have

$$|f(x)-f(0)| = \sum_{k=1}^{\infty}\lambda_k(1-\cos kx) = O(\omega_\alpha(x)). \tag{37}$$

By Lemma 3 the right-hand side equality of (37) is equivalent to (11). In virtue of Lemma 2 (11) is equivalent to (12). Thus the necessary part of Theorem 1 is proved. Now we suppose that (11) holds and put

$$|f(x+2h)-f(x)| = \Big|\sum_{k=1}^{\infty}\lambda_k[\cos k(x+2h)-\cos kx]\Big| = \tag{38}$$

$$= 2\Big|\sum_{k=1}^{\infty}\lambda_k \sin k(x+h)\cdot \sin kh\Big| \leqq 2\sum_{k=1}^{\infty}\lambda_k|\sin kh| \leqq 2\sum_{k=1}^{[1/h]}\lambda_k \sin kh+2\sum_{k=[1/h]}^{\infty}\lambda_k .$$

The second term in the last row of (38) is $O(\omega_\alpha(h))$ (see (11)). The first one can be handled as follows:

$$\sum_{k=1}^{[1/h]}\lambda_k\cdot \sin kh = h\sum_{k=1}^{[1/h]}k\lambda_k\frac{\sin kh}{kh} \leqq K\cdot h\Big(\sum_{k=1}^{[1/h]}k\lambda_k\Big) = O(\omega_\alpha(h)).$$

In the last step we used again (12) and Lemma 2. So from (38) we have $f\in H^{\omega_\alpha}$. The proof of Theorem 1 for cosine series is completed.

Proof of Theorem 2. Let $g\in H^{\omega_1}$. Using

$$|g(x)| \leqq K\omega_1(x),$$

term by term integration gives

$$\Big|\int_0^x g(t)\,dt\Big| = \sum_{k=1}^{\infty}k^{-1}\lambda_k(1-\cos kx) = O(x\omega_1(x)). \tag{39}$$

From (39) it follows that

$$\sum_{k=1}^{[1/x]}k^{-1}\lambda_k(1-\cos kx) = O(x\omega_1(x)). \tag{40}$$

(40) can be written as follows

$$x^2\sum_{k=1}^{[1/x]}k\lambda_k\frac{1-\cos kx}{k^2x^2} = O(x\omega_1(x)). \tag{41}$$

By using the same argument as before from (41) we get

$$\sum_{k=1}^{[1/x]} k\lambda_k = O\left(\frac{1}{x}\,\omega_1(x)\right). \tag{42}$$

Putting $x=1/n$ we have from (42)

$$\sum_{k=1}^{n} k\lambda_k = O\big(n\omega_1(1/n)\big)$$

which proves that from (13a) follows (13b). Now we suppose that (13b) is fulfilled. Put

$$|g(x+2h)-g(x)| = \Big|\sum_{k=1}^{\infty} \lambda_k\,[\sin k(x+2h)-\sin kx]\Big| = \tag{43}$$

$$= 2\Big|\sum_{k=1}^{\infty} \lambda_k \cos k(x+h)\sin kh\Big| \leqq 2\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k = I+II.$$

By using (13b) we have that

$$I = 2h\sum_{k=1}^{[1/h]} k\lambda_k\,\frac{\sin kh}{kh} = h\cdot O\left(\frac{1}{h}\,\omega_1(h)\right) = O\big(\omega_1(h)\big). \tag{44}$$

For II to be estimated by $K\omega_1(h)$ we can use the same argument as in the second part of the proof Lemma 2. Namely taking $\delta=1$ and $\delta-\beta=1$ we get that (13b) implies

$$\sum_{k=n}^{\infty} \lambda_k = O\big(\omega_1(1/n)\big)$$

which gives that

$$II = O\big(\omega_1(h)\big). \tag{45}$$

Thus (43), (44) and (45) give that $g\in H^{\omega_1}$. Theorem 2 is completed.

Proof of Theorem 3.

First we prove the theorem for cosine series. Suppose that (14a) holds, that is,

$$|f(x+h)+f(x-h)-2f(x)| \leqq K\omega_1(h)$$

from which we get

$$|f(h)-f(0)| \leqq K\omega_1(h) \tag{46}$$

in other words

$$\sum_{k=1}^{\infty} \lambda_k(1-\cos kh) = O\big(\omega_1(h)\big) \tag{47}$$

holds.

From (47) by using Lemma 3, (14b) follows, that was to be proved. Now we assume (14b) and estimate the following difference by using Lemma 3 at the last step:

(48) $$|f(x+2h)+f(x-2h)-2f(x)| = 4\Big|\sum_{k=1}^{\infty} \lambda_k \sin^2 kh \cos kx\Big| \leqq$$

$$\leqq 4 \sum_{k=1}^{\infty} \lambda_k \sin^2 kh = 2 \sum_{k=1}^{\infty} \lambda_k (1-\cos 2kh) = O(\omega_1(h)).$$

Thus the proof of Theorem 3 is completed for cosine series. The proof for sine series in direction from (14b) to (14a) can be done in the same way as for cosine series, since

(49) $$|g(x+2h)+g(x-2h)-2g(x)| = 4\Big|\sum_{k=1}^{\infty} \lambda_k \sin kx \sin^2 kh\Big|.$$

So we detail only the other direction. Suppose that

(50) $$g\in (H^{\omega_1})^*,$$

that is,

(51) $$|g(x+h)+g(x-h)-2g(x)| = O(\omega_1(h)).$$

Write (51) in the following form

(52) $$2\Big|\sum_{k=1}^{\infty} \lambda_k \sin kx (1-\cos kh)\Big| = O(\omega_1(h)).$$

By integrating term by term in (52) we get

(53) $$\sum_{k=1}^{\infty} \lambda_k \frac{1-\cos kx}{k} (1-\cos kh) = O(x\omega_1(h)).$$

From (53) we have

(54) $$\sum_{k=1}^{\infty} x^2 k\lambda_k \frac{1-\cos kx}{k^2x^2} (1-\cos kh) = O(x\omega_1(h)).$$

Using (54) it follows that

(55) $$\sum_{k=1}^{[1/x]} xk\lambda_k (1-\cos kh) = O(\omega_1(h)).$$

Putting $h=x$ in (55)

(56) $$\sum_{k=1}^{[1/h]} hk\lambda_k (1-\cos kh) = O(\omega_1(h))$$

can be obtained which gives

$$\sum_{k=1}^{[1/h]} h^3 k^3 \lambda_k \frac{1-\cos kh}{k^2 h^2} = O(\omega_1(h)). \tag{57}$$

From (57) taking $h=1/n$

$$\sum_{k=1}^{n} k^3 \lambda_k = O(n^3 \omega_1(1/n)) \tag{58}$$

follows.

By using Lemma 2 (58) implies (14b), which was to be proved. Thus Theorem 3 is completely proved.

Proof of Theorem 4. First we prove the necessity of the conditions, namely we suppose that (15a) holds.
From Theorem 3 using the relation $H^{\omega_1} \subset (H^{\omega_1})^*$ it follows that

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)). \tag{59}$$

So it remains just to prove

$$\Big\|\sum_{k=1}^{n} k\lambda_k \sin kx\Big\| = O(n\omega_1(1/n)). \tag{60}$$

Set

$$|f(x+h)-f(x)| = 2\sum_{k=1}^{[1/h]} \lambda_k \sin k(x+h) \sin kh + O\Big(\sum_{k=[1/h]}^{\infty} \lambda_k\Big).$$

From (15a) and (59) we get

$$\Big\|\sum_{k=1}^{[1/h]} \lambda_k \sin k(x+h) \sin kh\Big\| = O(\omega_1(h)). \tag{61}$$

Since $\sin kh = kh + O(k^3h^3)$ we have from (61) that

$$\Big\|h \sum_{k=1}^{[1/h]} \lambda_k k \sin k(x+h) + h^3 \sum_{k=1}^{[1/h]} \lambda_k k^3 \sin k(x+h)\Big\| = O(\omega_1(h)) \tag{62}$$

and having in view Lemma 2 we get

$$\Big\|h^3 \sum_{k=1}^{[1/h]} \lambda_k k^3 \sin k(x+h)\Big\| = O(\omega_1(h)). \tag{63}$$

Using (63) we have from (62)

$$\Big\|h \sum_{k=1}^{[1/h]} k\lambda_k \sin k(x+h)\Big\| = O(\omega_1(h)). \tag{64}$$

Since $\sin k(x+h)=\sin kx-O(k^2h^2)\sin kx+O(kh)\cos kx$, (64) has the following form

$$\left\|h\sum_{k=1}^{[1/h]}k\lambda_k\sin kx-h\sum_{k=1}^{[1/h]}k\lambda_k O(k^2h^2)\sin kx+\right.\tag{65}$$

$$\left.+h\sum_{k=1}^{[1/h]}k\lambda_k O(kh)\cos kx\right\|=O(\omega_1(h)).$$

Taking into account that from (59) by using Lemma 2

$$\sum_{k=1}^{n}k^3\lambda_k=O\big(n^3\omega_1(1/n)\big)\tag{66}$$

follows, the norm of the second term in the left-hand side of (65) can be estimated as follows

$$\left\|h\sum_{k=1}^{[1/h]}k\lambda_k O(k^2h^2)\sin kx\right\|\leqq Kh^3\sum_{k=1}^{[1/h]}k^3\lambda_k=O\big(\omega_1(h)\big).\tag{67}$$

Similarly by using

$$\sum_{k=1}^{n}k^2\lambda_k=O\big(n^2\omega_1(1/n)\big)$$

instead of (66) we can get that the magnitude of the third term of (65) in norm is $O\big(\omega_1(h)\big)$. Using this last estimation and (65), (67) we have (60).

The sufficiency of conditions (15b) can be proved in very similar way as the necessity, so we omit it. Thus Theorem 4 is completed.

Proof of Theorem 5. Let $f(x)=\sum_{k=1}^{\infty}\lambda_k\cos kx$ and suppose that $f\in H^{\omega_0}$. Then we have

$$|f(h)-f(0)|\leqq K\omega_0(h),$$

that is,

$$\sum_{k=1}^{\infty}\lambda_k(1-\cos kh)\leqq K\omega_0(h).$$

Integrating both sides on $(0,x)$ we have

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{k}(kx-\sin kx)\leqq Kx\omega_0(x).\tag{68}$$

Since $kx-\sin kx\geqq 0$ so we have from (68)

$$\sum_{k=2n}^{\infty}\frac{\lambda_k}{k}(kx-\sin kx)\leqq Kx\omega_0(x).\tag{69}$$

Putting $1/n$ for x and taking into account that

$$\frac{k}{n}-\sin\frac{k}{n} \geqq \frac{1}{2}\frac{k}{n} \quad \text{for} \quad k \geqq 2n$$

we get

$$\sum_{k=2n}^{\infty} \lambda_k \leqq K_1 \omega_0(1/n) \tag{70}$$

which gives (16b).

Now we suppose that (16b) holds and we prove $f\in H^{\omega_0}$. First we note that we can notice that the first part of the proof of Lemma 2 remains valid if we take $\delta-\beta=0$ and $\delta=1$. So from (16b) we have

$$\sum_{k=1}^{n} k\lambda_k = O\big(n\omega_0(1/n)\big). \tag{71}$$

And now estimate the following difference using (16b) and (71).

$$|f(x+2h)-f(x)| = \Big|\sum_{k=1}^{\infty} \lambda_k [\cos k(x+2h)-\cos kx]\Big| =$$

$$= 2\Big|\sum_{k=1}^{\infty} \lambda_k \sin k(x+h) \sin kh\Big| \leqq 2\Big|\sum_{k=1}^{\infty} \lambda_k \sin kh\Big| =$$

$$\leqq 2\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k = O\big(\omega_0(h)\big)$$

which proves that (16b) implies (16a).

Now we prove (18) from (17).

Suppose that $g\in H^{\omega_0}$. Using the estimation

$$|g(x)| \leqq K\omega_0(x), \tag{72}$$

term by term integration gives from (72) that

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k}(1-\cos kx) \leqq Kx\omega_0(x). \tag{73}$$

From (73)

$$\sum_{k=1}^{[1/x]} k\lambda_k \leqq K\frac{\omega_0(x)}{x}$$

follows as at the proof of (33) which taking $x=1/n$ gives (18). The proof of (20) from (19) can be done in the very same way as (16a) from (16b), so we omit it. Theorem 5 is completed.

References

[1] G. Alexits, *Convergence problems of orthogonal series*, Akadémiai Kiadó (Budapest, 1961).
[2] R. P. Boas, Fourier series with positive coefficients, *J. Math. Anal.*, **17** (1967), 463—483.
[3] L. Leindler, Strong approximation and classes of functions, *Mitteilungen Math. Seminar Giessen*, **132** (1978), 29—38.
[4] L. Leindler, Strong approximation and generalized Lipschitz classes, *Functional analysis and approximation* (Oberwohlfach, 1980), Birkhäuser (Basel—Boston, 1981), pp. 343—350.
[5] L. Leindler, Strong approximation and generalized Zygmund class, *Acta. Sci. Math.*, **43** (1981), 301—309.
[6] L. Leindler, *Strong approximation by Fourier series*, Akadémiai Kiadó (Budapest, 1985).
[7] G. G. Lorentz, Fourier-Koeffizienten und Funktionen Klassen, *Math. Z.*, **51** (1948), 135—149.
[8] A. Zygmund, *Trigonometric Series. I—II*, (Cambridge, 1968).

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY