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On imbedding theorems for weighted polynomial approximation and modulus of continuity of functions

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0. Introduction

Let φ and ψ be two measurable functions on (a, b), $(-\infty \leqq a < b \leqq \infty)$. Denote $\varphi(L)\psi(L) = \varphi(L)\psi(L)_{(a,b)}$ the set of those measurable functions f on (a, b) for which

$$\int_a^b \varphi(|f(x)|)\,\psi(|f(x)|)\,dx$$

exists. In the case $\psi \equiv 1$ and $\varphi(x) = |x|^p$ $(1 \leqq p < \infty)$ we usually write L^p instead of $\varphi(L)\psi(L)$.

The norm of $f \in L^p(a, b)$ is defined by

$$\|f\|_p = \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}.$$

The space L^p of all the functions of periodic 2π will be denoted by $L^p[2\pi]$. The modulus of continuity of a function $f \in L^p(a, b)$ is defined as follows

$$\omega(f, \delta)_p = \sup_{0 \leqq h \leqq \delta} \Big(\int_a^{b-h} |f(x+h) - f(x)|^p\,dx\Big)^{1/p} \quad (0 \leqq \delta \leqq b-a).$$

If $f \in L^p[2\pi]$ then let

$$\omega(f, \delta)_p = \sup_{0 \leqq h \leqq \delta} \Big(\int_0^{2\pi} |f(x+h) - f(x)|^p\,dx\Big)^{1/p} \quad (\delta \geqq 0).$$

A nondecreasing continuous function Ω on $[0, 1]$ is called a modulus of continuity if

$$\Omega(0) = 0, \quad \Omega(\delta_1 + \delta_2) \leqq \Omega(\delta_1) + \Omega(\delta_2) \quad (0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 1).$$

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For a modulus of continuity Ω and $1\leqq p<\infty$ let

$$H_p^{\Omega}=H_p^{\Omega,\omega}:=\{f\in L^p\colon\ \omega(f,\delta)\leqq c(f)\Omega(\delta),\ \delta>0\}$$

here and later $c(x, \dots)$ denotes a constant depending only on $x, \dots$, furthermore c will denote an absolute constant (not necessarily the same in different formulae).

Let $F=\{f_n\}_{n=0}^{\infty}$ be an orthonormal system on (a, b). Define for $n=0, 1, \dots$

$$\Pi_n(F):=\Big\{p_n=\sum_{k=0}^{n}\lambda_k f_k\colon\ \lambda_k \text{ are real numbers, } k=0,1,\dots\Big\}.$$

If for some $1\leqq p<\infty$, $F\subset L^p$, then let

$$E_n(F,f)_p:=\inf_{p_n\in\Pi_n(F)}\|f-p_n\|_p \quad (f\in L^p,\ n=0,1,\dots).$$

For a given decreasing sequence of real numbers tending to zero $\alpha=(\alpha_n)=(\alpha_n\downarrow 0)$, let

$$E(F,\alpha,p):=\{f\in L^p\colon\ E_n(F,f)_p\leqq c(f)\alpha_n,\ n=0,1,\dots\}.$$

Many authors have studied the so-called imbedding problems: What are sufficient conditions and what are necessary conditions (regarding Ω) for

$$H_p^{\Omega,\omega}\subset A, \tag{1}$$

where A is a given set of functions. A similar problem is to find sufficient conditions and necessary conditions (regarding α) for

$$E(F,\alpha,p)\subset B, \tag{2}$$

where B denotes some given set of functions. For example UL'JANOV [10] considered these problems in the case $A=B=L^q[2\pi]$ $(1\leqq p<q<\infty)$ and if F is the trigonometric system. TIMÁN [9] answering one of Ul'janov's questions proved that a certain sufficient condition due to Ul'janov is also necessary for imbedding (2) with $B=L^q[2\pi]$. L. Leindler generalized these results for $A=B=\varphi(L)\psi(L)$ (see e.g. [4], [5]). Some analogous results on the infinite interval due to J. NÉMETH [8].

Let $\lambda>0$. The orthonormal system F is called (by the present author) a $\{N,\lambda\}$-system if the inequality

$$\|p_n\|_q\leqq c n^{\lambda((1/p)-(1/q))}\|p_n\|_p \tag{3}$$

holds for every $p_n\in\Pi_n(F)$, $n=1,2,\dots$ and $1\leqq p<q<\infty$. In the case $\lambda=1$, inequality (3) is called Nikol'skiĭ-inequality.

The following statement is true, its proof is similar to that of TIMAN [9]. Let F be a $\{N,\lambda\}$-system and let $f\in L^p$ $(1\leqq p<\infty)$. If for some $1\leqq p<q<\infty$

$$\varepsilon:=\sum_{n=1}^{\infty}n^{\lambda(q/p-1)-1}E_n^q(F,f)_p<\infty \tag{4}$$

then $f \in L^q$ and $\|f\|_q \leqq c\{\|f\|_p^q + \varepsilon\}^{1/q}$. Consequently, for a $\{N, \lambda\}$-system, the condition

(5) $$\sum_{n=1}^{\infty} n^{\lambda(q/p-1)-1} \alpha_n^q < \infty$$

is sufficient for imbedding (2) with $B = L^q$. We can ask if this is also necessary.

On the other hand, many results of the approximation theory show that for a given system F there exist new moduli of continuity for which the analogues of Jackson and Bernstein theorems are true. Therefore the following problem seems to be natural: What can we say about imbedding (1) in the case if ω is also a modulus of continuity?

In this paper we give an answer to the first question in the case of the generalized Hermite functions and we consider the second problem for the modulus of continuity to be defined later on. Some results will be proved for $\varphi(L)\psi(L)$ as well.

1. The main results

Let

$$w(x) = (1+|x|^u)^{v/2u} e^{-|x|^u/2} \quad (-\infty < x < \infty), \quad u \geqq 2, \quad v \geqq 0$$

and let $\{h_n\}$ be the system of the orthonormal polynomials with respect to the weight w^2. Then the system $F_{u,v} = \{f_n w\}$ is orthonormal on $(-\infty, \infty)$. If $u=2$, $v=0$ then $F_{u,v}$ is the system of the orthonormal Hermite functions. The weight w was introduced by Freud [2] for all real v and $u \geqq 2$. In this paper, when no additional condition is required, we always assume that $v \geqq 0$, $u \geqq 2$.

We define the modulus of continuity of a function $f \in L^p(-\infty, \infty)$ as follows

(6) $$\omega^*(f, \delta)_p = \omega^*_{A,B}(f, \delta)_p = \omega^*_{A,B}(u, v, f, \delta)_p =$$

$$= \sup_{0 \leqq h \leqq \delta} \Big\{ \int_{-\infty}^{B} |f_p(x+h) - f_p(x)|^p w^p(x)\, dx \Big\}^{1/p} +$$

$$+ \sup_{0 < h \leqq \delta} \Big\{ \int_{A}^{\infty} |f_p(x-h) - f_p(x)|^p w^p(x)\, dx \Big\}^{1/p} \quad (\delta > 0, \; -\infty < A < B < \infty),$$

where

$$f_p := w^{-p} f.$$

The modulus of this type was introduced in [3].

For a given sequence of real numbers (φ_n) and $1 \leqq p, q < \infty$, let

(7) $$\Phi(x) = \Phi_{p,q,\lambda}(x) := \sum_{k=1}^{[x]} k^{\lambda(q/p-1)-1} \varphi_k.$$

In the case $\lambda = 1$ this function was introduced by LEINDLER [6].

Further on we simply write $\varphi(L)\psi(L)$ for $\varphi(L)\psi(L)_{(-\infty,\infty)}$.

The following theorems are true:

Theorem 1. *Let* $1 \leqq p \leqq q < \infty$ *and let* $\alpha = (\alpha_n \downarrow 0)$, (φ_n) *be given nonnegative monotonic sequences satisfying*

(8) $$n\alpha_n \leqq cm\alpha_m \quad \text{for} \quad 1 \leqq n < m$$

and $\varphi_{k^2} \leqq c\varphi_k$, *and if* $q > p$ *then let* (φ_n) *be decreasing. Let* $\Phi = \Phi_{p,q,\lambda}$ *be the function defined in* (7) *with* $\lambda = 1 - 1/u$. *Then a necessary condition for*

(9) $$E(F_{u,v}, \alpha, p) \subset L^{q+(1-1/u)(1-q/p)}\Phi(L)$$

is

(10) $$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \varphi_n \alpha_n^q < \infty.$$

Theorem 2. *Let* $1 \leqq p < q < \infty$ *and let* $\alpha = (\alpha_n \downarrow 0)$ *be a sequence having the properties required in Theorem* 1. *Let* $v_0 = 0$. *A necessary and sufficient condition for*

(11) $$E(F_{u,v_0}, \alpha, p) \subset L^q$$

is

(12) $$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \alpha_n^q < \infty.$$

Theorem 3. *Let* Ω *be a modulus of continuity,* $1 \leqq p \leqq q < \infty$ *and let* (φ_n) *be a sequence having the properties required in Theorem* 1. *Let* $\Phi = \Phi_{p,q,\lambda}$ *with* $\lambda = 1 - 1/u$. *A necessary condition for*

(13) $$H_p^{\Omega,\omega^*} \subset L^{q+(1-1/u)(1-q/p)}\Phi(L)$$

is

(14) $$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \varphi_n \Omega^q(n^{-(1-1/u)}) < \infty.$$

Theorem 4. *Let* $1 \leqq p < q < \infty$ *and let* $\omega_0^* = \omega_{A,B}(u, v_0, f, \delta)_p$ *with* $v_0 = 0$. *A necessary and sufficient condition for*

(15) $$H_p^{\Omega,\omega_0^*} \subset L^q$$

is

(16) $$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \Omega^q(n^{-(1-1/u)}) < \infty.$$

2. Lemmas

Lemma 1 ([6], Lemma 5). *Let* $p>0$ *and let* $(\alpha_n\downarrow 0)$ *be a sequence satisfying* (8). *Let* (φ_n) *be a nonnegative monotonic sequence having the property that for a certain* α

$$\sum_{k=m}^{\infty}\frac{\varphi_k}{k^{\alpha+p}}\leqq c\frac{m\varphi_m}{m^{\alpha+p}} \tag{17}$$

and

$$\sum_{k=1}^{\infty}\varphi_k k^{-\alpha}\alpha_k^p=\infty. \tag{18}$$

Then there exists a sequence $\{B_k\}$ *such that*

$$B_k\downarrow 0,\quad B_k\leqq\alpha_k,\quad \sum_{k=1}^{m}k^{\lambda p-1}B_k^p\leqq c(\lambda,p)\,m^{\lambda p}\alpha_m^p\quad \textit{for any}\quad \lambda>0 \tag{19}$$

and

$$\sum_{k=1}^{\infty}\varphi_k k^{-\alpha}B_k^p=\infty. \tag{20}$$

This lemma differs from Lemma 5 of [6] in the rate of λ, since the last inequality in (19) is true for any $\lambda>0$ (in [6] this inequality was proved for $\lambda=1$). Indeed, the sequence $\{B_k\}$ defined in [6] has property (19). The proof of this fact is similar to that of the last inequality in (2.4) of [6].

We have similar remark concerning the inequality (25) in the following lemma.

Lemma 2 ([6], Lemma 6). *Let* $p\geqq 1$, $\alpha<1$ *and let* $(\alpha_n\downarrow 0)$ *be a sequence satisfying* (8). *If for the positive increasing sequence* (φ_n),

$$\sum_{k=m}^{\infty}\varphi_k k^{-\alpha-p}\leqq c\varphi_m m^{1-\alpha-p} \tag{21}$$

and

$$\sum_{n=1}^{\infty}\varphi_n n^{-\alpha}\alpha_n^p=\infty \tag{22}$$

hold, then there exist a sequence $\{B_k\}$ *and a sequence of integers* $\{n_k\}$ *such that*

$$B_n\downarrow 0,\quad B_n\leqq\alpha_n \tag{23}$$

$$n_{k+1}>2n_k\quad \textit{and}\quad B_{n_{k+1}}\leqq\frac{1}{2}B_{n_k}\quad (k\geqq 1) \tag{24}$$

$$\sum_{n=1}^{m}n^{\lambda q-1}B_n^q\leqq c(q,\lambda)\,m^{\lambda q}\alpha_m^q\quad \textit{for any}\quad q,\lambda>0 \tag{25}$$

$$\sum_{n=1}^{\infty}\varphi_n n^{-\alpha}B_n^p=\infty\quad \textit{and}\quad \sum_{k=1}^{\infty}\varphi_{n_k}n_k^{1-\alpha}B_{n_k}=\infty \tag{26}$$

and

(27)
$$\sum_{k=1}^{\infty} \varphi_{2^n} 2^{n(1-\alpha)} (B_{2^n} - B_{2^{n-1}})^p = \infty.$$

Lemma 3. *Let* $1 \leqq p \leqq q < \infty$, $\lambda \geqq 1/2$ *and let* (α_n), (φ_n) *be sequences having the properties required in Theorem* 1. *If*

(28)
$$\sum_{n=1}^{\infty} n^{\lambda(q/p-1)-1} \varphi_n \alpha_n^q = \infty$$

then there exists a function $f_0 \in L^p[0, 1]$ *having the following properties:*

(29)
$$f_0(x) = 0, \quad x \in [2^{-\lambda}, 1]$$

(30)
$$\int_0^h |f_0(x)|^p \, dx \leqq c\alpha_{2^k}^p \quad (0 < h \leqq 2^{-\lambda(k+2)}, \ k = 1, 2, \ldots)$$

(31)
$$\omega(f_0, 2^{-\lambda k})_p \leqq c\alpha_{2^k}, \quad k = 1, 2, \ldots$$

and

(32)
$$f_0 \notin L^{q+\lambda(1-q/p)} \Phi(L),$$

where $\Phi = \Phi_{p,q,\lambda}$ *is defined by* (7).

Proof. First we remark that in the case $\lambda = 1$ this lemma was proved in [6], [7]. Here we use a similar method for the construction of f_0.

If $q=p$ then the conditions of Lemma 1 with $\alpha = 1$ are satisfied, so there exists a sequence $\{\bar{B}_k\}$ satisfying (19) and (20) with $\alpha = 1$.

If $q > p$ then the conditions of Lemma 2 are satisfied with $\alpha = 1 - \lambda\left(\frac{q}{p} - 1\right)$ and the exponent p appearing in Lemma 2 is chosen to be q. Therefore there exist $\{\hat{B}_k\}$ and $\{n_k\}$ satisfying (23)—(27).

Now we can define

(33)
$$f_0(x) = \begin{cases} \varrho_n & \text{if} \quad x = 3^\lambda 2^{-\lambda(n+2)} \\ 0 & \text{if} \quad x \in [2^{-\lambda}, 1], \quad x = 2^{-\lambda n} \\ \text{linear} & \text{on} \quad [2^{-\lambda(n+1)}, 3^\lambda 2^{-\lambda(n+2)}], \\ & \qquad [3^\lambda 2^{-\lambda(n+2)}, 2^{-\lambda n}], \quad n = 1, 2, \ldots, \end{cases}$$

where

(34)
$$\varrho_n := 2^{(n+1)\lambda/p} (B_{2^n}^p - B_{2^{n+1}}^p)^{1/p} \quad (n = 1, 2, \ldots)$$

with

$$B_n := \begin{cases} \bar{B}_n & \text{if} \quad p = q \\ \hat{B}_n & \text{if} \quad p < q. \end{cases}$$

Let

(35) $$\sigma := \frac{\sqrt{2}-1}{2}$$

and let

(36) $$h \in (\sigma 2^{-\lambda(k+3)}, \sigma 2^{-\lambda(k+2)}], \quad k \geqq 2.$$

Then it is easy to see that

$$0 < (4^\lambda - 1)h < 1-h.$$

We have

$$\int_0^{1-h} |f_0(t+h)-f_0(t)|^p\,dt = \int_0^{(4^\lambda-1)h} + \int_{(4^\lambda-1)h}^{1-h} =: I_1 + I_2.$$

By (19) and (23) we get

$$I_1 \leqq \int_0^{4^\lambda h} |f_0(x)|^p\,dx \leqq \int_0^{2^{-\lambda k}} |f_0(x)|^p\,dx = \sum_{n=k}^{\infty} \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(x)|^p\,dx =$$

$$= (1-2^{-\lambda}) \sum_{n=k}^{\infty} 2^{-\lambda(n+1)} \varrho_n^p = (1-2^{-\lambda}) \sum_{n=k}^{\infty} (B_p^{2^n} - B_{2^{n+1}}^p) \leqq$$

$$\leqq c(\lambda) B_{2^k}^p \leqq c(\lambda) \alpha_{2^k}^p.$$

To estimate I_2 we notice that by (35) and (36) we have for every $t \leqq 2^{-\lambda n}$, $1 \leqq n \leqq$ $\leqq k+1$

$$t+h \leqq 2^{-\lambda(n-1)}.$$

Therefore for those values of t

$$|f_0(t+h)-f_0(t)| \leqq (\varrho_n + \varrho_{n-1}) \leqq ch 2^{\lambda(n+2)} (\varrho_n + \varrho_{n-1}).$$

Now, using (19), (23) and (25) we have

$$I_2 = \int_{(4^\lambda-1)h}^{1-h} |f_0(t+h)-f_0(t)|^p\,dt \leqq \int_{2^{-\lambda(k+2)}}^{2^{-\lambda}} |f_0(t+h)-f_0(t)|^p\,dt =$$

$$= \sum_{n=1}^{k+1} \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(t+h)-f_0(t)|^p\,dt \leqq c \sum_{n=1}^{k+1} 2^{-\lambda n} [h 2^{\lambda(n+2)} (\varrho_n + \varrho_{n-1})]^p \leqq$$

$$\leqq ch^p \sum_{n=1}^{k+1} 2^{\lambda(p-1)n} (\varrho_n + \varrho_{n-1})^p \leqq ch^p \sum_{n=0}^{k+1} 2^{\lambda(p-1)n} \varrho_n^p =$$

$$= ch^p \sum_{n=0}^{k+1} 2^{\lambda pn} (B_{2^n}^p - B_{2^{n+1}}^p) \leqq ch^p \sum_{n=0}^{k+1} 2^{\lambda pn} B_{2^n}^p \leqq$$

$$\leqq ch^p \sum_{i=1}^{2^{k+1}} i^{\lambda p-1} B_i^p \leqq ch^p (2^{k+1})^{\lambda p} \alpha_{2^{k+1}}^p \leqq c\alpha_{2^k}^p.$$

So $I_1+I_2\leqq c\alpha_{2^k}^p$, from which (31) follows. (29) follows from the definition of f_0. We obtain (30) by the estimate of I_1.

Now let us prove (32). If $q=p$ then the function $\Phi_{p,q,\lambda}$ and the sequence $\{B_n\}$ do not depend on λ, therefore we can use the estimates on p. 61 of [6]. According to this, for $N=1, 2, \dots$, there exists μ depending on N such that

$$(37)\qquad \sum_{k=1}^{N} \varphi_k k^{-1} B_k^p \leqq c \sum_{n=1}^{\mu-1} \Phi(2^n)(B_{2^n}^p - B_{2^{n+1}}^p) + c \leqq$$

$$\leqq c \sum_{k=1}^{\mu} \Phi(2^n) \varrho_n^p 2^{-\lambda(n+1)} + c.$$

Since by our assumption and (20) the first sum in inequality (37) tends to infinity as $N\to\infty$, therefore

$$(38)\qquad \sum_{n=1}^{\mu} \Phi(2^n) \varrho_n^p 2^{-\lambda n} \to \infty \quad (\mu\to\infty).$$

On the other hand, by the assumption $\varphi_{k^2}\leqq c\varphi_k$ we have $\Phi(u^2)\leqq c\Phi(u)$. Consequently, since $\lambda\geqq 1/2$ we get

$$\Phi(2^n) \leqq c\Phi(2^{\lambda n}).$$

Hence by (38) we have

$$(39)\qquad \sum_{n=1}^{\infty} \Phi(2^{\lambda n}) \varrho_n^p 2^{-\lambda n} = \infty.$$

However,

$$\int_0^1 |f_0(x)|^p \Phi\left(\frac{1}{x}\right) dx = \sum_{n=0}^{\infty} \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(x)|^p \Phi\left(\frac{1}{x}\right) dx \geqq$$

$$\geqq \sum_{n=0}^{\infty} \Phi(2^{\lambda n}) \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(x)|^p dx \geqq c \sum_{n=0}^{\infty} \Phi(2^{\lambda n}) \varrho_n^p 2^{-\lambda n}.$$

So by (39)

$$(40)\qquad \int_0^1 |f_0(x)|^p \Phi\left(\frac{1}{x}\right) dx = \infty.$$

This together with the property $\Phi(u^2)\leqq c\Phi(u)$ implies by Lemma 13 of [10] that $f_0 \notin L^p\Phi(L)_{[0,1]}$, which proves (32) for $q=p$.

Let now $q>p$. Using the assumption $\varphi_{k^2}\leqq c\varphi_k$ we have

$$\Phi(x) = \sum_{k=1}^{[x]} k^{\lambda(q/p-1)-1}\varphi_k \geqq \sum_{k=[x/2]}^{[x]} k^{\lambda(q/p-1)-1}\varphi_k \geqq c x^{\lambda(q/p-1)} \varphi_{[x]}.$$

Therefore

$$\int_0^1 |f_0(x)|^{q+\lambda(1-q/p)} \Phi(|f_0(x)|)\, dx \geqq c \int_0^1 |f_0(x)|^q \varphi(|f_0(x)|)\, dx, \tag{41}$$

where

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ \varphi_n & \text{if } x = n \\ \text{linear on} & [n, n+1], \quad n = 1, 2, \dots \end{cases}$$

Using Lemma 2 with $\alpha = 1 - \lambda(q/p - 1)$ we have

$$\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)} (B_{2^n} - B_{2^{n+1}})^q = \infty. \tag{42}$$

On the other hand

$$\int_0^1 |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx = \sum_{n=1}^{\infty} \int_{2^{-n\lambda}}^{-(n-1)\lambda} |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx \geqq$$

$$\geqq c(\lambda) \sum_{n=1}^{\infty} \varphi(2^{n\lambda}) \varrho_n^q 2^{-n\lambda} \geqq c(\lambda) \sum_{n=1}^{\infty} \varphi_{2^n} \varrho_n 2^{-n\lambda} \geqq$$

$$\geqq c(\lambda) \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)} (B_{2^n}^p - B_{2^{n+1}}^p)^{q/p} \geqq c(\lambda) \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)} (B_{2^n} - B_{2^{n+1}})^q.$$

Hence by (42) we get

$$\int_0^1 |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx = \infty.$$

Therefore again using Lemma 13 of [10] we have

$$\int_0^1 |f_0(x)|^q \varphi(|f_0(x)|)\, dx = \infty$$

so, by (41)

$$f_0 \notin L^{q+\lambda(1-q/p)} \Phi(L).$$

This completes the proof of Lemma 3.

Lemma 4 ([7], Theorem 3.1). *Let* $v_0 = 0$, $u \geqq 2$, $n = 1, 2, \dots$. *Then for any* $p_n \in \Pi_n(F_{u, v_0})$ *and* $1 \leqq p < q < \infty$ *we have*

$$\|p_n\|_q \leqq c n^{(1-1/u)(1/p-1/q)} \|p_n\|_p. \tag{43}$$

Lemma 5 ([1], Lemma 3.6 and [2] Lemma 4.7). *Let* $1 \leqq p < \infty$. *Suppose that a function g is absolutely continuous on every finite interval and* $f := wg$, $wg' \in L^p$, *then*

$$E_n(F_{u,v}, f)_p \leqq \frac{c}{n^{1-1/u}} \|wg'\|_p \quad (n = 1, 2, \ldots). \tag{44}$$

Lemma 6. *Let* $1 \leqq p < \infty$. *For any* $f \in L^p$ *and* $-\infty < A < B < \infty$ *we have*

$$E_n(F_{u,v}, f)_p \leqq c(A, B)\, \omega^*_{A,B}(f, n^{-(1-1/u)})_p \quad (n = 1, 2, \ldots) \tag{45}$$

where ω^* *is defined in* (5).

Proof. The existence of ω^* indeed follows from the following inequalities

$$\begin{cases} w(x) \leqq c(B, \delta)\, w(x+h) & (-\infty < x \leqq B) \\ w(x) \leqq c(A, \delta)\, w(x-h) & (A \leqq x < \infty) \end{cases} \quad (\delta > 0,\ 0 < h \leqq \delta). \tag{46}$$

Let now

$$\lambda_n := n^{-(1-1/u)}, \quad f_p := w^{-p} f.$$

By Minkowskii-inequality we have

$$\Big\{ \int_A^B \Big| 2\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} [f_p(x+t) - f_p(x-t)]\, dt \Big|^p dx \Big\}^{1/p} \leqq$$

$$\leqq 2c\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} \Big\{ \int_A^B |f_p(x+t) - f_p(x-t)|^p w^p(x)\, dx \Big\}^{1/p} dt \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p.$$

Hence, it follows that there exists an $A \leqq x_n \leqq B$ such that

$$|d_n| \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p$$

with

$$d_n := 2\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} [f_p(x_n + t) - f_p(x_n - t)]\, dt.$$

Let

$$\varphi_n(x) := \begin{cases} 2\lambda_n^{-1} \displaystyle\int_{\lambda_n/2}^{\lambda_n} f_p(x+t)\, dt & \text{if} \quad x \leqq x_n \\ 2\lambda_n^{-1} \displaystyle\int_{\lambda_n/2}^{\lambda_n} f_p(x-t)\, dt + d_n & \text{if} \quad x > x_n. \end{cases}$$

Then it is easy to see that

$$\|(f_p - \varphi_n) w\|_p \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p$$

and

$$\|\varphi_n' w\|_p \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p \cdot \lambda_n^{-1}.$$

Since φ_n is absolutely continuous on every finite interval, by the last two inequalities, using Lemma 6 we get

$$E_n(F_{u,v},f)_p \leqq \|(f_p-\varphi_n)w\|_p+E_n(F_{u,v},\varphi_n w) \leqq$$
$$\leqq c(A,B)\,\omega^*_{A,B}(f,\lambda_n)_p+c(A,B)\lambda_n\|\varphi_n' w\|_p \leqq$$
$$\leqq c(A,B)\,\omega^*_{A,B}(f,\lambda_n)_p,$$

which proves (45).

4. Proofs of the theorems

Proof of Theorem 1. Suppose that

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1}\varphi_n\alpha_n^q=\infty. \tag{47}$$

Then by Lemma 3 there exists a function $f_0\in L^p[0,1]$ satisfying (29)—(32) with $\lambda=1-\frac{1}{u}\left(\geqq\frac{1}{2}\right)$.

We define

$$f(x)=\begin{cases} df_0(x)\,w^p(x) & \text{if} \quad x\in[0,1] \quad \text{with} \quad d=e^{p/2},\\ 0, & \text{if} \quad x\notin[0,1],\end{cases}$$

and estimate $\omega^*_{A,B}(f,\delta)_p$ with $A=2, B=3$. By $\lambda:=1-\frac{1}{u}$ we have for $1-h\geqq 2^{-\lambda}$

$$I_1(h):=\int_{-\infty}^{3}|f_p(x+h)-f_p(x)|^p w^p(x)\,dx\leqq$$
$$\leqq c\int_0^h|f_0(x)|^p\,dx+c\int_0^{1-h}|f_0(x+h)-f_0(x)|^p\,dx.$$

Hence by (30), (31) we get

$$I_1(h)\leqq c\alpha_{2^k}^p \quad \text{if} \quad h\leqq 2^{-\lambda(k+2)}, \quad k=1,2,\ldots.$$

Therefore by the definition of ω^* we have

$$\omega^*_{2,3}(f,2^{-\lambda k})_p\leqq c\alpha_{2^k} \quad (k=1,2,\ldots),$$

from which it follows by (45) that

$$E_{2^k}(F_{u,v},f)_p\leqq c\alpha_{2^k} \quad (k=1,2,\ldots).$$

Since $n\alpha_n \leqq cm\alpha_m$ for $1 \leqq n < m$, we obtain

$$E_n(F_{u,v}, f)_p \leqq c\alpha_n \quad n = 1, 2, \ldots),$$

too. This proves that $f \in E(F_{u,v}, \alpha, p)$.

On the other hand, since $f(x) \geqq f_0(x)$ $(x \in [0, 1])$, by (32) we have

$$f \notin L^{q+\lambda(1-q/p)} \Phi(L).$$

The proof of Theorem 1 is completed.

Remark 1. In the proof of Theorem 1 the chosen values of constants A and B in ω^* indeed are not essential. For any $-\infty < A < B < \infty$ by similar method we can construct a function f such that $f \notin L^{q+\lambda(1-q/p)} \Phi(L)$ and

$$\omega^*_{A,B}(f, 2^{-\lambda k})_p \leqq c\alpha_{2^k} \quad (k = 1, 2, \ldots).$$

Proof of Theorem 2. If $\varphi_n = 1$ $(n = 0, 1, \ldots)$ then

$$L^{q+\lambda(1-q/p)} \Phi_{p,q,\lambda}(L) = L^q.$$

Therefore the necessary part in Theorem 2 follows from Theorem 1. The sufficient part is a consequence of the statement summarized in the introduction, since by (43) F_{u,v_0} is a $\left\{N, \left(1 - \frac{1}{u}\right)\right\}$-system.

Proof of Theorem 3. Assume that series (14) is divergent. Then by virtue of Remark 1, with $\alpha_n := \Omega(n^{-(1-1/u)})$, we can construct a function $f \in L^p$ such that $f \notin L^{q+\lambda(1-q/p)} \Phi(L)$ and

$$\omega^*_{A,B}(f, 2^{-(1-1/u)k})_p \leqq c\Omega(2^{-(1-1/u)k}) \quad (k = 1, 2, \ldots).$$

Hence by the properties of the modulus of continuity it follows that

$$\omega^*_{A,B}(f, \delta)_p \leqq c\Omega(\delta) \quad (\delta > 0).$$

So, we have $f \in H_p^{\Omega, \omega^*}$.

Proof of Theorem 4. The necessary part of Theorem 4 is a consequence of Theorem 3. The sufficient part follows from Theorem 2 and Lemma 6.

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