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# Uniform boundedness theorems for $k$-triangular set functions

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In a recent paper, we have obtained a generalization of the classical boundedness Dieudonné theorem ([9], Prop. 9), in the setting of finitely additive group valued-functions ([14], (3.2)).

The purpose of this paper is to obtain an analogous result ((3.3)), in the setting of semigroup valued $k$-triangular functions. For this, firstly we establish that Nikodym's boundedness theorem holds for $k$-triangular exhaustive functions on a ring with the Subsequential Interpolation Property ((1.6)). This proposition yields some recent results of E. Pap as special cases (see [21], [23], [24]). We apply (3.3) to obtain again a Dieudonné type theorem for finitely additive group-valued functions (Corollary (3.8), see also [14], (4.2)).

**1.** Let $X$ be a commutative semigroup with neutral element 0; let $p$ be a semi-invariant pseudometric on $X$, namely a pseudometric satisfying the inequality

$$p(x+z, y+z) \leqq p(x, y) \quad \forall x, y, z \in X,$$

or, equivalenty, the inequality

$$p(x+x', y+y') \leqq p(x, y)+p(x', y') \quad \forall x, x', y, y' \in X.$$

Let $\mathbf{R}^+=[0, +\infty[$, $\overline{\mathbf{R}}^+=[0, +\infty]$. To $p$ there corresponds the function

$$|\ |\colon x \in X \to p(x, 0) \in \mathbf{R}^+$$

for which

$$|0| = 0$$

$$\big||x|-|y|\big| \leqq |x+y| \leqq |x|+|y| \quad \forall x, y \in X,$$

([22], [23]).

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We will denote by $(X, |\ |)$ the uniform semigroup $(X, \mathscr{U}_p)$, where $\mathscr{U}_p$ is the uniformity of $X$ generated by the pseudometric $p$ ([11], [27]). We say that a subset $Y$ of $X$ is *bounded* if $\sup_{y \in Y} |y| < +\infty$.

Let $\mathscr{R}$ be a ring of subsets of a set $S$ and $\varphi$ a function from $\mathscr{R}$ to $(X, |\ |)$. We say that $\varphi$ is *bounded* if the set $\varphi(\mathscr{R})$ is a bounded subset of $X$.

Let $k \in \mathbf{R}^+$. We say that $\varphi$ is *k-triangular* if $\varphi(\emptyset) = 0$ and for any disjoint sets $A$ and $B$ from $\mathscr{R}$,

$$|\varphi(A)| - k|\varphi(B)| \leqq |\varphi(A \cup B)| \leqq |\varphi(A)| + k|\varphi(B)|.$$

It is easy to see that $\varphi$ is $k$-triangular if and only if

$$\varphi(\emptyset) = 0$$

and, for any sets $C$, $D$ from $\mathscr{R}$, we have

$$\big||\varphi(C)| - |\varphi(D)|\big| \leqq k|\varphi(C \setminus D)| + k|\varphi(D \setminus C)|$$

([16]).

Moreover, a function $\varphi$ $k'$-triangular is $k$-triangular for each $k \geqq k'$ and $\varphi$ $k$-triangular for $k \in ]0, 1[$ implies $|\varphi(X)| = 0$ for each $X \in \mathscr{R}$. Hence below we will consider $k$-triangular functions with $k \geqq 1$.

Let $\mathscr{G}$ be a lattice contained in $\mathscr{R}$; we say that a function $\varphi$ from $\mathscr{R}$ to $(X, |\ |)$ is *$\mathscr{G}$-exhaustive* if, for every disjoint sequence $(G_n)_{n \in N}$ in $\mathscr{G}$, we have

$$\lim_n \varphi(G_n) = 0;$$

an $\mathscr{R}$-exhaustive function is called exhaustive.

We say that a function $\varphi$ from $\mathscr{R}$ to $(X, |\ |)$ is *order continuous* if for every decreasing sequence $(A_n)_{n \in N}$ in $\mathscr{R}$ such that $\bigcap_{n \in N} A_n = \emptyset$

$$\lim_n \varphi(A_n) = 0.$$

We write, for every $\mathscr{H} \subseteqq \mathscr{R}$ and $A \in \mathscr{R}$,

$$\mathscr{H}_A = \{H \in \mathscr{H} \colon H \subseteqq A\}$$

$$\mathscr{H} \cap A = \{H \cap A,\ H \in \mathscr{H}\}.$$

Let $\varphi$ be a function from $\mathscr{R}$ to $(X, |\ |)$; its *semivariation* (suprematition or supremacy in [15], [16]) is the function

$$\tilde{\varphi}\colon A \in \mathscr{R} \to \sup_{B \in \mathscr{R}_A} |\varphi(B)| \in \overline{\mathbf{R}}^+$$

([11], [12], [19]).

We have

$$\tilde{\varphi}(\emptyset) = 0 \quad \text{if} \quad |\varphi(\emptyset)| = 0,$$

$$|\varphi(A)| \leqq \tilde{\varphi}(A) \quad \forall A \in \mathscr{R},$$

$$A \subseteqq B \Rightarrow \tilde{\varphi}(A) \leqq \tilde{\varphi}(B).$$

Moreover $\tilde{\varphi}$ is $k$-subadditive if the function

$$A \in \mathscr{R} \to |\varphi(A)| \in \mathbf{R}^+$$

is $k$-subadditive [1]); $\tilde{\varphi}$ is exhaustive iff $\varphi$ is exhaustive ([12], Lemma (2.2)).

Now, we give the proof of:

(1.1). *Let $\mathscr{R}$ be a ring of subsets of $S$ and $\varphi$ a $k$-triangular and exhaustive function from $\mathscr{R}$ to $(X, |\ |)$. Then $\varphi$ (and therefore $\tilde{\varphi}$) is bounded*[2]).

Suppose the contrary. Then by Lemma (2.1) of [19] we can find $A_0 \in \mathscr{R}$ such that for every $A \in \mathscr{R}$

$$|\varphi(A \setminus A_0)| \leqq 1.$$

Therefore the set $\varphi(\mathscr{R}_{A_0})$ is not bounded[3]) and we can find $B_1 \in \mathscr{R}_{A_0}$ such that

$$|\varphi(B_1)| > k + |\varphi(A_0)|.$$

Hence we have also

$$|\varphi(A_0 \setminus B_1)| \geqq \frac{1}{k} \big| |\varphi(B_1)| - |\varphi(A_0)| \big| > 1$$

and or $\varphi(\mathscr{R}_{B_1})$ or $\varphi(\mathscr{R}_{A_0 \setminus B_1})$ is not bounded.

Then we write $A_1 = B_1$ and $C_1 = A_0 \setminus B_1$ if $\varphi(\mathscr{R}_{B_1})$ is not bounded; on the contrary, we write $A_1 = A_0 \setminus B_1$ and $C_1 = B_1$.

It is clear now that we can obtain, as in [19], Theorem (2.2), a sequence $(C_n)_{n \in N}$ of mutually disjoint sets of $\mathscr{R}$ such that

$$|\varphi(C_n)| > 1 \quad \forall n \in \mathbf{N},$$

a contradiction with the assumption that $\varphi$ is exhaustive.

(1.2). *Let $\mathscr{R}$ be a ring of subsets of $S$ and let $\varphi$ an order continuous function from $\mathscr{R}$ to $(X, |\ |)$. If the function*

$$A \in \mathscr{R} \to |\varphi(A)|$$

*is $k$-subadditive, this function and the semivariation of $\varphi$, $\tilde{\varphi}$, are also countably $k$-subadditive*[4]).

Let $(A_n)_{n\in N}$ be a disjoint sequence of elements of $\mathscr{R}$ such that $\bigcup_{n\in\mathbf{N}} A_n\in\mathscr{R}$. Then, for every $n\in N$ $(n\geqq 2)$ and for every $A\in\mathscr{R}$,

$$\left|\varphi\left(\bigcup_{n\in\mathbf{N}} A_n\cap A\right)\right| \leqq \left|\varphi\left(\bigcup_{i\leqq n} A_i\cap A\right)\right| + k\left|\varphi\left(\bigcup_{i>n} A_i\cap A\right)\right| \leqq$$

$$\leqq \varphi(A_1\cap A) + k\sum_{1<i\leqq n}|\varphi(A_i\cap A)| + k\left|\varphi\left(\bigcup_{i>n} A_i\cap A\right)\right|.$$

Taking limits in the above inequality, we obtain, for each $A\in\mathscr{R}$,

$$\left|\varphi\left(\bigcup_{n\in\mathbf{N}} A_n\cap A\right)\right| \leqq |\varphi(A_1\cap A)| + k\sum_{n\geqq 2}|\varphi(A_n\cap A)|,$$

and also

$$\tilde{\varphi}\left(\bigcup_{n\in\mathbf{N}} A_n\right) \leqq \tilde{\varphi}(A_1) + k\sum_{n\geqq 2}\tilde{\varphi}(A_n);$$

this completes the proof.

Corollary (1.2). *If $\varphi$ is an order continuous k-subadditive function defined on the ring $\mathscr{R}$ with values in $\mathbf{R}^+$, then $\varphi$ and its semivariation $\tilde{\varphi}$ are also countably k-subadditive.*

(1.3). *Let $\mathscr{R}$ be a quasi $\sigma$-ring of subsets of $S$ and let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of exhaustive functions from $\mathscr{R}$ to $(X, |\ |)$. Then, for each disjoint sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathscr{R}$, there exist a subsequence $(A_{n_r})_{r\in\mathbf{N}}$ of $(A_n)_{n\in\mathbf{N}}$ and a quasi $\sigma$-ring $\mathscr{S}$ contained in $\mathscr{R}$ such that $A_{n_r}\in\mathscr{S}$ for each $r\in\mathbf{N}$, such that for every $n\in\mathbf{N}$ the restriction of $\varphi_n$ to $\mathscr{S}$ is order continuous*[5]).

Let, for each $n\in\mathbf{N}$, $\tilde{\varphi}_n$ be the semivariation of $\varphi_n$ and let

$$\eta\colon A\in\mathscr{R} \to \sum_{n\in\mathbf{N}}\frac{1}{2^n}\inf\{1, \tilde{\varphi}_n(A)\}\in\mathbf{R}^+;$$

it is easy to see that $\eta$ is an exhaustive function such that

$$\eta(A)\leqq\eta(B) \quad \text{if} \quad A\subseteqq B, \quad A, B\in\mathscr{R}\ ^{6)}.$$

Let $(A_n)_{n\in\mathbf{N}}$ be a disjoint sequence of sets of $\mathscr{R}$; then we can find a subsequence $(A_{n_r})_{r\in\mathbf{N}}$ of $(A_n)_{n\in\mathbf{N}}$ and a quasi $\sigma$-ring $\mathscr{S}$ contained in $\mathscr{R}$ such that $A_{n_r}\in\mathscr{S}$, for each $r\in\mathbf{N}$, and the restriction of $\eta$ to $\mathscr{S}$ is order continuous[7]).

Hence, if $(B_p)_{p\in\mathbf{N}}$ is a decreasing sequence of sets of $\mathscr{S}$ such that $\bigcap_{p\in\mathbf{N}} B_p=\emptyset$, we have, for each $n\in\mathbf{N}$,

$$\lim_p \varphi_n(B_p\cap B) = 0,$$

uniformly with respect to $B\in\mathscr{S}$; namely, for each $n\in\mathbf{N}$, the restriction of $\varphi_n$ to $\mathscr{S}$ is order continuous.

(1.4). *Let $\mathscr{R}$ be a ring of subsets of $S$ and let $\Phi$ be a set of k-triangular functions from $\mathscr{R}$ to $(X, |\ |)$, such that*

a) *$\Phi(A)$ is bounded for every $A \in \mathscr{R}$,*

b) *for every sequence $(\varphi_n)_{n\in\mathbf{N}}$ of elements of $\Phi$ and for every disjoint sequence $(A_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{R}$ there exists an infinite subset $M$ of* $\mathbf{N}$ *such that $\bigcup_{n\in M}\{\varphi_n(A_n)\}$ is bounded.*

*Then $\Phi(\mathscr{R})$ is bounded.*[8]

Suppose the contrary. Then there are two possibilities.

*Case* I: There exists $A \in \mathscr{R}$ such that $\Phi(\mathscr{R}_A)$ is not bounded.
In this case, firstly we prove:

c) *for every $A \in \mathscr{R}$ such that $\Phi(\mathscr{R}_A)$ is not bounded and for every $n \in \mathbf{N}$, there exists $(\varphi, B) \in \Phi \times \mathscr{R}_A$ such that*

$$|\varphi(B)| > n \quad \textit{and} \quad \Phi(\mathscr{R}_B) \textit{ is not bounded.}$$

In fact, suppose that there exist $n_0 \in \mathbf{N}$ and $A_0 \in \mathscr{R}$ such that $\Phi(\mathscr{R}_{A_0})$ is not bounded such that for every $(\varphi, B) \in \Phi \times \mathscr{R}_{A_0}$, $|\varphi(B)| > n_0$ implies that $\Phi(\mathscr{R}_B)$ is bounded. Let $(\bar{\varphi}, B) \in \Phi \times \mathscr{R}_{A_0}$ such that $|\bar{\varphi}(B)| > 2kn_0$; therefore

$$|\bar{\varphi}(B)| > n_0 \quad \text{and} \quad |\bar{\varphi}(A_0 \setminus B)| > n_0.$$

Hence, $\Phi(\mathscr{R}_B)$ and $\Phi(\mathscr{R}_{A_0\setminus B})$ being bounded, $\Phi(\mathscr{R}_{A_0})$ is bounded, a contradiction.

Let now $A_1 \in \mathscr{R}$ such that $\Phi(\mathscr{R}_{A_1})$ is not bounded and $r(1)$ such that

$$|\varphi(A_1)| \leqq r(1) \quad \forall \varphi \in \Phi;$$

by c) there exists $(\varphi_1, A_2) \in \Phi \times \mathscr{R}_{A_1}$ such that

$$|\varphi_1(A_2)| > k + r(1) \quad \text{and} \quad \Phi(\mathscr{R}_{A_2}) \text{ is not bounded.}$$

Continuing by induction, we can find a decreasing sequence $(A_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{R}_n$, a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions of $\Phi$ and a sequence $\big(r(n)\big)_{n\in\mathbf{N}}$ of natural numbers such that for every $n \in \mathbf{N}$,

$$|\varphi_n(A_n)| \leqq r(n), \quad |\varphi_n(A_{n+1})| > kn + r(n), \quad \Phi(\mathscr{R}_{A_n}) \text{ are not bounded.}$$

Finally, if we write $C_n = A_n \setminus A_{n+1}$ for each $n \in \mathbf{N}$, $(C_n)_{\in\mathbf{N}}$ is a disjoint sequence of sets of $\mathscr{R}$ such that

$$|\varphi_n(C_n)| > n \quad \forall n \in \mathbf{N},$$

a contradiction with b).

*Case* II: For every $A \in \mathscr{R}$ the set $\Phi(\mathscr{R}_A)$ is bounded.

In this case, if we denote by $\mathscr{R}'_A$ the ring of sets of $\mathscr{R}$ disjoint from $A$, we have that $\Phi(\mathscr{R}'_A)$ is not bounded, for each $A \in \mathscr{R}$. Then, we put $A_0 = \emptyset$ and we choose $\varphi_2 \in \Phi$ and $A_1 \in \mathscr{R}$ such that $|\varphi_1(A_1)| \geqq 1$. Continuing by induction, we find for every $n \in \mathbf{N}$, $\varphi_n \in \Phi$ and $A_n \in \mathscr{R}'_{\bigcup\limits_{1 \leqq i \leqq n-1} A_i}$ such that $|\varphi_n(A_n)| > n$, a contradiction with b). This completes the proof.

(1.5). *Let $\mathscr{R}$ be a quasi $\sigma$-ring of subsets of $S$ and let $\Phi$ be a set of $k$-triangular and exhaustive functions from $\mathscr{R}$ to $(X, |\ |)$. If for every $A \in \mathscr{R}$ the set $\Phi(A)$ is bounded, then $\Phi(\mathscr{R})$ is bounded.*

Suppose that $\Phi(\mathscr{R})$ is not bounded. Then, by (1.4), there exist a sequence $(\varphi_n)_{n \in \mathbf{N}}$ of functions of $\Phi$ and a disjoint sequence $(A_n)_{n \in \mathbf{N}}$ of sets of $\mathscr{R}$ such that for every infinite subset $M$ of $\mathbf{N}$ the set $\bigcup\limits_{n \in M} \{\varphi_n(A_n)\}$ is not bounded.

Let now, by (1.3), $(A_{n_i})_{i \in \mathbf{N}}$ be a subsequence of $(A_n)_{n \in \mathbf{N}}$ and $\mathscr{S}$ a quasi $\sigma$-ring contained in $\mathscr{R}$ such that $A_{n_i} \in \mathscr{S}$ $(\forall i \in \mathbf{N})$ and the restriction of $\varphi_n$ to $\mathscr{S}$ is order continuous, for each $n \in \mathbf{N}$.

Let $p_1$ be a positive real number and let $i_1 \in \mathbf{N}$ such that

$$|\varphi_{n_{i_1}}(A_{n_{i_1}})| > 2p_1;$$

$\varphi_{n_{i_1}}$ being exhaustive, we can find $h_1 > i_1$ such that

$$|\varphi_{n_{i_1}}(A_{n_m})| < p_1/2k^3 \quad \forall m > h_1.$$

We write, $\forall i \in \mathbf{N}$, $\alpha_i = k^2 \sup\limits_{\varphi \in \Phi} |\varphi(A_{n_i})| < +\infty$ and we put $p_2 = \max\{2p_1, \alpha_{i_1}\}$. Then there exist $i_2 > h_1$ and $h_2 > i_2$ such that

$$|\varphi_{n_{i_2}}(A_{n_{i_2}})| > 3p_2$$

and

$$|\varphi_{n_{i_1}}(A_{n_m})| \leqq p_1/2^2k^3, \quad |\varphi_{n_{i_2}}(A_{n_m})| \leqq p_1/2^2k^3 \quad \forall m \geqq h_2.$$

Similarly, if we write $p_s = \max\{sp_{s-1}, \alpha_{i_{s-1}}\}$ for each $s \in \mathbf{N}$, $(s > 1)$ we can find $h_{s-1} < i_s < h_s$ such that

$$|\varphi_{n_{i_s}}(A_{n_{i_s}})| \geqq (s+1)p_s$$

and, for each $r \in \{1, \ldots, s\}$,

$$|\varphi_{n_{i_r}}(A_{n_m})| < p_1/2^s k^3 \quad \forall m \geqq h_s.$$

Let now $(A_{n_{i_{s_q}}})_{q\in\mathbf{N}}$ be a subsequence of $(A_{n_{i_s}})_{s\in\mathbf{N}}$ such that $A_0=\bigcup_{q\in\mathbf{N}} A_{n_{i_{s_q}}}\in\mathcal{S}$; we obtain $(\forall q>1)$ from (1.2)

$$|\varphi_{n_{i_{s_q}}}(A_0)| \geqq |\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_q}}})| - k^2 \sum_{l<q} |\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_l}}})| - k^3 \sum_{l>q} |\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_l}}})| \geqq$$

$$\geqq (s_q+1)p_{s_q} - \sum_{l<q} \alpha_{i_{s_l}} - \sum_{l>q} p_l/2^{s_l-1} \geqq (s_q+1)p_{s_q} - \sum_{l<q} p_{s_l+1} - p_1 \geqq$$

$$\geqq s_q p_{s_q} - (q-1)p_{s_{q-1}} \geqq qp_1,$$

a contradiction with the boundedness of $\Phi(A_0)$.

(1.6). *Let $\mathcal{R}$ be a ring of subsets of $S$ with the Subsequential Interpolation Property*[9]) *and let $\Phi$ be a set of k-triangular and exhaustive functions from $\mathcal{R}$ to $(X, |\ |)$. If for every $A\in\mathcal{R}$ the set $\Phi(A)$ is bounded, then $\Phi(\mathcal{R})$ is bounded.*

We have to prove that b) of (1.4) is verified. Let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of functions of $\Phi$ and $(A_n)_{n\in\mathbf{N}}$ a disjoint sequence of sets of $\Phi$.

It is easy to prove that $\mathcal{N}=\{A\in\mathcal{R}\colon\ \tilde{\varphi}_n(A)=0\ \ \forall n\in\mathbf{N}\}$ is an ideal of $\mathcal{R}$[10]) and $\mathcal{R}/\mathcal{N}$ satisfies the countable chain condition[11]); therefore by the (7.1.1) of [28] $\mathcal{R}/\mathcal{N}$ is a quasi $\sigma$-ring.

Now, we denote for each $n\in\mathbf{N}$ by $\hat{\varphi}_n$ the function

$$[A]\in\mathcal{R}/\mathcal{N} \to |\varphi_n(A)|$$

and we note that, $\forall n\in\mathbf{N}$, $\hat{\varphi}_n$ is a $k$-triangular and exhaustive function from $\mathcal{R}/\mathcal{N}$ to $\mathbf{R}^+$[12]).

Therefore, by (1.5) the set $\bigcup_{n\in\mathbf{N}} \{\varphi_n(A_n)\}\subseteqq \bigcup_{n\in\mathbf{N}} \hat{\varphi}_n(\mathcal{R}/\mathcal{N})$ is bounded. The proof is complete.

Remark 1. We remark that (1.4) contains Theorem 1, p. 30 of [16] and (1.6) contains the Nikodym's boundedness Theorem of [20], Theorem $N$ of [18], Corollaries 4, 5, 6 p. 29 of [16].

We remark also that from (1.6) we obtain Corollary (Nikodym) of [1] and Theorem 2 of [21][13]).

2. We shall denote below by $\mathcal{A}$ a field of subsets of $S$ and bv $\mathcal{F}$ and $\mathcal{G}$ two lattices contained in $\mathcal{A}$ such that $S\setminus F\in\mathcal{G}$, for each $F\in\mathcal{F}$.

Let $\varphi$ be a function from $\mathcal{A}$ to $(X, |\ |)$; we say that $\varphi$ is *inner regular (with respect to $\mathcal{F}$)* in $A$, $A\in\mathcal{A}$, if for every $\varepsilon>0$ there exists $F\in\mathcal{F}$ such that $F\subseteqq A$ and $\tilde{\varphi}(A\setminus F)<\varepsilon$.

We say that $\varphi$ is *inner regular (with respect to $\mathcal{F}$)* on $\mathcal{H}$, $\mathcal{H}\subseteqq\mathcal{A}$, if $\varphi$ is inner regular (with respect to $\mathcal{F}$) in each $A\in\mathcal{H}$; $\varphi$ is said *inner regular* if it is inner regular on $\mathcal{A}$.

We note that:

(2.1). *Let $\varphi$ be a function from $\mathscr{A}$ to $(X, |\ |)$ such that the function*

$$A \in \mathscr{R} \to |\varphi(A)|$$

*is k-subadditive. Then $\varphi$ is inner regular if and only if it satisfies the condition*

(°) *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$ there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteq A \subseteq G$ and $\tilde{\varphi}(G \setminus F) < \varepsilon$*[14]).

It follows easily from the properties of $\tilde{\varphi}$[15]).

Remark 2. This proposition is valid, in particular, for an inner regular $k$-subadditive function defined on $\mathscr{A}$ with values in $\mathbf{R}^+$.

If $S$ is a Hausdorff locally compact topological space, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of $S$, $\mathscr{A}$ is a field containing $\mathscr{G}$, a function $\varphi$ from $\mathscr{A}$ to $(X, |\ |)$, such that the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is $k$-subadditive, is inner regular (with respect to $\mathscr{F}$) iff the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is regular $(R)$, in the sense of [8].

(2.2). *Let $\mathscr{F}$ be a semicompact lattice, so a lattice with the property:*

($*$) *For every sequence $(F_n)_{n \in \mathbf{N}}$ in $\mathscr{F}$ such that $\bigcap_{n \in \mathbf{N}} F_n = \emptyset$, there exists $n_0 \in \mathbf{N}$ such that $\bigcap_{n \leqq n_0} F_n = \emptyset$.*

*Let $\varphi$ be an inner regular (with respect to $\mathscr{F}$) function from $\mathscr{A}$ to $(X, |\ |)$; then,*

1) *if the function*

$$A \in \mathscr{A} \to |\varphi(A)|$$

*is k-subadditive, $\varphi$ is order continuous and therefore the function*

$$A \in \mathscr{A} \to |\varphi(A)|$$

*and the semivariation of $\varphi$, $\tilde{\varphi}$, are also countably k-subadditive;*

2) *if $\varphi$ is a k-triangular function, $\varphi$ is $\mathscr{H}$-exhaustive, for every lattice $\mathscr{H} \subseteq \mathscr{A}$ such that for every disjoint sequence $(H_n)_{n \in \mathbf{N}}$ in $\mathscr{H}$ the $\sigma$-ring generated by $\{H_n, n \in \mathbf{N}\}$ is contained in $\mathscr{A}$.*

To prove 1), by (1.2), it suffices to prove that $\varphi$ is order continuous. For this, if $(A_n)_{n \in \mathbf{N}}$ is a decreasing sequence of sets of ' such that $\bigcap_{n \in \mathbf{N}} A_n = \emptyset$, for any $\varepsilon > 0$ and $n \in \mathbf{N}$, let $F_n \in \mathscr{F}$ such that

$$F_n \subseteq A_n \quad \text{and} \quad \tilde{\varphi}(A_n \setminus F_n) < \varepsilon / 2^n k.$$

Then by $(*)$ there exists $m_0 \in \mathbf{N}$ such that

$$\bigcap_{i \leqq m} F_i = \emptyset \quad \forall m \geqq m_0;$$

hence for each $m \geqq m_0$,

$$|\varphi(A_m)| \leqq \tilde{\varphi}(A_m) = \tilde{\varphi}\big(A_m \setminus \bigcap_{i \leqq m} F_i\big) = \tilde{\varphi}\big(\bigcup_{i \leqq m} (A_i \setminus F_i)\big) \leqq$$

$$\leqq \tilde{\varphi}(A_1 \setminus F_1) + k \sum_{1 < i \leqq m} \tilde{\varphi}(A_i \setminus F_i) \leqq k \sum_{n \in \mathbf{N}} \varepsilon/2^n k = \varepsilon.$$

To prove 2), it suffices to remark that, by 1), $\varphi$ is order continuous and, if $(H_n)_{n \in \mathbf{N}}$ is a disjoint sequence in $\mathscr{H}$, for every $n \in \mathbf{N}$,

$$|\varphi(H_n)| \leqq \big|\varphi\big(\bigcup_{i \geqq n} H_i\big)\big| + k \big|\varphi\big(\bigcup_{i \geqq n+1} H_i\big)\big|.$$

This completes the proof.

(2.3). *Let $\mathscr{F}$ and $\mathscr{G}$ satisfy the property:*

$(**)$ *For each $F \in \mathscr{F}$ and for each sequence $(G_n)_{n \in \mathbf{N}}$ in $\mathscr{G}$ such that $F \subseteqq \bigcup_{n \in \mathbf{N}} G_n$ there exists $n_0 \in \mathbf{N}$ such that $F \subseteqq \bigcup_{n \leqq n_0} G_n$*[16]*), and let $\mathscr{G}$ be closed under the countable union of mutually disjoint sets.*

*If $\varphi$ is a function from $\mathscr{A}$ to $(X, |\ |)$ inner regular (with respect to $\mathscr{F}$) on $\mathscr{G}$, the semivariation of $\varphi$, $\tilde{\varphi}$, (and therefore $\varphi$) is $\mathscr{G}$-exhaustive.*

Let $(G_n)_{n \in \mathbf{N}}$ be a disjoint sequence in $\mathscr{G}$. For every $\varepsilon > 0$, let $F \in \mathscr{F}$ such that

$$F \subseteqq \bigcup_{n \in \mathbf{N}} G_n \quad \text{and} \quad \tilde{\varphi}\big(\bigcup_{n \in \mathbf{N}} G_n \setminus F\big) < \varepsilon;$$

hence, if $n_0 \in \mathbf{N}$ is such that

$$F \subseteqq \bigcup_{n \leqq n_0} G_n$$

for every $m \geqq n_0 + 1$

$$|\varphi(G_m)| \leqq \tilde{\varphi}(G_m) \leqq \tilde{\varphi}\big(\bigcup_{n \in \mathbf{N}} G \setminus F\big) < \varepsilon;$$

the proof is complete.

We remark also:

(2.4). *Let $\varphi$ be a k-triangular function inner regular (with respect to $\mathscr{F}$) defined on $\mathscr{A}$ with values in $(X, |\ |)$. Then $\varphi$ satisfies the condition*

$(^{\circ\circ})$ *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteqq A \subseteqq G$ and for every $A' \in \mathscr{A}$ such that $F \subseteqq A' \subseteqq G$ we have*

$$\big||\varphi(A)| - |\varphi(A')|\big| < \varepsilon.$$

Let $A \in \mathscr{A}$ and let $\varepsilon > 0$. Then, by (2.1), we can find $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that

$$F \subseteqq A \subseteqq G \quad \text{and} \quad |\varphi(B)| < \varepsilon/2k \quad \forall B \subseteqq G \setminus F.$$

If $A' \in \mathscr{A}$ and $F \subseteq A' \subseteq G$, then obviously

$$(A \setminus A') \cup (A' \setminus A) \subseteq G \setminus F$$

and therefore

$$\big| |\varphi(A)| - |\varphi(A')| \big| \leq k\, |\varphi(A \setminus A')| + k\, |\varphi(A' \setminus A)| < \varepsilon.$$

In particular, we have:

Corollary (2.4). *Let $\varphi$ be a $k$-triangular function defined on $\mathscr{A}$ with values in $\mathbf{R}^+$. If $\varphi$ is inner regular, $\varphi$ satisfies the condition*

($^{\circ\,\circ}_{\circ}$) *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, there exist $F \in \mathscr{F}_A$ and $A \subseteq G \in \mathscr{G}$ such that for every $A \in \mathscr{A}$ such that $F \subseteq A' \subseteq G$ we have*

$$|\varphi(A) - \varphi(A')| < \varepsilon.$$

Remark 3. If $S$ is a Hausdorff locally compact topological space, $\mathscr{A}$ is the $\sigma$-field of the Borel sets of $S$, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of $S$, from (2.4) (resp. from Corollary (2.4)) we obtain that, if $\varphi$ is an inner regular $k$-triangular function from $\mathscr{A}$ to $(X, |\ |)$ (resp. to $\mathbf{R}^+$), the function

$$A \in \mathscr{A} \to |\varphi(A)| \in \mathbf{R}^+$$

(resp. $\varphi$) is regular on $\mathscr{A}$ in the sense of [7], p. 303; see also Remark 2.

(2.5). *Let $(\Gamma, |\ |)$ be a quasi-normed abelian group and let $\varphi$ a finitely additive function from $\mathscr{A}$ to $(\Gamma, |\ |)$. Then $\varphi$ is inner regular if and only if, for every $A \in \mathscr{A}$ and for every $\varepsilon > 0$ there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteq A \subseteq G$ and for every $A' \in \mathscr{A}$ with $F \subseteq A' \subseteq G$ we have*

$$|\varphi(A) - \varphi(A')| < \varepsilon.$$

Obviously we can use the same arguments of the proof of Prop. 1, p. 304 of [7].

We say that a function $\varphi$ from $\mathscr{A}$ to $(X, |\ |)$ is *regular* if

(a) $\varphi$ is inner regular,

(b) for every $F \in \mathscr{F}$ and for every $\varepsilon > 0$ there exist $E \in \mathscr{G}$, $H \in \mathscr{F}$, $G \in \mathscr{G}$ such that $F \subseteq E \subseteq H \subseteq G$ and $\tilde{\varphi}(G \setminus F) < \varepsilon$[17]).

Remark 4. If we suppose that $\mathscr{F}$ and $\mathscr{G}$ have the property:

($\bullet$)for every $F \in \mathscr{F}$ and for every $G \in \mathscr{G}$ such that $F \subseteq G$, there exist $E \in \mathscr{G}$, $H \in \mathscr{F}$, such that $F \subseteq E \subseteq H \subseteq G$, clearly a function $\varphi$ from $\mathscr{A}$ to $(X, |\ |)$ such that the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is $k$-subadditive (in particular a $k$-triangular function) is regular iff it is inner regular (see (2.1)).

In particular, if $S$ is a Hausdorff locally compact (resp. normal) topological space, $\mathscr{F}$ is the lattice of the compact (resp. closed) sets of $S$, $\mathscr{G}$ is the lattice of the open sets of $S$, $\mathscr{A}$ is a field containing $\mathscr{G}$, a $k$-triangular function is regular iff it is inner regular.

We remark also that in the case $S$ Hausdorff locally compact topological space, $\mathscr{A}$ the $\sigma$-field of the Borel sets of $S$, $\mathscr{F}$ the lattice of the compact sets, $\mathscr{G}$ the lattice of the open sets of $S$, a $k$-triangular function $\varphi$ from $\mathscr{A}$ to $(X, |\ |)$ with regular variation[18]) (regular in the sense of [7], p. 303) satisfies the condition ($\bullet$) of the (2.1) and therefore it is regular (see also [24], Theorem 1 and Corollary 1).

**3.** (3.1) *Let $\Phi$ be a set of k-triangular inner regular functions from $\mathscr{A}$ to $(X, |\ |)$. Then, for every $A\in\mathscr{A}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and for every $n\in\mathbf{N}$ there exists $(\varphi, B)\in\Phi\times((\mathscr{F}\cup\mathscr{G})\cap A)$ such that*

$$|\varphi(B)| > n \quad \textit{and} \quad \Phi(\mathscr{A}_B) \quad \textit{is} \quad \textit{not bounded.}$$

Assume that there exist $A_0\in\mathscr{A}$ such that $\Phi(\mathscr{A}_{A_0})$ is not bounded and $n_0\in\mathbf{N}$ such that for every $(\varphi, B)\in\Phi\times((\mathscr{F}\cup\mathscr{G})\cap A_0)$

$$|\varphi(B)| > n_0 \quad \text{implies that} \quad \Phi(\mathscr{A}_B) \quad \text{is bounded.}$$

Let now $F\in\mathscr{F}_{A_0}$ and $\bar\varphi\in\Phi$ such that $|\bar\varphi(F)|>(1+k)n_0$; therefore we have

$$F\in\mathscr{F}_{A_0},\quad A_0\setminus F\in\mathscr{G}\cap A_0,\quad \bar\varphi(F) > n_0,\quad \bar\varphi(A_0\setminus F) > n_0.$$

Then, both $\Phi(\mathscr{A}_F)$ and $\Phi(\mathscr{A}_{A_0\setminus F})$ are bounded, a contradiction with the assumption that $\Phi(\mathscr{A}_{A_0})$ is not bounded.

Now we can give the proof of:

(3.2). *Let $\Phi$ be a set of k-triangular and regular functions from $\mathscr{A}$ to $(X, |\ |)$ such that*

*$\alpha$) for every $G\in\mathscr{G}$, $\Phi(G)$ is bounded,*

*$\beta$) for every sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions of $\Phi$ and for every disjoint sequence $(G_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{G}$, there exists an infinite subset $M$ of $\mathbf{N}$ such that $\bigcup_{n\in\mathbf{M}}\{\varphi_n(G_n)\}$ is bounded.*

*Then $\Phi(\mathscr{A})$ is bounded.*

Assume that $\Phi(\mathscr{A})$ is not bounded.

We will show firstly that $\Phi$ satisfies the following property:

$\gamma$) For every $A\in\mathscr{G}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and for every $n\in\mathbf{N}$ there exists $(\bar\varphi, G, A')\in\Phi\times\mathscr{G}_A\times\mathscr{G}_A$ such that

$$(***)\qquad |\bar\varphi(G)| > n,\quad G\cap A' = \emptyset,\quad \Phi(\mathscr{A}_{A'}) \quad \text{is not bounded.}$$

Let $A\in\mathscr{G}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and let $n\in\mathbf{N}$; let $h\in\mathbf{N}$ such that $|\varphi(A)|\leqq h$, for every $\varphi\in\Phi$.

There are two possibilities.

*Case I:* There exists $(\varphi_*, H)\in\Phi\times(\mathscr{F}\cap A)$ such that

$$|\varphi_*(H)|>2(h+kn) \quad\text{and}\quad \Phi(\mathscr{A}_H) \quad\text{is not bounded.}$$

In this case, let $G'\in\mathscr{G}$, $F'\in\mathscr{F}$, $G''\in\mathscr{G}$ such that

$$H\subseteqq G'\cap A\subseteqq F'\cap A\subseteqq G''\cap A \quad\text{and}\quad \tilde{\varphi}_*(G''\cap A\setminus H)<(k+hn)/k,$$

so $|\varphi_*(F'\cap A)|>h+kn$. Then, if we put

$$G=A\setminus F'\cap A,\quad A'=G'\cap A$$

it is easy to see that $(\varphi_*, G, A')\in\Phi\times\mathscr{G}_A\times\mathscr{G}_A$ verifies $(***)$.

*Case II:* For every $(\varphi, H)\in\Phi\times(\mathscr{F}\cap A)$, $|\varphi(H)|>2(h+kn)$ implies that $\Phi(\mathscr{A}_H)$ is bounded.

In this case, let $F\in\mathscr{F}_A$ and $\varphi_*\in\Phi$ such that $|\varphi_*(F)|>4(h+kn)$; let $G'\in\mathscr{G}$, $F'\in\mathscr{F}$, $G''\in\mathscr{G}$ such that

$$F\subseteqq G'\subseteqq F'\subseteqq G'' \quad\text{and}\quad \tilde{\varphi}_*(G''\setminus F)<(h+kn)/k,$$

so

$$|\varphi_*(F'\cap A)|>2(h+kn) \quad\text{and}\quad |\varphi_*(G'\cap A)|>2(h+kn).$$

Finally, if we put

$$G=G'\cap A,\quad A'=A\setminus F'\cap A,$$

obviously $(\varphi_*, G, A')$ verifies the $(***)$.

It is clear now that, by the same argument as that of (3.2) of [14], we obtain a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions of $\Phi$ and a disjoint sequence $(G_n)_{n\in\mathbf{N}}$ in $\mathscr{G}$ such that $|\varphi_n(G_n)|>n$, for every $n\in\mathbf{N}$; a contradiction with $\beta$).

(3.3) *Let $\mathscr{G}$ be a SIP-lattice*[19]) *and let $\Phi$ be a set of k-triangular functions from $\mathscr{A}$ to $(X, |\ |)$, $\mathscr{G}$-exhaustive and regular, such that for every $G\in\mathscr{G}$, $\Phi(G)$ is bounded; then $\Phi(\mathscr{A})$ is bounded.*

It suffices to prove that $\Phi$ satisfies contion $\beta$) of (2.2).

For this, let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of functions of $\Phi$ and let $(G_n)_{n\in\mathbf{N}}$ be a disjoint sequence of sets of $\mathscr{G}$. We denote respectively by $(G_{n_i})_{i\in\mathbf{N}}$ and by $\mathscr{S}$ a subsequence of $(G_n)_{n\in\mathbf{N}}$ and a ring with the Subsequential Interpolation Property contained in $\mathscr{G}$, such that $G_{n_i}\in\mathscr{S}$ for every $i\in\mathbf{N}$.

Clearly the restriction of $\varphi_n$ to $\mathscr{S}$ is exhaustive for each $n\in\mathbf{N}$, and the set $\bigcup_{n\in\mathbf{N}}\{\varphi_n(G)\}$ is bounded for each $G\in\mathscr{G}$; therefore, by the (1.6) the set $\bigcup_{i\in\mathbf{N}}\varphi_{n_i}(G_{n_i})\subseteqq$ $\subseteqq\bigcup_{n\in\mathbf{N}}\varphi_n(\mathscr{S})$ is bounded.

The proof is complete.

Corollary (3.4). *Let $\mathscr{G}$ be a SIP-lattice and suppose that $\mathscr{F}$ and $\mathscr{G}$ have the property:*

($\bullet$) *for every $F\in\mathscr{F}$ and every $G\in\mathscr{G}$ such that $F\subseteq G$, there exist $E\in\mathscr{G}$, $H\in\mathscr{F}$ such that $F\subseteq E\subseteq H\subseteq G$.*

*If $\Phi$ is a set of k-triangular, $\mathscr{G}$-exhaustive and inner regular functions from $\mathscr{A}$ to $(X,|\ |)$ such that for every $G\in\mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

It follows immediately from (3.3) (see Remark 4). In particular we have:

Corollary (3.5). *Let $S$ be a normal topological space, $\mathscr{G}$ the lattice of the open sets, $\mathscr{F}$ the lattice of the closed sets of $S$, $\mathscr{A}$ a field containing $\mathscr{G}$. If $\Phi$ is a set of k-triangular, $\mathscr{G}$-exhaustive and inner regular functions from $\mathscr{A}$ to $(X,|\ |)$ such that for every $G\in\mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

Corollary (3.6). *Let $S$ be a Hausdorff locally compact topological space, $\mathscr{F}$ the lattice of the compact sets, $\mathscr{G}$ the lattice of the open sets, $\mathscr{A}$ a field containing $\mathscr{G}$. If $\Phi$ is a set of k-triangular and inner regular functions from $\mathscr{A}$ to $(X,|\ |)$, such that, for every $G\in\mathscr{G}$, $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

It follows immediately from Corollary (3.4) and (2.2).

Corollary (3.7). *Let $S$ be a Hausdorff topological space, $\mathscr{G}$ the lattice of the open sets, $\mathscr{F}$ the lattice of the compact sets of $S$, $\mathscr{A}$ a field containing $\mathscr{G}$. If $\Phi$ is a set of k-triangular and regular functions from $\mathscr{A}$ to $(X,|\ |)$ such that for every $G\in\mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

It follows immediately from (3.3) and (2.2).

Remark 5. Clearly (see (2.1) and Remark 4), Corollary (3.6) contains Theorem 2 and Theorem 3 of [24] (see also [23], [6] Proposition 9, [2] Remark 2, p. 168).

We note that, if we put $\mathscr{F}=\mathscr{G}=\mathscr{A}$, (3.3) yields a Nikodym's boundedness theorem for $k$-triangular functions defined in a field which is a SIP-lattice. Moreover, from (3.3) we can obtain a Dieudonné boundedness type theorem for finitely additive functions from $\mathscr{A}$ with values in a topological commutative group $\Gamma$ (see [14]). In fact, if $\Gamma$ is a topological commutative group with neutral element 0, a finitely additive function $\varphi$ from $\mathscr{A}$ to $\Gamma$ is $\mathscr{G}$-exhaustive (resp. inner regular, regular (in the sense of [14])) iff, for every continuous real-valued quasi-norm $\varrho$ on $\Gamma$, the $\mathbf{R}^+$-valued 1-triangular function $\varrho\circ\varphi$ is $\mathscr{G}$-exhaustive (resp. inner regular, regular)[20]).

Therefore:

Corollary (3.8). *Let $\Gamma$ be a topological commutative group and let $\mathscr{G}$ be a*

*SIP-lattice. If $\Phi$ is a set of finitely additive and $\mathcal{G}$-exhaustive regular functions from $\mathcal{A}$ to $\Gamma$, such that for every $G\in\mathcal{G}$, $\Phi(G)$ is $\mathcal{U}$-bounded, then $\Phi(\mathcal{A})$ is $\mathcal{U}$-bounded*[21]).

For every continuous real-valued quasi-norm $\varrho$ on $\Gamma$, apply (3.3) to the set $\bigcup_{\varphi\in\Phi}(\varrho\circ\varphi)$.

## Notes

[1]) A function $\psi$ from $\mathcal{R}$ to $\overline{\mathbf{R}}^+$ is said *k-subadditive* (resp. *countably k-subadditive*) if, for any disjoint sets $A$, $B$ from $\mathcal{R}$, $\psi(A\cup B)\leqq\psi(A)+k\psi(B)$ (resp. for any disjoint sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{R}$ such that $\bigcup_{n\in\mathbf{N}} A_n\in\mathcal{R}$, $\psi\left(\bigcup_{n\in\mathbf{N}} A_n\right)\leqq\psi(A_1)+k\sum_{n>1}\psi(A_n)$) (see [15], [16]).

[2]) See [15], Corollary 1 for the case $\varphi$ $k$-triangular with values in an abelian quasi-normed group and [19], Corollary (2.3) for the case $\varphi$ finitely additive.

[3]) In fact, for each $A\in\mathcal{R}$,

$$|\varphi(A)|\leqq|\varphi(A\cap A_0)|+k|\varphi(A\setminus A_0)|;$$

therefore $\varphi(\mathcal{R}_{A_0})$ bounded implies $\varphi(\mathcal{R})$ bounded.

[4]) See [15], Lemma 2, for the case $\varphi$ $k$-triangular function with values in an abelian quasi-normed group.

[5]) For the definition of quasi $\sigma$-ring see [3], [9], [13], [28]; see also [25], Lemma 1.

[6]) We note that function a $\varphi$ from $\mathcal{R}$ to $(X,|\ |)$ is exhaustive iff for every disjoint sequence $(A_n)_{n\in\mathbf{N}}$ of set of $\mathcal{R}$

$$\lim_n \varphi(A_n\cap A)=0,$$

uniformly with respect to $A\in\mathcal{R}$ (see the proof of (1.1), Ch. II of [4]) see also note [10]), p. 134 of [4].

[7]) It follows from (1.1) of [13]; in fact, it is easy to see that it is true also for an exhaustive function $\eta$ from $\mathcal{R}$ to $\mathbf{R}^+$ such that

$$\eta(X)\leqq\eta(Y)\quad\text{if}\quad X,Y\in\mathcal{R}\colon\ X\subseteqq Y.$$

[8]) We refer to [16], remark p. 29, for an example of a set $\Phi$ of (real) 1-triangular exhaustive functions verifying a), for which the set $\Phi(\mathcal{R})$ is not bounded. We write $\Phi(A)=\bigcup_{\varphi\in\Phi}\varphi(A)$ $\forall A\in\mathcal{R}$ and $\Phi(\mathcal{H})=\bigcup_{A\in\mathcal{H}}\Phi(A)$ $\forall\mathcal{H}\subseteqq\mathcal{R}$.

[9]) See [10], [5] for the definition of rings with the Subsequential Intersolation Property (rings with the (P2) property in [28], satisfying condition $(E_2)$ in [9]).

[10]) We note that, for each $n\in\mathbf{N}$, $\{A\in\mathcal{R}\colon\ \tilde\varphi_n(A)=0\}$ is an ideal of $\mathcal{R}$; see also [17], [27].

[11]) In fact, let $\mathcal{A}$ be a disjoint set of non-zero elements of $\mathcal{R}/\mathcal{N}$; we write, $\forall\,(n,k)\in\mathbf{N}\times\mathbf{N}$, $\mathcal{A}_k^{(n)}=\{[A]\in\mathcal{R}/\mathcal{N}\colon\ \tilde\varphi_n(A)>1/k\}$. Then $\mathcal{A}=\bigcup_{(n,k)\in\mathbf{N}\times\mathbf{N}}\mathcal{A}_k^{(n)}$ and, $\varphi_n$ being exhaustive $\forall n\in\mathbf{N}$, $\mathcal{A}_k^{(n)}$ is or empty or finite set, $\forall(n,k)\leqq\mathbf{N}\times\mathbf{N}$.

[12]) In fact, $\forall n \in \mathbf{N}$, we have $|\varphi_n(A)| = |\varphi_n(B)|$ if $[A]=[B]$, $\hat{\varphi}_n([\emptyset])=0$,
$|\hat{\varphi}_n([A]) - \hat{\varphi}_n([B])| \leqq k\varphi_n(A \setminus B) + k\varphi_n(B \setminus A) = k\hat{\varphi}_n([A] \setminus [B]) + k\hat{\varphi}_n([B]-[A])$ $\forall [A], [B] \in \mathscr{R}/\mathscr{N}$; for every disjoint sequence $([A_p])_{p \in \mathbf{N}}$ we put $A_1' = A_1$ and, $\forall p>1, A_p' = A_p - \bigcup_{i<p} A_p \cap A_i$ and we have

$$\lim_p \hat{\varphi}_n([A_p]) = \lim_p |\varphi_n(A_p')| = 0.$$

[13]) We note that, if $\mathscr{R}$ is a $\sigma$-ring, a $k$-triangular and order continuous function is exhaustive. Moreover, if $X$ is a commutative semigroup with a family $F$ of non-negative real valued functions $f$ which have the property

$$f(x) - f(y) \leqq f(x+y) \leqq f(x) + f(y), \quad \text{for each} \quad x, y \in X,$$

for every triangle set function ([21]) order continuous $\mu$ from $\mathscr{R}$ to $(X, | \ |)$ the function

$$\nu \colon A \in \mathscr{R} \to f(\mu(A)) \in [0, +\infty[$$

is, for every $f \in F$, a 1-triangular and order continuous function.

[14]) If $S$ is a Hausdorff locally compact topological space, $\mathscr{A}$ is the $\sigma$-field of the Borel sets of $S$, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of $S$, the (°) is the condition $(R)$ of [23], [24].

[15]) For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, let $F \in \mathscr{F}_A$ and $H \in \mathscr{F}_{S \setminus A}$ such that $\tilde{\varphi}(A \setminus F) < \varepsilon/2$ and $\tilde{\varphi}(S \setminus A \setminus H) < \varepsilon/2$ and put $G = S \setminus H$.

[16]) For instance, if $S$ is a Hausdorff topological space, $\mathscr{A}$ is the $\sigma$-field of the Borel sets of $S$, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets, $\mathscr{F}$ and $\mathscr{G}$ satisfy the property $(**)$ (and therefore $\mathscr{F}$ has the $(*)$).

[17]) If $(X, | \ |)$ is a quasi-normed group and $\varphi$ is a finitely additive function from $\mathscr{A}$ to $(X, | \ |)$, this is the definition of regular finitely additive function of [14].

[18]) The variation $|\varphi|$ of $\varphi$ is defined in the usual way;

$$|\varphi|(A) = \sup_{\Pi} \sum_{B \in \Pi} |\varphi(B)| \quad A \in \mathscr{A},$$

where the supremum is taken over all partitions of $A$ into a finite number of disjoint sets in $\mathscr{A}$.

[19]) We say that a lattice $\mathscr{G}$ is a SIP-lattice if for each disjoint sequence $(G_n)_{n \in \mathbf{N}}$ of sets of $\mathscr{G}$ there exist a subsequence $(G_{n_i})_{i \in \mathbf{N}}$ of $(G_n)_{n \in \mathbf{N}}$ and a ring $\mathscr{S}$ with the SIP contained in $\mathscr{G}$, such that $G_{n_i} \in \mathscr{S}$, for each $i \in \mathbf{N}$ ([14]).

[20]) See [14] for the definitions of finitely additive inner regular and regular functions from $\mathscr{A}$ to $\Gamma$. Recall that for every neighbourhood $U$ of 0, there exist an $\varepsilon > 0$ and a continuous (real-valued) quasi-norm $\varrho$ on $\Gamma$ such that $\{x \in \Gamma \colon \varrho(x) < \varepsilon\} \subseteqq U$.

[21]) See [14] for the definition of $\mathscr{U}$-bounded subset of $\Gamma$; recall that a subset $Y$ of $\Gamma$ is $\mathscr{U}$-bounded iff, for every continuous real-valued quasi-norm on $\Gamma$, $\sup_{y \in Y} \varrho(y) < +\infty$ ([28], Th. (6.8) (a)). See [14], (3.2).

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