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On the lattice of complete congruences of a complete lattice: On a result of K. Reuter and R. Wille

G. GRÄTZER, H. LAKSER and B. WOLK

1. Introduction. For a complete lattice L, let Com L denote the lattice of *complete* congruence relations of L. Obviously, Com L is a complete lattice; however, unlike Con L, the lattice of congruence relations of a lattice L, it is not distributive in general. In fact, in [4], K. REUTER and R. WILLE raise the question whether every complete lattice K can be represented in the form Com L for some complete lattice L.

K. REUTER and R. WILLE [4] prove the following

Theorem. *Let K be a complete distributive lattice in which every element is the (infinite) join of (finitely) join-irreducible elements. Then K is isomorphic to the lattice of complete congruences of some complete lattice L.*

They quote [1, pp. 69 and 58]: the condition of the Theorem holds for every distributive dually continuous lattice, and in particular, for every completely distributive complete lattice.

The proof of K. Reuter and R. Wille is based on an earlier paper of R. WILLE [5] on complete congruence relations of concept lattices. In this note we show how the approaches of [2] and [3] apply.

In Sec. 2 and 3, we present two essentially equivalent proofs of the Theorem. The first uses sequences and it is purely computational; it assumes no background in lattice theory. The second is based on ideals of partial lattices and uses some knowledge of lattice theory; this approach may help visualize the proof.

In Sec. 4, we show that the complete lattice L of the Theorem can be chosen to be sectionally complemented. We also compare the constructions of [4], Sec. 2, and 3. Finally, we find the "simplest" complete lattice L such that Com L is not distributive.

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2. Construction with sequences. Let K be a complete distributive lattice; let J denote the set of join-irreducible elements of K. We assume that every element u of K is a join of join-irreducible elements, that is, $u=\bigvee_K((u]\cap J)$, where $(u]=\{p\in K\mid p\leqq u\}$.

To construct the lattice L, take the lattice $Q=M_3^J$, the J-th power of the lattice M_3. (In forming the direct power, J is regarded as an unordered set.) The elements of the lattice M_3 will be denoted by o, a, b, c, i, where o is the zero, a, b, c are the atoms, and i is the unit. For $s\in Q$ and $p\in J$, $s(p)$ will denote the p-th component of s. For $s\in Q$, let $T(s)=\{p\in J\mid s(p)=i\}$ and $\tau(s)=\bigvee_K T(s)$. We define $\bar{s}\in Q$ as follows:

$$\bar{s}(p)=\begin{cases} i, & \text{if } p\leqq\tau(s) \text{ in } K \text{ and } s(p)>o \text{ in } M_3;\\ s(p), & \text{otherwise.}\end{cases} \tag{1}$$

We call s *closed* iff $s=\bar{s}$. We construct L as the set of all closed $s\in Q$, partially ordered componentwise.

Claim 1. *Let* $S\subseteqq L$. *Then* $u=\bigwedge_Q S$ *is again closed.*

Proof. Take a $p\in J$ such that $u(p)>o$. Since $u(p)=\bigwedge_Q(s(p)\mid s\in S)$ and $u(p)$ is completely meet-irreducible in M_3, it follows that $u(p)=s(p)$ for some $s\in S$. Now $u\leqq s$, hence $\bar{u}\leqq\bar{s}=$(since s is closed)$=s$, hence $\bar{u}(p)\leqq\bar{s}(p)=s(p)=u(p)$, and therefore u is closed.

Thus L is a $\bigwedge$-sublattice of Q. It follows that L is a complete lattice, in which

$$\bigvee_L S=\overline{\bigvee_Q S} \quad \text{for} \quad S\subseteqq L. \tag{2}$$

For $z\in K$, we define a congruence, θ^z, on Q as follows:

$$u\equiv v\ (\text{mod}\ \theta^z) \quad \text{iff} \quad u(p)=v(p) \quad \text{for all} \quad p\nleqq z. \tag{3}$$

Obviously, θ^z is the kernel of the projection of $Q=M_3^J$ onto $M_3^{J-(z]}$.

Claim 2. *Let* $u, v\in Q$. *Then*

$$u\equiv v\ (\text{mod}\ \theta^z) \quad \textit{implies that} \quad \bar{u}\equiv\bar{v}\ (\text{mod}\ \theta^z).$$

Proof. Let $u\equiv v(\text{mod}\ \theta^z)$. We want to prove that $\bar{u}(p)=\bar{v}(p)$ for $p\nleqq z$. Since $u(p)=v(p)$, we can assume that $u(p)\neq\bar{u}(p)$, by symmetry. By (1), in K,

$$p\leqq\tau(u)=\bigvee_K T(u)=\bigvee_K(T(u)-(z])\vee_K\bigvee_K(T(u)\cap(z]).$$

Since p is join-irreducible in K, this implies that $p\leqq\bigvee_K(T(u)-(z])$ or $p\leqq\bigvee_K(T(u)\cap(z])$. The latter would imply that $p\leqq z$, contradicting that $p\nleqq z$. Hence $p\leqq\bigvee_K(T(u)-(z])$.

By (3), $u(q)=v(q)$ for $q\notin(z]$. Hence $T(u)-(z]=T(v)-(z]$. Since $u(p)\neq\bar{u}(p)$, therefore, by (1), $u(p)>o$ and so $v(p)>o$. Finally,

$$p \leqq \bigvee_K (T(u)-(z]) = \bigvee_K (T(v)-(z]) \leqq \bigvee_K T(v) = \tau(v),$$

hence $\bar{v}(p)=i$ by (1). Therefore, $\bar{u}(p)=\bar{v}(p)$.

By Claim 2, the restriction, θ_L^z, of θ^z to L is a complete congruence relation on L. To complete the proof of the Theorem we have to prove that every complete congruence relation of L is of this form.

Let θ be a complete congruence of L. Set

$$P = \{p\in J \mid \text{there exist } u^p, v^p\in L,\ u^p \equiv v^p \ (\text{mod}\ \theta),\ u^p(p) \neq v^p(p)\}.$$

We claim that $\theta=\theta_L^z$ with $z=\bigvee_K P$.

Obviously, $\theta\leqq\theta_L^z$.

For $x\in M_3$ and $Y\subseteq J$, let x_Y denote the element of Q defined by

$$x_Y(p) = \begin{cases} x, & \text{for } p\in Y, \\ o, & \text{otherwise.} \end{cases}$$

Note that $\bar{x}_Y=x_Y$, since $x_Y(p)$ is either o or x, and so $x_Y\in L$.

For convenience of notation, if $x\in M_3$ and $Y\subseteq K$, then we write x_Y for $x_{Y\cap J}$.

For $Y=\{y\}$, we write x_y for $x_{\{y\}}$. Note that $\{x_Y \mid x\in M_3\}$ is a sublattice of L isomorphic to M_3. For all $Y\subseteq J$, $o_Y=0$, the zero of L.

Since, for all $p\in P$,

$$u^p \equiv v^p \ (\text{mod}\ \theta),$$

it follows, by taking the meet of both sides with i_p, that

$$u^p(p)_p \equiv v^p(p)_p \ (\text{mod}\ \theta),$$

and so

$$i_p \equiv o_p \ (\text{mod}\ \theta).$$

By the completeness of θ,

(4) $$i_P \equiv 0 \ (\text{mod}\ \theta).$$

Now consider $s=i_P\vee_L b_{(z]-P}$. Obviously, $\tau(s)=z$, hence, $s=i_{(z]}$. Thus joining both sides of (4) with $b_{(z]-P}$ yields

$$i_{(z]} \equiv b_{(z]-P} \ (\text{mod}\ \theta).$$

Thus

$$i_{(z]-P} \equiv b_{(z]-P} \ (\text{mod}\ \theta),$$

so

$$i_{(z]-P} \equiv 0 \ (\text{mod}\ \theta).$$

Consequently,

$$i_{(z]} \equiv 0 \pmod{\theta},$$

completing the proof of $\theta_L^z \leqq \theta$, and the proof of the Theorem.

3. Construction with ideals. We are given K and J as in Sec. 2. First, we construct a partial lattice, M, as in [3, pp. 81—84]: the elements of M are 0, for every $p \in J$, the elements p, p_1, and p_2, and for $p, q \in J$, $p > q$, the element $p(q)$; if p is a maximal element of J, we set $p = p_1 = p_2$. For $p > q$, we form the six-element lattice, $M(p, q)$, with elements $0, p_2, q_1, q_2, q$, and $p(q)$; the operations are defined by

$$q_1 \wedge q_2 = 0, \quad q_1 \vee q_2 = q, \quad p_2 \wedge q = 0,$$
$$p_2 \vee q_1 = p(q), \quad p_2 \wedge q_2 = p(q), \quad p_2 \vee q = p(q).$$

In the partial lattice M, all the elements p_1 and p_2 $(p \in J)$ are atoms; any two elements have a meet; two elements have a join iff they belong to an $M(p, q)$ and then their join is the join in $M(p, q)$. Note that $J \subseteq M$.

The partial lattice M is *atomic* (every element is a join of atoms), hence every complete congruence relation is determined by its *kernel*, i.e., by the congruence class containing 0.

Every congruence of M extends uniquely to a congruence of the lattice, $\mathrm{Id}\, M = Q$, of ideals of M. Since Q is atomic, it follows from [3, p. 147] that an element S of Q is standard iff for any atom u of Q such that $u \not\leqq S$, the atoms of M in $S \vee u$ are the atoms of S and u.

For an ideal I of M, we define $\tau(I) = \bigvee_K (I \cap J)$. We call I *closed* iff for all $p \in J$ and $p \leqq \tau(I)$, if p_1 or $p_2 \in I$, then $p \in I$. Using the fact that all $p \in J$ are join-irreducible, it is easy to verify that if $I \cap J$ is finite, then I is closed. Every ideal I has a closure $\bar{I}$, the smallest closed ideal containing I, and the closed ideals of M form a lattice $\mathrm{Cd}\, M = L$.

For $a \in K$, let I_a be the ideal of M generated by $J \cap (a]$. Obviously, I_a is a closed ideal. We claim that I_a is standard. Indeed, let u be an atom of L, such that $u \not\leqq I_a$; then there is a $p \in J$ with $u = (p_1]$ (or $(p_2]$) and $p \not\leqq a$. Obviously, $\tau(I_a \vee_Q u) = a$, hence $I_a \vee_Q u$ is closed, implying that the only atom of $I_a \vee_L u$ not in I_a is u.

Let θ_a be the standard congruence relation associated with the standard element I_a of L. We claim that $a \to \theta_a$ is an isomorphism between K and $\mathrm{Com}\, L$. Since $a \to \theta_a$ is obviously order preserving, it is sufficient to prove that it is one-to-one and onto.

θ_a is a complete congruence on L. Indeed, $I \equiv J \pmod{\theta_a}$ iff the atoms of $I - I_a$ and $J - I_a$ are the same; thus θ_a preserves $\bigcap (= \bigwedge_Q = \bigwedge_L)$ and it also preserves $\bigvee_L$ since the kernel, $(I_a]$, is principal. Conversely, let θ be a complete congruence on L. Since θ is complete, the kernel must be principal, generated by an ideal S of M.

Let $a=\bigvee_K J\cap S$. We claim that $\theta=\theta_a$; equivalently, that $S=I_a$. The ideals S and I_a are equal iff they contain the same atoms. So let $p_i\in S$ ($i=1$ or 2), i.e., $p_i\equiv 0$ (mod θ); then $p\equiv 0$ (mod θ) also holds: if p is maximal in J, it holds by virtue of $p=p_i$; otherwise, take a $q>p$ in J and compute in $M(q,p)$ that $p_i\equiv 0$ (mod θ) implies that $p\equiv 0$ (mod θ). Conversely, let $p_i\in I_a$ ($i=1$ or 2), i.e., $p\leqq a=\bigvee_K(J\cap S)$. Then $p\leqq\tau(S)$, and S is closed, hence $p_i\in S$. This, again, completes the proof of Theorem.

4. Concluding remarks. A lattice L with zero is *sectionally complemented* if every interval $[0, a]$ is complemented. See, e.g., [3, Sec. III. 3 and III. 4] for the significance of this property. Using our first proof we can somewhat strengthen the Theorem.

Addendum to Theorem. *The complete lattice L of the Theorem can be chosen to be sectionally complemented.*

Proof. Let $u, t\in L$, and let $u<t$. We have to construct a $v\in L$ with $u\wedge_L v=0$ and $u\vee_L v=t$. Set

$$A=\{p\in J\mid t(p)=i \text{ and } u(p)=o\}.$$

For $p\in J$, define $u^*(p)$ as a complement of $u(p)$ in $[o, t(p)]$. Now we describe v; for $p\in J$, define

$$v(p)=\begin{cases} u^*(p), & \text{if } u^*(p) \text{ is the unique complement of } u(p) \text{ in } [o, t(p)],\\ u^*(p), & \text{if } p\not\leqq\bigvee_K A,\\ o, & \text{otherwise.}\end{cases}$$

Obviously, $v\in Q$. Furthermore, $T(v)=A$. Hence v is closed, and so $v\in L$. Now, $u(p)\wedge v(p)=o$ holds in M_3 by definition for all $p\in J$, so $u\wedge_L v=0$. Finally, $(u\vee_Q v)(p)=u(p)\vee u^*(p)=t(p)$ except if $u^*(p)$ is not the unique complement of $u(p)$ in $[o, t(p)]$ and $p\leqq\bigvee_K A$; in this case, $(u\vee_Q v)(p)=u(p)\vee o=u(p)$. However, $T(u\vee_Q v)\supseteqq T(v)=A$ and $u(p)\in\{a, b, c\}$ (otherwise, $u^*(p)$ would be the unique complement of $u(p)$ in $[o, t(p)]$), hence by (1), $\overline{u\vee_Q v}(p)=i=t(p)$, proving that $u\vee_L v=t$.

It is reasonable to ask how the constructions of [4], Sec. 2, and 3 compare. Let K be the three-element chain. It can be computed that the construction of [4] yields a lattice isomorphic to L_1 which can be represented as M_3^2 with the elements $\langle a, a\rangle$, $\langle a, b\rangle$, and $\langle a, i\rangle$ removed. Sec 2 yields a lattice L_2 which can be represented as M_3^2 with the elements $\langle a, i\rangle$, $\langle b, i\rangle$, and $\langle c, i\rangle$ removed. Note that L_1 and L_2 both have 22 elements but they are not isomorphic. Finally, Sec. 3 produces the six-element lattice $M(p, q)$.

Finally, in [4, Section 4], K. REUTER and R. WILLE produce examples of complete lattices L such that Com L is not distributive. We think the following example is the simplest.

Let L be N (the set of nonnegative integers with the usual partial ordering) with two additional elements: a, i. Let 0 be the zero, and i the unit of L. Let $a \wedge n = 0$ and $a \vee n = i$ for all $n \in N$, $n \neq 0$. Obviously, L is a complete lattice. We define three complete congruences, α, β, and γ on L:

nontrivial classes

$$\alpha: [2n+1, 2n+2], \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$$\beta: [2n+1, 2n+2], \quad \text{for} \quad n = 1, 2, \ldots$$

$$\gamma: [2n, 2n+1], \quad \text{for} \quad n = 1, 2, \ldots.$$

It is easy to check that α, β, γ generate a sublattice isomorphic to N_5 in Com L.

Observe that L is a "minimal" example. If Com L is nondistributive, then L must contain a chain, C, of the type $\omega+1$ or its dual, otherwise Com L is isomorphic to Con L, and hence distributive. $L-C$ is nonempty; indeed, if $L=C$, then Com L=Com C, and Com C is isomorphic to Con ω, which is distributive. We conclude that $L-C$ must contain at least one element. In our example, it contains exactly one element.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF MANITOBA
WINNIPEG, MAN. R3T 2N2,
CANADA