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Endomorphism monoids in small varieties of bands

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Introduction

The study of the relationship between an algebra A and its endomorphism monoid End (A) has gradually, in the course of two decades, crystallized into a general framework, which we find worthwhile to outline here.

As soon as we have a class $\mathscr{K}$ of algebras, the assignment $A \to \mathrm{End}(A)$ for $A \in \mathscr{K}$ defines the class $\mathscr{M}$ of monoids M isomorphic to End (A) for some $A \in \mathscr{K}$, i.e. the monoids *representable* in $\mathscr{K}$. The class $\mathscr{K}$ is said to be *monoid universal* if all monoids are representable in $\mathscr{K}$. If every finite monoid M is representable by a finite algebra in $\mathscr{K}$ then we say that $\mathscr{K}$ is *finite monoid universal.*

The *problem of representability* of a given monoid M in a given class $\mathscr{K}$ of algebras is just one aspect of the relationship between $A \in \mathscr{K}$ and End (A). Another, in a way complementary aspect of this relationship is the *problem of determinancy* of $A \in \mathscr{K}$ by End (A): to what extent the knowledge of End (A) (up to isomorphism) determines the structure of A (within the class $\mathscr{K}$)? The class $\mathscr{K}$ is said to be *k-determined*, for a cardinal k, if any set of pairwise non-isomorphic algebras from $\mathscr{K}$ with the same (up to isomorphism) endomorphism monoid has the cardinality strictly less than k.

Since both representability and determinacy are tied to the algebraic structure of the algebras of a given class, it is natural to consider in the first place the varieties of algebras (of a given similarity type); the lattice of subvarieties can serve as a sort of a structural hierarchy in which universality is an increasing property and determinacy a decreasing property.

When we try to elucidate the nature of universality/determinacy in this lattice of subvarieties setting, we are naturally led to the notion of (categorical) universality:

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A variety $\mathscr{K}$ is said to be *universal* if the category of all graphs and compatible mappings can be fully embedded into $\mathscr{K}$. If, moreover, there exists a full embedding from the category of all graphs and compatible mappings into $\mathscr{K}$ such that it maps finite graphs into finite algebras then $\mathscr{K}$ is called *finite-to-finite universal*.

All known monoid universal varieties are also universal but, in general, it does not hold. It is an open problem whether for varieties the monoid universality and the categorical universality are equivalent. The categorical universality of a variety $\mathscr{V}$ excludes any k-determinacy of $\mathscr{V}$, for the reason that simply for any cardinal k, the discrete category of k graphs can be fully embedded into $\mathscr{V}$. More generally, any monoid M has a proper class of pairwise non-isomorphic representing objects in $\mathscr{V}$ (see [7] or [9]).

So much for the general framework of the present study.

Our subject proper — endomorphism monoids of bands (i.e., idempotent semigroups) — does not ideally fit into the above general scheme for the obvious reason that bands admit all constant self maps as endomorphisms, thus the endomorphism monoid of any band has left zeros, thus no variety of bands is monoid universal (and also not universal). However, as it is shown in the previous work [3] of the authors, monoid universality (and even more, universality) is there, only as if buried by a layer of superfluous morphisms. A natural way how to dispose of the undesirable morphisms is to strengthen the structure of the representing objects — the bands in our case. It may come as a surprise that even very small varieties of bands can be made universal by enriching their operational type by two or three nullary operation symbols, i.e. by turning the bands in question into 2 or 3-pointed bands (1-pointed would not do).

Every band variety is determined, within the variety of all bands, by a single equation $u=v$, a useful means to refer to the variety as $[u=v]$ (especially if there is no other commonly accepted name for its members).

Figure 1 visualizes the meet semilattice T_0 which is isomorphic to the bottom of the lattice of band varieties, see [1, 4, 5]. The nodes of T_0 represent the following band varieties:

$a_0 = [x=y]$ — trivial bands

$a_1 = [xy=x]$ — left zero semigroups

$a_2 = [xy=yx]$ — semilattices

$a_3 = [yx=x]$ — right zero semigroups

$a_4 = [xyz=xzy]$ — left normal bands

$a_5 = [xyz=xz]$ — rectangular bands

$a_6 = [yzx = zyx]$ — right normal bands

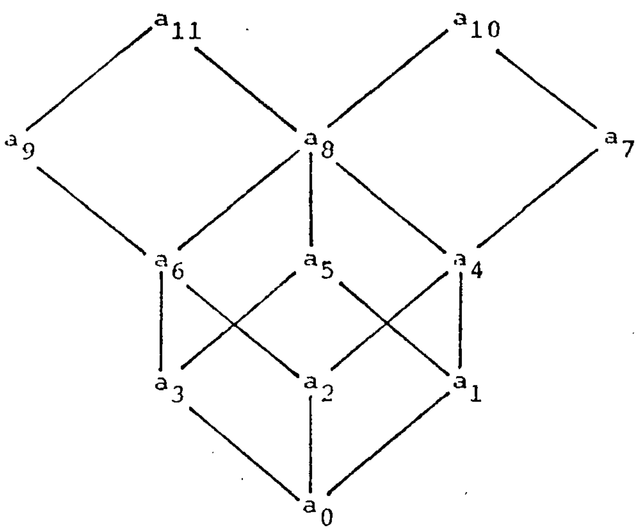
$a_7 = [xyx = xy]$ — semilattices of left zero semigroups

$a_8 = [xyzu = xzyu]$ — normal bands

$a_9 = [xyx = yx]$ — semilattices of right zero semigroups

$a_{10} = [xyz = xyxz]$ — left distributive bands

$a_{11} = [xyz = xzyz]$ — right distributive bands.



The meet semilattice T_0
Figure 1

It is readily seen that no number of nullary operations added to semilattices or rectangular bands makes them monoid universal.

The aim of this paper is to prove

Theorem 1.1. *The variety of rectangular bands and the variety of semilattices with an arbitrary number of nullary operations added is not universal.*

Theorem 1.2. *A variety $\mathscr{V}$ of bands with two nullary operations added is universal if and only if $\mathscr{V}$ contains either the variety of semilattices of left zero semigroups or the variety of semilattices of right zero semigroups.*

Theorem 1.3. *A variety $\mathscr{V}$ of bands with three nullary operations added is universal if and only if the variety of semilattices is a proper subvariety of $\mathscr{V}$.*

It should be said that the very "undesirable" morphisms, removed by the additional nullary operations in order to achieve universality, are very precious for the determinacy of small band varieties: semilattices are 3-determined [10], normal bands are 5-determined [11], semilattices of left (or right) zero semigroups are 3-determined and left (or right) distributive bands are 5-determined [3].

The results of this paper raise the question whether there exist other strengthenings of the structure of bands to obtain a universal category. The authors [3] showed that also the variety of bands with a unary operation * satisfying the identities $xx^*x=x$ and $x^{**}=x$ is universal. It is an open question whether we can restrict ourselves to the *-bands (here the unary operation, moreover, satisfies the identity $x^*y^*=(yx)^*$), or to a subvariety of *-bands.

The semigroup theoretical notions used in this paper can be found in the monographs [2] or [8].

The rest of the paper is devoted to the proof of Theorems 1.1, 1.2, and 1.3. The proof is divided into three parts. The proof of the universality of the 2-pointed variety $[xyx=xy]$ (or $[xyx=yx]$) is contained in Section 2, and the proof of the universality of the 3-pointed variety $[xyz=xzy]$ (or $[yzx=zyx]$) is the aim of Section 3. Common to both parts is the use of unary varieties. Denote by $I(1, 1)$ the variety of algebras with two unary idempotent operations and $I(1, 1, 0)$ its 1-pointed version.

It is known

Theorem 1.4 [9]. *$I(1, 1)$ and $I(1, 1, 0)$ are finite-to-finite universal.*

Our universality proofs construct a full embedding of $I(1, 1)$ or $I(1, 1, 0)$ into the variety in question.

The final section is devoted to the proof of non-universality of some pointed varieties of bands. This finishes the proof of Theorems 1.1, 1.2, and 1.3.

2. Universality of 2-pointed semilattices of left zero semigroups

Denote by $(S, *)$ the groupoid given by the following table (see on the next page).

Then the following holds:

Proposition 2.1. *The groupoid $(S, *)$ is a semigroup belonging to the variety $[xyx=xy]$ of semilattices of left zero semigroups. Moreover, $B=\{b_i;\ i\in 2\}$, $C=\{c_i;\ i\in 2\}$, $D=\{d_i;\ i\in 2\}$, $E=\{e_i;\ i\in 4\}$ are all non-singleton $\mathcal{D}$-classes of $(S, *)$.*

Proof by a direct inspection.

Assume that $(X, \varphi_0, \varphi_1)$ is an algebra from $I(1, 1)$ such that $X\cap S=\emptyset$. Denote by X_0, X_1, X_2 three disjoint copies of X, the element $x\in X$ in the copy X_i, $i\in 3$

$(S, *)$	a_0	a_1	t_0	t_1	t_2	b_0	b_1	c_0	c_1	d_0	d_1	e_0	e_1	e_2	e_3
a_0	a_0	b_0	t_0	b_0	t_0	b_0	b_0	c_0	c_1	d_1	d_1	e_0	e_1	e_3	e_3
a_1	b_1	a_1	b_1	t_1	t_1	b_1	b_1	c_1	c_1	d_0	d_1	e_1	e_1	e_2	e_3
t_0	t_0	b_0	t_0	b_0	t_0	b_0	b_0	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
t_1	b_1	t_1	b_1	t_1	t_1	b_1	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
t_2	t_0	t_1	t_0	t_1	t_2	b_0	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
b_0	b_0	b_0	b_0	b_0	b_0	b_0	b_0	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
b_1	b_1	b_1	b_1	b_1	b_1	b_1	b_1	c_1	c_1	d_1	d_1	e_1	e_1	e_3	e_3
c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	c_0	e_0	e_0	e_0	e_0	e_0	e_0
c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1	e_1	e_1	e_1	e_1	e_1	e_1
d_0	d_0	d_0	d_0	d_0	d_0	d_0	d_0	e_2	e_2	d_0	d_0	e_2	e_2	e_2	e_2
d_1	d_1	d_1	d_1	d_1	d_1	d_1	d_1	e_3	e_3	d_1	d_1	e_3	e_3	e_3	e_3
e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0	e_0
e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1	e_1
e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2	e_2
e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3	e_3

Figure 2

is denoted by x_i. We shall define a groupoid $\Phi'(X, \varphi_0, \varphi_1)=(Y, \cdot)$ which is a co-extension of S (i.e. there exists a surjective homomorphism $f\colon (Y, \cdot)\to(S, *)$) as follows:

$$Y = (S\setminus\{t_i;\ i\in 3\})\cup(\cup\{X_i;\ i\in 3\})$$

and if $y, z\in Y$ then:

$y\cdot z=u*v$ if the following hold:

$y\in S$ and $y=u$ or $y\in X_i$ and $u=t_i$ for some $i\in 3$,

$z\in S$ and $z=v$ or $z\in X_i$ and $v=t_i$ for some $i\in 3$, $u*v\in Y$;

$y\cdot z=u_k$ if there exist $u, v\in X$ with $y=u_i$, $z=v_j$, and $t_i^* t_j=t_k$ for some $i, j, k\in 3$;

$y\cdot z=u_k$ if $y=u_i\in X_i$, $z\in\{a_0, a_1\}$ and $t_i*z=t_k$ for some $i, k\in 3$;

$y\cdot z=(\varphi_k(u))_k$ if $y=a_i$, $z=u_j\in X_j$ and $a_i*t_j=t_k$ for some $i, j, k\in 3$.

Denote by ψ a mapping from Y to S such that

$$\psi(y) = y \quad \text{for} \quad y \in S, \qquad \psi(y) = t_i \quad \text{for} \quad y \in X_i, \; i \in 3.$$

We have

Proposition 2.2. *$\Phi'(X, \varphi_0, \varphi_1) = (Y, \cdot)$ is a semigroup belonging to the variety $[xyx = xy]$ for every $(X, \varphi_0, \varphi_1) \in I(1, 1)$. Furthermore, $\psi\colon (Y, \cdot) \to (S, *)$ is a surjective homomorphism and $B, C, D, E, X_i, i \in 3$ are all non-singleton $\mathcal{D}$-classes of $(Y, \cdot)$.*

Proof. That ψ is a homomorphism is straightforward. We show that $(Y, \cdot)$ is a semigroup. Let $x, y, z \in Y$ and we investigate the equality

$$(*) \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Since ψ is a homomorphism and $(S, *)$ is a semigroup we obtain that $(*)$ holds for every $x, y, z \in Y$ with $(x \cdot y) \cdot z \in S$ or $x \cdot (y \cdot z) \in S$. If $(x \cdot y) \cdot z \in Y \setminus S$ then $(x \cdot y) \cdot z \in \in \bigcup \{X_i;\, i \in 3\}$, and moreover, $(x \cdot y) \cdot z \in X_i$ if and only if $x \cdot (y \cdot z) \in X_i$. Assume that $(x \cdot y) \cdot z \in X_0$ then $x, y, z \in X_0 \cup X_2 \cup \{a_0\}$. If $x \in X_0$ then $(*)$ holds because x is a left zero with respect to the set $X_0 \cup X_2 \cup \{a_0\}$. If $x = u_2$ for some $u \in X$ then for every $v \in X_0 \cup X_2 \cup \{a_0\}$ we have $u_2 \cdot v \in \{u_0, u_2\}$ and hence we again obtain $(*)$. Finally, assume $x = a_0$. If $y = u_i$ for some $u \in X$, $i \in \{0, 2\}$ then we have $(x \cdot y) \cdot z = = (\varphi_0(u)_0) \cdot z = \varphi_0(u)_0$ and $y \cdot z \in \{u_0, u_2\}$, hence $x \cdot (y \cdot z) = \varphi_0(u)_0$ and $(*)$ hold. If $y = a_0$ then $z = u_i$ for some $u \in X$, $i \in \{0, 2\}$ and hence $(x \cdot y) \cdot z = a_0 \cdot z = \varphi_0(u)_0$ and $x \cdot (y \cdot z) = x \cdot \varphi_0(u)_0 = \varphi_0(\varphi_0(u))_0 = \varphi_0(u)_0$ because φ_0 is idempotent. Analogously we prove $(*)$ if $(x \cdot y) \cdot z \in X_1$. Finally, if $(x \cdot y) \cdot z \in X_2$, then $x, y, z \in X_2$ and because X_2 is a left zero subsemigroup of $(Y, \cdot)$ we conclude that $(*)$ holds and hence $(Y, \cdot)$ is a semigroup. The rest is obvious.

Define a functor Φ from $I(1, 1)$ into the 2-pointed variety $[xyx = xy]$ of all 2-pointed semilattices of left zero semigroups. For an algebra $(X, \varphi_0, \varphi_1)$ from $I(1, 1)$ define $\Phi(X, \varphi_0, \varphi_1) = (Y, \cdot, c_0, d_0)$ where $\Phi'(X, \varphi_0, \varphi_1) = (Y, \cdot)$. For a homomorphism $f\colon (X, \varphi_0, \varphi_1) \to (X', \varphi_0', \varphi_1')$ in $I(1, 1)$ define a mapping Φf:

$$\Phi f(x_i) = f(x)_i \quad \text{for every} \quad x \in X, \; i \in 3, \quad \text{and} \quad \Phi f(s) = s \quad \text{for every} \quad s \in S.$$

If $u \in \{a_0, a_1\}$, $v \in \bigcup \{X_i;\, i \in 3\}$ then $\Phi f(uv) = \Phi f(u) \cdot \Phi f(v)$ because f is a homomorphism, for the remaining case we obtain by a direct inspection that Φf is a homomorphism. Thus we can summarize:

Proposition 2.3. *Φ is an embedding of $I(1, 1)$ into the 2-pointed $[xyx = xy]$.*

We prove that Φ is full. Assume that $\Phi(X, \varphi_0, \varphi_1) = (Y, \cdot, c_0, d_0)$, $\Phi(X', \varphi_0',$ $\varphi_1') = (Y', \cdot, c_0, d_0)$ are algebras from the 2-pointed variety $[xyx = xy]$ and let $f\colon (Y, \cdot, c_0, d_0) \to (Y', \cdot, c_0, d_0)$ be a homomorphism. Then we have

Lemma 2.4. *For every* $u \in S \cap Y$ *we have* $f(u)=u$.

Proof. Since f preserves the nullary operations we have $f(c_0)=c_0, f(d_0)=d_0$. Hence $f(C) \subseteqq C$, $f(D) \subseteqq D$. Furthermore, an arbitrary $\mathscr{D}$-class containing an arbitrary element of the set $\bigcup \{X_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}$ is greater than the $\mathscr{D}$-classes C and D. Thus $f(\bigcup \{X_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}) \subseteqq \bigcup \{X'_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}$. Moreover, a_0 is a unique element of the set $\bigcup \{X_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}$ with $a_0 \cdot c_0 = c_0$ and a_1 is a unique element of the set $\bigcup \{X_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}$ with $a_1 \cdot d_0 = d_0$. Hence $f(a_0) = a_0, f(a_1) = a_1$. Since the subsemigroup generated by $\{a_0, a_1, c_0, d_0\}$ is $S \cap Y$ we obtain that f is identical on the set $S \cap Y$.

Lemma 2.5. *There exists* $g: X \to X'$ *such that for every* $x \in X$, $i \in 3$ *we have* $f(x_i) = g(x)_i$.

Proof. Choose $x \in X$. By Lemma 2.4 we conclude that $f(x_2) \in \bigcup \{X'_i;\ i \in 3\} \cup B \cup \{a_0, a_1\}$. If $f(x_2) \in X'_0 \cup \{a_0, b_0\}$ then $b_1 = f(b_1) = f(x_2 \cdot a_1 \cdot a_0) = f(x_2) \cdot f(a_1) \cdot f(a_0) = b_0$ — a contradiction, if $f(x_2) \in X'_1 \cup \{a_1, b_1\}$ then $b_0 = f(b_0) = f(x_2 \cdot a_0 \cdot a_1\} = f(x_2) \cdot f(a_0) \cdot f(a_1) = b_1$ — a contradiction. Thus $f(X_2) \subseteqq X'_2$. Set $g: X \to X'$ with $f(x_2) = g(x)_2$ for every $x \in X$. Then we have $f(x_0) = f(x_2 \cdot a_0) = f(x_2) \cdot f(a_0) = g(x)_2 \cdot a_0 = g(x)_0$ and $f(x_1) = f(x_2 \cdot a_1) = f(x_2) \cdot f(a_1) = g(x)_2 \cdot a_1 = g(x)_1$.

Lemma 2.6. *The mapping g of Lemma* 2.5 *is a homomorphism from* $(X, \varphi_0, \varphi_1)$ *into* $(X', \varphi'_0, \varphi'_1)$.

Proof. Consider $x \in X$, then $g(\varphi_0(x))_0 = f(\varphi_0(x)_0) = f(a_0 \cdot x_2) = f(a_0) \cdot f(x_2) = a_0 \cdot g(x)_2 = \varphi'_0(g(x))_0$ and hence $g \circ \varphi_0 = \varphi'_0 \circ g$. By the dual argument we obtain $g \circ \varphi_1 = \varphi'_1 \circ g$, whence g is a homomorphism.

Since for a homomorphism g from Lemma 2.5 we have $\Phi g = f$ we have proved that Φ is a full embedding and thus Theorem 1.4 completes the proof of the following

Theorem 2.7. *The variety* $[xyx = xy]$ *with two nullary operations added is finite-to-finite universal.*

Hence we immediately obtain

Corollary 2.8. *The variety* $[xyx = yx]$ *with two nullary operations added is finite-to-finite universal.*

Proof. Obviously, a semigroup $(T, \cdot)$ belongs to the variety $[xyx = xy]$ if and only if the semigroup $(T, \oplus)$ belongs to the variety $[xyx = yx]$ where $t \oplus u = u \cdot t$ for every $t, u \in T$. Hence Corollary 2.8 immediately follows from Theorem 2.7.

3. The universality of the 3-pointed variety $[xyz=xzy]$

For an algebra $A=(X, \varphi, \psi, q)\in I(1,1,0)$ denote by X_i, $i\in 2$ two disjoint copies of the set X, for an element $x\in X$ denote by x_i the corresponding element in the copy X_i, $i\in 2$. Define an algebra ΦA in the 3-pointed variety $[xyz=xzy]$. The underlying set of ΦA is $(X_0\times\{a_1, a_2, a_7, a_{10}\})\cup(X_1\times\{a_3, a_9, a_{11}\})\cup((X_0\cup X_1)\times\times\{a_4, a_5, a_6, a_8\})\cup\{(0, a_0)\}$. For $x, y\in X$, $m, n\in 2$, $i, j\in 12$ if $a_i\wedge a_j=a_k$ in the semilattice T_0 and if (x_m, a_i), (y_n, a_j) are elements of the underlying set of ΦA then

$$(x_m, a_i)(y_n, a_j)=\begin{cases}(x_m, a_k) & \text{if } k>3,\\ (\bar{\psi}(x_m)_1, a_3) & \text{if } k=3,\\ (x_0, a_2) & \text{if } k=2,\\ (\bar{\varphi}(x_m)_0, a_1) & \text{if } k=1,\\ (0, a_0) & \text{if } k=0,\end{cases}$$

moreover $(0, a_0)$ is a zero in ΦA, where $\bar{\varphi}, \bar{\psi}\colon (X_0\cup X_1)\to X$ are the mappings defined $\bar{\varphi}(x_0)=x$, $\bar{\varphi}(x_1)=\varphi(x)$, $\bar{\psi}(x_0)=\psi(x)$, $\bar{\psi}(x_1)=x$ for every $x\in X$. By a direct inspection we obtain that the definition of the binary operation is correct and that ΦA is a strong semilattice of left zero semigroups, thus by [8] it is a left normal band. The three added nullary operations are (q_0, a_5), (q_0, a_7), (q_1, a_9).

For a homomorphism $f\colon A\to B$ where $A=(X, \varphi, \psi, q)$, $B=(Y, \varphi', \psi', r)$ denote by f' the mapping defined as follows: $f'(x_m, a_i)=(f(x)_m, a_i)$ for every $x\in X$, $m\in 2$, $i\in 12\setminus\{0\}$, $f'(0, a_0)=(0, a_0)$. By a direct inspection we obtain that f' maps the underlying set of ΦA into the underlying set of ΦB, furthermore the restriction of f' to ΦA is a homomorphism. Thus if the restriction of f' to ΦA and ΦB is denoted by Φf then we obtain

Proposition 3.1. *Φ is an embedding of $I(1,1,0)$ into the 3-pointed variety $[xyz=xzy]$.*

Proof. By a direct inspection.

We prove that Φ is a full embedding. For the purpose assume that $A, B\in I(1,1,0)$ where $A=(X, \varphi, \psi, q)$, $B=(Y, \varphi', \psi', r)$ and that $f\colon \Phi A\to\Phi B$ is a homomorphism in the 3-pointed variety $[xyz=xzy]$.

Lemma 3.2. *The structural homomorphism of f is the identity.*

Proof: Since T_0 is the structural semilattice of ΦA and ΦB we get that the structural homomorphism g of f is an endomorphism of T_0. Since f preserves the nullary operations we conclude that $g(a_i)=a_i$ for every $i\in\{5, 7, 9\}$. Moreover, g preserves the order and thus $g(a_i)=a_i$ for $i\in\{10, 11\}$. Since g is an endomorphism and $\{a_i;\, i\in\{5, 7, 9, 10, 11\}\}$ generates T_0 we conclude that g is the identity.

Define mappings $f_0, f_1\colon X \to Y$ such that

$$f(x_0, a_{10}) = \bigl(f_0(x)_0, a_{10}\bigr) \quad \text{for every} \quad x \in X,$$

$$f(x_1, a_{11}) = \bigl(f_1(x)_1, a_{11}\bigr) \quad \text{for every} \quad x \in X.$$

Lemma 3.3. *For every* $i \in \{1, 2, 4, 5, 6, 7, 8, 10\}$, *we have* $f(x_0, a_i) = (f_0(x)_0, a_i)$ *for every* $x \in X$.

For every $i \in \{3, 4, 5, 6, 8, 9, 11\}$, *we have* $f(x_1, a_i) = (f_1(x)_1, a_i)$ *for every* $x \in X$.

Proof. For every $x \in X$ and $i \in \{1, 2, 4, 5, 6, 7, 8\}$ we have $(x_0, a_i) = (x_0, a_{10}) \cdot \cdot (x_0, a_i)$ and hence

$$f(x_0, a_i) = f(x_0, a_{10}) f(x_0, a_i) = \bigl(f_0(x)_0, a_{10}\bigr) f(x_0, a_i) = \bigl(f_0(x)_0, a_i\bigr).$$

Hence we obtain the first assertion. The proof of the second assertion is dual.

Corollary 3.4. $f_0 = f_1$.

Proof. We apply Lemma 3.3 and the fact that

$$\bigl(f_0(x)_0, a_2\bigr) = f(x_0, a_2) = f\bigl((x_1, a_{11})(x_0, a_2)\bigr) = f(x_1, a_{11}) f(x_0, a_2) =$$
$$= \bigl(f_1(x)_1, a_{11}\bigr)\bigl(f_0(x)_0, a_2\bigr) = \bigl(f_1(x)_0, a_2\bigr)$$

for every $x \in X$.

Lemma 3.5. f_0 *is a homomorphism of* $I(1, 1, 0)$ *from* A *into* B.

Proof. Obviously $f_0(q) = r$. We have

$$\bigl(\varphi(f_0(x))_0, a_1\bigr) = \bigl(f_0(x)_1, a_5\bigr)\bigl(f_0(x)_0, a_1\bigr) = f\bigl((x_1, a_5)(x_0, a_1)\bigr) =$$
$$= f\bigl(\varphi(x)_0, a_1\bigr) = \bigl(f_0(\varphi(x))_0, a_1\bigr).$$

Thus f_0 commutes with φ. By duality we obtain that f_0 commutes with ψ. Hence f_0 is a homomorphism.

Since $\Phi f_0 = f$ we conclude that Φ is a full embedding from $I(1, 1, 0)$ into the 3-pointed variety $[xyz = xzy]$. Theorem 1.4 completes the proof of the following:

Theorem 3.6. *The variety* $[xyz = xzy]$ *with three nullary operations added is finite-to-finite universal.*

If we apply the same idea as in the proof of Corollary 2.8 we obtain

Corollary 3.7. *The variety* $[yzx = zyx]$ *with three nullary operations added is finite-to-finite universal.*

4. Non-universality of pointed varieties of bands

First we investigate the variety of rectangular bands, and the variety of semilattices. If B is a rectangular band then B is a product of a left zero semigroup L and a right zero semigroup R. It is well known that $f: B \to B$ is an endomorphism of B if and only if $f = g \times h$ where $g: L \to L$, $h: R \to R$. Hence we obtain:

Proposition 4.1. *For any cardinal α, no α-pointed rectangular band B represents a non-trivial group as* End (B).

We prove an analogous result for semilattices:

Proposition 4.2. *For any cardinal α, no α-pointed semilattice S represents a non-trivial group of a finite order as* End (S).

Proof. Assume the contrary, let S be an α-pointed semilattice such that its endomorphism monoid is isomorphic to a non-trivial group G of finite order. First, for every $g \in \mathrm{End}(S)$ and for every $x \in S$ if $g(x) \neq x$ then x and $g(x)$ are incomparable because there exists a natural number n with $g^n(x) = x$. For every endomorphism $g \in \mathrm{End}(S)$ define $f: S \to S$ such that $f(x) = xg(x)$ for every $x \in S$. Obviously, $f \in \mathrm{End}(S)$ and $f(x)$ and x are comparable for every $x \in S$. Moreover, f is identical if and only if g is identical and this is a contradiction with the fact that G is non-trivial.

Propositions 4.1 and 4.2 complete the proof of Theorem 1.1. Moreover, Theorems 1.1 and 3.6, and Corollary 3.7 complete the proof of Theorem 1.3. Thus it suffices to finish the proof of Theorem 1.2. For this purpose we shall investigate the 2-pointed variety of normal bands.

Proposition 4.3. *Let B be a normal band with a structural semilattice S. If f is an endomorphism of S such that $f(s) \leqq s$ for every $s \in S$ then there exists an endomorphism $g: B \to B$ with a structural morphism f such that for every $\mathscr{D}$-class D of B with $f(D) = D$ and for every $x \in D$ we have $g(x) = x$.*

Proof. By [8], B is a strong semilattice of rectangular bands, i.e. for every $s \in S$ there exists a rectangular band $D(s)$ (it is the $\mathscr{D}$-class corresponding to s) and for every pair $s, t \in S$ with $s \leqq t$ there exists a homomorphism $\mu_{t,s}: D(t) \to D(s)$ such that

a) for every $s \in S$, $\mu_{s,s}$ is the identity;

b) for every triple $s, t, u \in S$ with $s \leqq t \leqq u$ we have

$$\mu_{t,s} \circ \mu_{u,t} = \mu_{u,s};$$

c) $B = \bigcup \{D(s); s \in S\}$ and $\{D(s); s \in S\}$ are pairwise disjoint;

d) for every $s, t \in S$, $x \in D(s), y \in D(t)$ we have

$$xy = \mu_{s,s\wedge t}(x)\,\mu_{t,s\wedge t}(y)$$

where the former product is in B and the latter one is in $D(s \wedge t)$.

For every $x \in D(s)$, $s \in S$ define $g(x) = \mu_{s, f(s)}(x)$. By a) through d) we easily obtain that g is an endomorphism of B with the required properties.

Lemma 4.4. *Let S be a semilattice with an element $d \in S$. If $f \in \mathrm{End}\,(S_d)$ where $S_d = \{s \in S;\ s \geqq d\}$ is a subsemilattice of S then $g\colon S \to S$ defined by $g(s) = f(s)$ for $s \in S_d$, $g(s) = d \wedge s$ for $s \in S \setminus S_d$ is an endomorphism of S.*

Proof. Clearly g is correctly defined. Let $x, y \in S$. If $x, y \in S_d$ then also $x \wedge y \in S_d$ and since $f \in \mathrm{End}\,(S_d)$ we obtain $g(x) \wedge g(y) = g(x \wedge y)$. If $y \in S \setminus S_d$ then $g(x \wedge y) = x \wedge y \wedge d$. If $x \in S_d$ then $x, f(x) \geqq d$, whence $x \wedge y \wedge d = f(x) \wedge y \wedge d = g(x) \wedge g(y)$; if $x \in S \setminus S_d$ then obviously $g(x) \wedge g(y) = x \wedge y \wedge d$. If $x \in S \setminus S_d$ the proof is analogous.

Theorem 4.5. *No 2-pointed normal band B represents a nontrivial group as* End (B).

Proof. Assume that B is a normal band with two added nullary operations a_i, $i \in 2$ such that End (B) is a group (i.e. every endomorphism of B is an automorphism). Let S be the structural semilattice of B, assume that elements b_i, $i \in 2$ of S correspond to the $\mathcal{D}$-classes containing a_i, $i \in 2$. If there exists $s \in S$ such that $s \geqq b_i$ for $i \in 2$ and s is not the unity of S then consider the endomorphism h of S such that $h(x) = s \wedge x$ for every $x \in S$. Since $s \geqq b_i$ we have $h(b_i) = b_i$. By Proposition 4.3 there exists a band endomorphism g of B with structural endomorphism h and $g(a_i) = a_i$ for $i \in 2$. Thus g is an endomorphism of B and because neither h nor g is an automorphism, this is a contradiction. Hence we can assume that only the unity 1 in S is greater than b_i, $i \in 2$. Set $c = b_0 \wedge b_1$ and let $d \in S$ with $d \leqq c$. Denote $S_d = \{s \in S;\ s \geqq d\}$ and define $f\colon S_d \to S_d$ as follows:

$f(x) = x$ if $x = 1$,

$f(x) = b_i$ if $x \neq 1$ and $x \geqq b_i$ for an $i \in 2$,

$f(x) = c$ if $x \ngeqq b_i$ for any $i \in 2$ and $x \geqq c$,

$f(x) = d$ if $x \ngeqq c$ and $x \geqq d$.

By a direct inspection we obtain that f is an endomorphism of S_d with $f(x) \leqq x$ for every $x \in S_d$ and $f(b_i) = b_i$ for $i \in 2$. If we use Lemma 4.4 we obtain an endomorphism h of S with $h(x) \leqq x$ for every $x \in S$ and $h(b_i) = b_i$ for $i \in 2$. Finally, if we apply Proposition 4.3 we obtain a 2-pointed band endomorphism g of B with structural endomorphism h. Since g is an automorphism we conclude that h is an automorphism of S, thus $S_d=S\subseteq\{1, b_0, b_1, c, d\}$ where 1 is the unity of S (if it exists). It is routine to verify that B is rigid.

The proof of Theorem 1.2 follows from Theorems 2.7, 4.5, and from Corollary 2.8. In fact, we have proved stronger results than Theorems 1.1, 1.2, and 1.3:

Corollary 4.6. *For a variety $\mathscr{V}$ of bands with k nullary operations the following are equivalent:*

a) $\mathscr{V}$ is finite-to-finite universal;

b) $\mathscr{V}$ is universal;

c) $\mathscr{V}$ is monoid universal;

d) $\mathscr{V}$ is finite monoid universal;

e) there exist a non-trivial group G of finite order and an algebra $A\in\mathscr{V}$ with $\mathrm{End}\,(A)\cong G$;

f) either $k\geqq 2$ and $\mathscr{V}\supseteqq[xyx=xy]$ or $k\geqq 2$ and $\mathscr{V}\supseteqq[xyx=yx]$ or $k\geqq 3$ and the variety of all semilattices is a proper subvariety of $\mathscr{V}$.

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