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Loops with and without subloops

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In the theory of local loops, those loops which have one-parameter subloops are of considerable importance. M. A. AKIVIS [1] gave an interesting example of a quasigroup which has r-parameter subloops in every direction [2]. At the "*Web Geometry Conference*" held at Szeged, 1987, K. H. Hofmann raised the problem of exhibiting a simple example in which *there is no* one-parameter subloop in any direction.

Searching for a satisfactory answer to this problem we shall investigate the question: *Is the existence of one-parameter subloops a general property of local loops?*

The aim of our considerations are as follows:

1. Firstly, we exhibit a class of loops which have subloops in every direction, and examine associativity conditions for this loop-class.

2. Secondly, we give (analytic) examples of elastic loops in our class which are not groups on the one hand, and a further example for elastic loops whose one-parameter subloops are not one-parameter subgroups on the other hand. Hence we separate analytic elastic loops from right alternative analytic loops since the latter ones are necessarily power-associative (see [6]).

3. Thirdly, we exhibit a class of loops which have one-parameter subloops only in one direction.

4. Finally, we give an example of a loop without one-parameter subloops at all.

All the results of the present paper are based on the existence of *canonical coordinate systems* [3] and the following main feature of loops ([4], Theorem 1). If f is a local loop of class C^k $(k \geqq 2)$ and D is a canonical coordinate system then every one-parameter subloop is locally a straight line through the origin in D.

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I. Preliminaries

Definition 1. Let $\mathscr{F}$ be an n-dimensional differentiable manifold. A partial mapping f *of* class C^k

$$f\colon \mathscr{F}\times\mathscr{F}\to\mathscr{F}\colon (\mathbf{x}, \mathbf{y}) \to \mathbf{z} \quad (\mathbf{x}, \mathbf{y}, \mathbf{z}\in\mathscr{F})$$

is called *local loop-multiplication of class* C^k and $(\mathscr{F}; f)$ is called a *local loop of class* C^k if the following conditions are satisfied.

a) The multiplication is a local quasigroup, that is, there exist open neighbourhoods $\mathscr{V}, \mathscr{U} (\mathscr{V}\subset\mathscr{U}\subset\mathscr{F})$ such that $f\colon \mathscr{V}\times\mathscr{V}\to\mathscr{U}$ and $f(\mathbf{x}, \mathbf{y})=\mathbf{z}\in\mathscr{U}$ for all $\mathbf{x}, \mathbf{y}\in\mathscr{V}$. Furthermore for arbitrary elements $\mathbf{x}\in\mathscr{V}$, $\mathbf{z}\in\mathscr{V}$ (respectively, $\mathbf{y}\in\mathscr{V}$, $\mathbf{z}\in\mathscr{V}$) there exists one and only one $\mathbf{y}\in\mathscr{U}$ (respectively, $\mathbf{x}\in\mathscr{V}$) for which $f(\mathbf{x}, \mathbf{y})=\mathbf{z}$.

b) The loop has a unit element, that is, there is an element $\mathbf{e}\in\mathscr{V}$ such that $f(\mathbf{x}, \mathbf{e})=f(\mathbf{e}, \mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in\mathscr{V}$.

c) The loop-multiplication is of class C^k.

We shall consider charts $(\mathscr{U}_i, \varphi_i)$ for which $\varphi_i\colon \mathscr{U}\to\mathscr{W}_i\subseteq\mathbf{R}^n$; $\mathbf{e}\mapsto\mathbf{0}$, where $\mathbf{0}$ is the origin of $\mathbf{R}^n$.

A loop on an m-dimensional manifold $\mathscr{F}$ is called an *m-parameter loop*. Instead of $(\mathscr{F}; f)$ we shall frequently write f.

Since the canonical coordinate-system defined in [3] plays an important role in our considerations further on, we recall its definition.

Definition 2. Let us consider a loop f of class C^k $(k\geqq 2)$. We shall say that a coordinate-system φ given by the chart $(\mathscr{U}, \varphi)$, $f\colon \mathscr{U}\to\mathbf{R}^n$, $\varphi(\mathbf{e})=\mathbf{0}$, is a *canonical coordinate system (CCS)* with respect to f if in these coordinates we have

$$F(\mathbf{x}, \mathbf{x}) = 2\mathbf{x}$$

for all $\mathbf{x}\in\varphi(\mathscr{V})$, where

$$F = \varphi\circ f\circ(\varphi^{-1}\times\varphi^{-1}).$$

Further on, by a loop we mean a local loop of class C^k $(k\geqq 2)$.

Definition 3. Let $(\mathscr{F}; f)$ and $(\mathscr{G}; g)$ be two loops, and let ξ be a local map $\xi\colon \mathscr{F}\to\mathscr{G}$ of class C^k. If ξ is a local embedding, then $(\mathscr{F}; f)$ is called a *local m-parameter subloop of* $(\mathscr{G}; g)$.

II. Maximal families of one-parameter subloops

Our purpose is now to describe a class of loops having subloops in every direction.

Definition 4. A local multiplication

$$F\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n;\ \mathscr{V} \times \mathscr{V} \to \mathscr{U}$$

(where $\mathscr{V}$, $\mathscr{U}$ are appropriate neighbourhoods of $\mathbf{0}$) is called an *(α, β)-multiplication* if for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$

$$F(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \alpha(\mathbf{y})\mathbf{x} + \beta(\mathbf{x})\mathbf{y} \tag{1}$$

where $\alpha, \beta\colon \mathbf{R}^n \to \mathbf{R}\colon \mathbf{0} \to 0$ are real-valued functions of class C^k.

Proposition 1. *An (α, β)-multiplication is a local loop-multiplication on a neighbourhood $\mathscr{V}$ of the origin $\mathbf{0}$ of $\mathbf{R}^n$, the unit element is the origin $\mathbf{0}$.*

Proof. First of all let us show that (1) defines a local loop-multiplication of class C^k.

a) As α and β are defined on $\mathbf{R}^n$, $F(\mathbf{x}, \mathbf{y})$ is well-defined. The multiplication is locally solvable from left and right since

$$D_1 F(\mathbf{0}, \mathbf{0}) = I, \quad D_2 F(\mathbf{0}, \mathbf{0}) = I$$

where D_1 and D_2 denotes the partial derivative with respect to the first, respectively, second variable belonging to $\mathbf{R}^n$, and where I is the identity map in $\mathbf{R}^n$.

b) The origin $\mathbf{0}$ of $\mathbf{R}^n$ is the unit element since we have

$$F(\mathbf{x}, \mathbf{0}) = \mathbf{x} + \mathbf{0} + \alpha(\mathbf{0})\mathbf{x} + \beta(\mathbf{x})\mathbf{0} = \mathbf{x},$$

and similarly

$$F(\mathbf{0}, \mathbf{y}) = \mathbf{y}.$$

c) The loop-multiplication is of class C^k because α and β are of class C^k, as well.

A loop $(\mathbf{R}^n, F)$ with (α, β)-multiplication is called an $(\mathbf{R}^n; \alpha, \beta)$-loop.

Now we are going to show that these loops have subloops in every direction.

Theorem 1. *For every vector subspace $\mathscr{A}$ of $\mathbf{R}^n$ the restriction $F|_{\mathscr{A} \times \mathscr{A}}$ of an (α, β)-multiplication is a loop-multiplication on a neighbourhood of $\mathbf{0} \in \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a vector subspace of $\mathbf{R}^n$. Then for $\mathbf{x}, \mathbf{y} \in \mathscr{A}$ the element

$$\mathbf{z} = F(\mathbf{x}, \mathbf{y}) = [1 + \alpha(\mathbf{y})]\mathbf{x} + [1 + \beta(\mathbf{x})]\mathbf{y}$$

obviously belongs to $\mathscr{A}$, as well. Similarly if $\mathbf{x}, \mathbf{z}$ (respectively $\mathbf{y}, \mathbf{z}$) are elements of

$\mathscr{A}$, then $\mathbf{y}$ (respectively $\mathbf{x}$) also belongs to $\mathscr{A}$. This implies that the restriction of F to $\mathscr{A}\times\mathscr{A}$ is a local subloop of $(\mathbf{R}^n; F)$.

The subloop $(\mathscr{A}; F)$ is called an $(\mathscr{A}; \alpha, \beta)$-loop.

Let us emphasize two details of the above result.

Corollary. *The loop* $(\mathbf{R}^n; \alpha, \beta)$ *has r-parameter subloops in every r-dimensional subspace of* $\mathbf{R}^n$. *In particular, it has one-parameter subloops in every direction.*

Theorem 2. *Let F be an (α, β)-multiplication defined on a neighbourhood* $\mathscr{V}\subset\mathbf{R}^n$. *Then the loop* $(\mathbf{R}^n; \alpha, \beta)$ $(n\geqq 2)$ *is a local group on a sufficiently small neighbourhood* $\mathscr{W}$ *of* $\mathbf{0}$ *if and only if*

$$\alpha(F(\mathbf{x},\mathbf{y})) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

(*for every* $\mathbf{x},\mathbf{y}\in\mathscr{W}$).

Proof. 1) For the necessity we show that $(\mathbf{R}^n; \alpha, \beta)$ is a group if and only if for all $\mathbf{x},\mathbf{y}\in\mathscr{W}$ the relations

$$\alpha(F(\mathbf{x},\mathbf{y})) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) \tag{2a}$$

$$\beta(F(\mathbf{x},\mathbf{y})) = \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y}) \tag{2b}$$

are satisfied. Indeed, $(\mathbf{R}^n; \alpha, \beta)$ is a group if and only if F is associative, that is,

$$F(F(\mathbf{x},\mathbf{y}),\mathbf{z}) = F(\mathbf{x}, F(\mathbf{y},\mathbf{z}))$$

holds for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathscr{W}$ ($\mathscr{W}$ is a sufficiently small neighbourhood of $\mathbf{0}$). By a straightforward calculation we see that this identity is equivalent to the following one:

$$[\alpha(F(\mathbf{y},\mathbf{z}))-[\alpha(\mathbf{y})+\alpha(\mathbf{z})+\alpha(\mathbf{y})\alpha(\mathbf{z})]]\,\mathbf{x} = [\beta(F(\mathbf{x},\mathbf{y}))-[\beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})]]\,\mathbf{z}$$

for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathscr{W}$, which is equivalent to (2a) and (2b).

2) In order to show that condition (2a) is sufficient we shall prove that (2a) implies (2b). For this purpose we show that (2a) implies the linearity of $\alpha-\beta$ via the commutativity of the one-parameter subloops.

By (2a) we have

$$\alpha(F(\mathbf{x},\mathbf{y})) = \alpha(F(\mathbf{y},\mathbf{x})) \tag{3}$$

for all $\mathbf{x},\mathbf{y}\in\mathscr{W}$. Whenever for a direction $\mathbf{a}$ $D_{\mathbf{a}}\alpha(\mathbf{0})\neq 0$, we get from (3) that

$$F(\mathbf{x},\mathbf{y}) = F(\mathbf{y},\mathbf{x})$$

for all $\mathbf{x},\mathbf{y}$ belonging to the one-parameter subloop of direction $\mathbf{a}$. That is, this one-parameter subloop is commutative. However, if for the direction $\mathbf{a}$ $D_{\mathbf{a}}\alpha(\mathbf{0})=0$, we introduce new defining functions

$$\alpha^*(\mathbf{x}) = \alpha(\mathbf{x})+\langle \mathbf{a}, \mathbf{x}\rangle, \qquad \beta^*(\mathbf{x}) = \beta(\mathbf{x})-\langle \mathbf{a}, \mathbf{x}\rangle$$

to obtain the $(\mathbf{R}^n;\ \alpha^*, \beta^*)$-loop on $\mathcal{W}$. Since

$$u\mathbf{a}+v\mathbf{a}\alpha^*(v\mathbf{a})\,u\mathbf{a}+\beta^*(u\mathbf{a})v\mathbf{a} = u\mathbf{a}+v\mathbf{a}\alpha(v\mathbf{a})\,u\mathbf{a}+\beta(u\mathbf{a})v\mathbf{a}$$

for all sufficiently small real number u and v the one-parameter subloops of the direction $\mathbf{a}$ of the loops $(\mathbf{R}^n;\ \alpha^*, \beta^*)$ and $(\mathbf{R}^n;\ \alpha, \beta)$ are the same. Since $D_{\mathbf{a}}\alpha^*(\mathbf{0})\neq 0$ when $D_{\mathbf{a}}\alpha(\mathbf{0})=0$, we obtain that this one-parameter subloop is commutative.

From the commutativity it immediately follows that

$$\alpha(v\mathbf{x})\,u\mathbf{x}+\beta(u\mathbf{x})\,v\mathbf{x} = \alpha(u\mathbf{x})\,v\mathbf{x}+\beta(v\mathbf{x})\,u\mathbf{x}$$

holds in every direction $\mathbf{x}$ for all sufficiently small nonzero u and v in $\mathbf{R}$, which yields

$$\frac{\alpha(v\mathbf{x})-\beta(v\mathbf{x})}{v} = \frac{\alpha(u\mathbf{x})-\beta(u\mathbf{x})}{u}.$$

The left hand side does not depend on u, so we get

(4)

$$\alpha(v\mathbf{x})-\beta(v\mathbf{x}) = v\cdot\lim_{u\to 0}\frac{\alpha(u\mathbf{x})-\alpha(\mathbf{0})}{u} - v\cdot\lim_{u\to 0}\frac{\beta(u\mathbf{x})-\beta(\mathbf{0})}{u} = v\cdot D_{\mathbf{x}}\big(\alpha(\mathbf{0})-\beta(\mathbf{0})\big)$$

for each $\mathbf{x}\in\mathcal{W}$. Thus

$$\alpha(\mathbf{z})-\beta(\mathbf{z}) = \langle D(\alpha-\beta)(\mathbf{0}), \mathbf{z}\rangle.$$

That is, $\alpha-\beta$ is a linear function on the appropriate neighbourhood.

Now we are ready to prove (2b) from (2a). If in the left and right hand side of (2a) we substitute $\lambda+\beta$ for α (where λ is a linear function), we obtain firstly

$$\alpha\big(F(\mathbf{x},\mathbf{y})\big) = \lambda\big(F(\mathbf{x},\mathbf{y})\big)+\beta\big(F(\mathbf{x},\mathbf{y})\big) = \lambda(\mathbf{x})+\lambda(\mathbf{y})+\alpha(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{x})\lambda(\mathbf{y})+$$
$$+\beta\big(F(\mathbf{x},\mathbf{y})\big) = [\lambda(\mathbf{x})+\lambda(\mathbf{y})+\lambda(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{y})\lambda(\mathbf{x})+\beta(\mathbf{x})\lambda(\mathbf{y})]+\beta\big(F(\mathbf{x},\mathbf{y})\big),$$

secondly

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) =$$
$$= [\lambda(\mathbf{x})+\lambda(\mathbf{y})+\lambda(\mathbf{x})\lambda(\mathbf{y})+\lambda(\mathbf{x})\beta(\mathbf{y})+\beta(\mathbf{x})\lambda(\mathbf{y})]+[\beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})],$$

hence

$$\beta\big(F(\mathbf{x},\mathbf{y})\big) = \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})$$

which is just (2b) and the theorem is proved.

Remarks 1. In accordance with the proof above, the $(\mathbf{R}^n;\ \alpha, \beta)$-loop is not a group if $\alpha-\beta$ is not linear (for example if β is identically 0, and α is not linear). Thus relation (1) defines a non-trivial loop, in general. 2. By Theorem 1, r-parameter subloops $(\mathscr{A};\alpha,\beta)$ of $(\mathbf{R}^n;\alpha,\beta)$ exist for all subspace $\mathscr{A}\subseteqq\mathbf{R}^n$. Since Theorem 2 holds for these loops, too, we obtain that the r-parameter $(r\geqq 2)$ subloops of $(\mathbf{R}^n;\alpha,\beta)$ are not r-parameter subgroups if $\alpha-\beta$ is not linear.

We now consider some other algebraic identities which are weaker than associativity. Let us recall some definitions.

Definition 5. A loop has the *left-inverse property (right-inverse property)* if for each $\mathbf{x}\in\mathscr{V}$ there is an element $\bar{\mathbf{x}}_l\in\mathscr{V}$ $(\bar{\mathbf{x}}_r\in\mathscr{V})$ such that for every $\mathbf{y}\in\mathscr{V}$

(5) $$F(\bar{\mathbf{x}}_l, F(\mathbf{x},\mathbf{y}))=\mathbf{y},\quad\text{respectively,}\quad F(F(\mathbf{y},\mathbf{x}),\bar{\mathbf{x}}_r)=\mathbf{y},$$

in particular, for $\mathbf{y}=\mathbf{0}$

(6) $$F(\mathbf{x}_l,\mathbf{x})=\mathbf{0},\quad\text{respectively,}\quad F(\mathbf{x},\bar{\mathbf{x}}_r)=\mathbf{0}.$$

The loop F said to be *left alternative (right alternative)* if for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$

(7) $$F(F(\mathbf{x},\mathbf{x}),\mathbf{y})=F(\mathbf{x},F(\mathbf{x},\mathbf{y})),\quad\text{respectively,}\quad F(\mathbf{y},F(\mathbf{x},\mathbf{x}))=F(F(\mathbf{y},\mathbf{x}),\mathbf{x}).$$

The loop has the *property of elasticity* if for all $\mathbf{x},\mathbf{y}\in\mathscr{V}$

(8) $$F(\mathbf{x},F(\mathbf{y},\mathbf{x}))=F(F(\mathbf{x},\mathbf{y}),\mathbf{x}).$$

Theorem 3. *An* $(\mathbf{R}^n;\alpha,\beta)$*-loop* $(n\geqq 2)$ *is a group if and only if it*
a) *possesses the left-inverse property (right-inverse property),*
b) *possesses the left alternative property (right alternative property).*

Theorem 4. *Whenever* $\alpha=\beta$ *and* α *is linear then the corresponding* $(\mathbf{R}^n;\alpha,\beta)$*-loops* $(n\geqq 2)$ *are elastic. Such an* $(\mathbf{R}^n;\alpha,\beta)$*-loop of dimension* n $(n\geqq 2)$ *is a group if and only if* $\alpha\equiv 0$.

Proof of Theorem 3. a) If we use expression (1) to reformulate (5) and (6) we get

(5′) $$[1+\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]]\cdot\bar{\mathbf{x}}_l+[1+\alpha(\mathbf{y})+\beta(\bar{\mathbf{x}}_l)+\alpha(\mathbf{y})\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{x}+ +[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{y}=\mathbf{0},$$

(6′) $$[1+\alpha(\mathbf{x})]\cdot\bar{\mathbf{x}}_l+[1+\beta(\bar{\mathbf{x}}_l)]\cdot\mathbf{x}=\mathbf{0}$$

for all $\mathbf{x}$, $\mathbf{y}$ and $\bar{\mathbf{x}}_l\in\mathscr{V}$. Subtracting (6′) from (5′) we get

$$[\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]-\alpha(\mathbf{x})]\,\bar{\mathbf{x}}_l+\alpha(\mathbf{y})[1+\beta(\bar{\mathbf{x}}_l)]\mathbf{x}+ +[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\mathbf{y}=\mathbf{0}$$

for all $\mathbf{x}$, $\mathbf{y}$ and $\bar{\mathbf{x}}_l\in\mathscr{V}$. In view of (6′) we obtain

$$[\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]-\alpha(\mathbf{x})-\alpha(\mathbf{y})-\alpha(\mathbf{x})\alpha(\mathbf{y})]\,\bar{\mathbf{x}}_l+[\beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}_l)]\mathbf{y}=\mathbf{0}$$

for all $\mathbf{x}, \mathbf{y}$ and $\bar{\mathbf{x}}_l \in \mathscr{V}$. Notice that $\bar{\mathbf{x}}_l = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ (see (6)). Hence relation (5) holds for the loop $(\mathbf{R}^n; \alpha, \beta)$ if and only if the following two identities are satisfied

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}) \tag{9a}$$

$$0 = \beta(\mathbf{x})+\beta(\bar{\mathbf{x}}_l)+\beta(\mathbf{x})\beta(\bar{\mathbf{x}}) \tag{9b}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. Since condition (2a) is the same as (9a), this local loop is a local group. As a group always possesses left-inverse property, part a) of the theorem is proved in the case of left inverse loops.

Notice that from (9b) and (6′) we can express $\bar{\mathbf{x}}_l$ as follows:

$$\bar{\mathbf{x}}_l = \frac{1}{[1+\alpha(\mathbf{x})]\cdot[1+\beta(\mathbf{x})]}\,\mathbf{x}. \tag{$*$}$$

For right inverse $(\mathbf{R}^n; \alpha, \beta)$-loops the proof can be carried out in the same way as above. Furthermore, the right and left inverse of an element are clearly the same by $(*)$ above.

b) Expressing identity (7) in terms of (1) we obtain that (7) is equivalent to the following two identites

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y}), \tag{10a}$$

$$\beta[2\mathbf{x}+\alpha(\mathbf{x})\mathbf{x}+\beta(\mathbf{x})\mathbf{x}] = 2\beta(\mathbf{x})+\beta(\mathbf{x})\beta(\mathbf{x}). \tag{10b}$$

We see that (10a) and (2a) are equivalent. Thus from the left alternative property it follows that an $(\mathbf{R}^n; \alpha, \beta)$-loop is a local group. The converse is obvious, furthermore the right alternative case is similar, therefore part b) is proved.

Proof of Theorem 4. From expression (1) we obtain that (8) is equivalent to the following equality;

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})-\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{x})\mathbf{y}+\beta(\mathbf{y})\mathbf{x}] =$$
$$= \beta(\mathbf{x})+\beta(\mathbf{y})+\beta(\mathbf{x})\beta(\mathbf{y})-\beta[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}+\beta(\mathbf{x})\mathbf{y}]$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. It is clear that if $\alpha \equiv \beta$, then this equality is an identity for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. In other words, for $\alpha \equiv \beta$ the loop $(\mathbf{R}^n; \alpha, \beta)$ is elastic. Let now $\alpha \equiv \beta$ be linear function. Let us suppose that $(\mathbf{R}^n; \alpha, \beta)$ is a group, and $n \geqq 2$. Then (2a) is fulfilled and has the form

$$\alpha(F(\mathbf{x}), \mathbf{y}) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha[\mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{z}+\alpha(\mathbf{x})\mathbf{y}] = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{y})\alpha(\mathbf{x})+\alpha(\mathbf{x})\alpha(\mathbf{y}) = \alpha(\mathbf{x})+\alpha(\mathbf{y})+\alpha(\mathbf{x})\alpha(\mathbf{y})$$

$$\alpha(\mathbf{x})\alpha(\mathbf{y}) = 0$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. However, this means that $\alpha \equiv \beta \equiv 0$. Thus for the linear function $\alpha \equiv \beta$ which is not identically zero the loop $(\mathbf{R}^n; \alpha, \beta)$ is elastic, but not a group.

Another (less obvious) example is an $(\mathbf{R}^n; \alpha, \beta)$-loop $(n \geqq 1)$ for which $\alpha(\mathbf{x})$ is a quadratic function, e.g. $\alpha(\mathbf{x}) = \|\mathbf{x}\|^2$. It is easy to see that in this case the one-parameter subloops are not one-parameter subgroups. Indeed, for arbitrary $s, t, u \in \mathbf{R}$ we have

$$F(F(s\mathbf{x}, t\mathbf{x}), u\mathbf{x}) - F(s\mathbf{x}, F(t\mathbf{x}, u\mathbf{x})) =$$

$$= stu \cdot \|\mathbf{x}\|^2 \cdot \left[(3t+2s+2u)(s-u) + \|\mathbf{x}\|^2 t[t(s^3-u^3) + 2t^2(s^2-u^2) + t^3(s-u)]\right] \mathbf{x}.$$

Hence these subloops are not subgroups because the difference vector has a positive norm provided $0 < u < s < t$. Consequently, these loops are not groups.

Remark. L. V. Sabinin and P. O. Mikheev [5] proved that analytical right-alternative loops are power associative (that is, $F(\mathbf{x}^m, \mathbf{x}^n) = \mathbf{x}^{m+n}$ for arbitrary $\mathbf{x} \in \mathscr{F}$ and any integers m and n). However, we can show that the analogous statement for elastic loops is false. As a power-associative loop has one-parameter subgroups in every direction (see Kuz'min [6]), our analytical elastic $(\mathbf{R}^n; \alpha, \alpha)$-loops $(\alpha(\mathbf{x}) = \|\mathbf{x}\|^2)$ can not be power associative.

III. Loops without one-parameter subloops

In final part of this paper we exhibit loops which do not have any non-trivial one-parameter subloops whatsoever. However, let us start with another class of loops which have one-parameter subloops in one unique direction. For this purpose we give a slight modification of the loops given by (1).

Definition 6. Let us define a local multiplication $(\alpha, -\alpha; \varrho, \mathbf{a})$ in $\mathbf{R}^n$ as follows

$$F_2\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n\colon \mathscr{V} \times \mathscr{V} \to \mathscr{U}$$

where $\mathscr{V}$, $\mathscr{U}$ are appropriate neighbourhoods of 0, furthermore for all $x, y \in \mathscr{V}$ we have

$$F_2(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} + \alpha(\mathbf{y})\mathbf{x} - \alpha(\mathbf{x})\mathbf{y} + \varrho(\mathbf{x}, \mathbf{y})\mathbf{a} \tag{11}$$

for all $\mathbf{x}, \mathbf{y} \in \mathscr{V}$. Here α is the same as in Definition 4; furthermore $\mathbf{a} \in \mathbf{R}^n$ is a point (sufficiently close to $\mathbf{0}$) different from $\mathbf{0}$, and ϱ is a real function of class C^k:

$$\varrho\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}\colon \mathscr{V} \to \mathscr{U}^* \subset \mathbf{R}$$

such that $\varrho(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \mathbf{y}$. (That is, $\varrho(\mathbf{x}, \mathbf{0}) = \varrho(\mathbf{0}, \mathbf{y}) = \varrho(\mathbf{z}, \mathbf{z}) = 0$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathscr{V}$, and ϱ does not vanish in any other case).

Such functions ϱ exist, e.g. the function ϱ defined as follows:

(12) $$\varrho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \|\mathbf{x}-\mathbf{y}\|^2.$$

Proposition 2. *The local* $(\alpha, -\alpha;\ \varrho, \mathbf{a})$*-multiplication given by* (13) *defines a local loop of class* C^k *which has one-parameter subloop in the direction of* $\mathbf{a}$, *exclusively.*

Proof. We show that F_2 is a loop-multiplication.

a) Since α and ϱ are of class C^k, F_2 is well defined in $\mathscr{V}$. $F_2(\mathbf{x}, \mathbf{y})=\mathbf{z}$ can be solved from left and right because the derivative of F_2 with respect to the first and second variable (belonging to $\mathbf{R}^n$) is the identity map.

b) The origin $\mathbf{0}$ in $\mathbf{R}^n$ is the unit element since

$$F_2(\mathbf{x}, \mathbf{0}) = \mathbf{x}+\mathbf{0}+\alpha(\mathbf{0})\mathbf{x}-\alpha(\mathbf{x})\mathbf{0}+\varrho(\mathbf{x}, \mathbf{0})\mathbf{a} = \mathbf{x}$$

and similarly $F_2(\mathbf{0}, \mathbf{y})=\mathbf{y}$, for all $\mathbf{x}, \mathbf{y}\in\mathscr{V}$.

c) α and ϱ are of class C^k, therefore F_2 is also of class C^k.

Following this, let us notice that the coordinate system in which F_2 is defined, is a CCS. Indeed, according to Definition 2 we have

$$F_2(\mathbf{x}, \mathbf{x}) = \mathbf{x}+\mathbf{x}+\alpha(\mathbf{x})\mathbf{x}-\alpha(\mathbf{x})\mathbf{x}+\varrho(\mathbf{x}, \mathbf{x})\mathbf{a} = 2\mathbf{x}$$

for all $\mathbf{x}\in\mathscr{V}$. So, owing to Theorem 1 in [4], if there exists a one-parameter subloop, then it is locally a straight line. It is obvious that there exists one-parameter subloop in the direction of $\mathbf{a}$.

Let us suppose that there exists one-parameter subloop in the direction of $\mathbf{d}$. Then the elements of this subloop are of form $t\cdot\mathbf{d}$, $t\in(-T_0, T_0)$, at least locally. Let $\mathbf{x}=s_1\mathbf{d}\neq\mathbf{0}$ and $\mathbf{y}=s_2\mathbf{d}\neq\mathbf{0}$ $(s_1\neq s_2)$ be two different elements of it. According to (11) we have

$$F_2(\mathbf{x}, \mathbf{y}) = F_2(s_1\mathbf{d}, s_2\mathbf{d}) = [s_1+s_2+s_1\alpha(s_2\mathbf{d})-s_2\alpha(s_1\mathbf{d})]\mathbf{d}+\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+r\mathbf{d}$$

for some $r\in\mathbf{R}$. Since for these s_1 and s_2: $\varrho(s_1\mathbf{d}, s_2\mathbf{d})\neq 0$, directions $\mathbf{a}$ and $\mathbf{d}$ must be the same, which completes the proof.

This loop is called an $(\mathbf{R}^n;\ \alpha, -\alpha;\ \varrho, \mathbf{a})$-loop.

Our next and last example is that of a loop without nontrivial one-parameter subloops. For this purpose we shall modify the previous construction.

Definition 7. Let us define a local multiplication $(\alpha, -\alpha;\ \varrho, \mathbf{a};\ \sigma, \mathbf{b})$ in $\mathbf{R}^n$ $(n\geqq 2)$ in the following manner

$$F_3\colon \mathbf{R}^n\times\mathbf{R}^n \to \mathbf{R}^n\colon \mathscr{V}\times\mathscr{V}\to\mathscr{U}$$

where $\mathscr{V}$, $\mathscr{U}$ are appropriate neighbourhoods of the origin $\mathbf{0}$, furthermore for all

$\mathbf{x}, \mathbf{y} \in \mathscr{V}$ we have

$$F_3(\mathbf{x}, \mathbf{y}) = \mathbf{x}+\mathbf{y}+\alpha(\mathbf{y})\mathbf{x}-\alpha(\mathbf{x})\mathbf{y}+\varrho(\mathbf{x}, \mathbf{y})\mathbf{a}+\sigma(\mathbf{x}, \mathbf{y})\mathbf{b} \tag{13}$$

where α and σ are the same as in Definition 6, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent (fixed) points (directions) in $\mathbf{R}^n$ (sufficiently close to $\mathbf{0}$). Further we suppose that σ possesses all the properties of ϱ and that, in additional, from $\varrho(\mathbf{x}_0, \mathbf{y}_0)=\sigma(\mathbf{x}_0, \mathbf{y}_0)$ it follows that $\varrho(\mathbf{x}_0, \mathbf{y}_0)=\sigma(\mathbf{x}_0, \mathbf{y}_0)=\mathbf{0}$ (that is that σ and ϱ are different, except where they vanish).

Functions σ with the required properties exist, e.g. the one defined by the following formula:

$$\sigma(\mathbf{x}, \mathbf{y}) = h(\mathbf{x})\|\mathbf{y}\|^2\|\mathbf{x}-\mathbf{y}\|^2, \tag{14}$$

where $h\colon \mathbf{R}^n \to \mathbf{R}^n$ and $h(\mathbf{x})$ vanishes if and only if $\mathbf{x}=\mathbf{0}$, and $h(\mathbf{x}) \neq \|\mathbf{x}\|^2$. Furthermore h is assumed to be of class C^k.

Proposition 2. *Any multiplication given by* (13) *defines a local loop of class* C^k *which has no one-parameter subloop at all.*

This loop is called an $(\mathbf{R}^n;\ \alpha,\ -\alpha;\ \varrho,\ \mathbf{a};\ \sigma,\ \mathbf{b})$-loop.

Proof. In the same way as above we can prove that conditions a), b) and c) for loops are fulfilled.

It can be stated, again, that F_3 is defined in a CCS. Thus if there exists a one-parameter subloop of F_3 in the direction of $\mathbf{d}$, then for two different elements $\mathbf{x}=s_1\mathbf{d}$, $\mathbf{y}=s_2\mathbf{d}$ $(0\neq s_1\neq s_2\neq 0)$ we have

$$F_3(\mathbf{x}, \mathbf{y}) = F_3(s_1\mathbf{d}, s_2\mathbf{d}) =$$
$$= [s_1+s_2+s_1\alpha(s_2\mathbf{d})-s_2\alpha(s_1\mathbf{d})]\mathbf{d}+\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b} = r\mathbf{d}.$$

Thus we obtain that

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b} = t\mathbf{d}$$

for some $t=t(s_1, s_2)\in(-T_0, T_0)$. As for $\mathbf{d}$ we have unique expression $\mathbf{d}=\varkappa_1\mathbf{a}+\varkappa_2\mathbf{b}$, we get

$$t\mathbf{d} = (t\varkappa_1)\mathbf{a}+(t\varkappa_2)\mathbf{b} = \varrho(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{a}+\sigma(s_1\mathbf{d}, s_2\mathbf{d})\mathbf{b}$$

for all allowable $s_1\neq s_2$, hence we obtain that

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d}) = t\varkappa_1 \quad\text{and}\quad \sigma(s_1\mathbf{d}, s_2\mathbf{d}) = t\varkappa_2. \tag{15}$$

Let us first suppose that $\mathbf{d}=\mathbf{a}$. That is, there exists a one-parameter subloop in the direction of $\mathbf{a}$. Then we have

$$\varrho(s_1\mathbf{a}, s_2\mathbf{a})\mathbf{a}+\sigma(s_1\mathbf{a}, s_2\mathbf{a})\mathbf{b} = t\mathbf{a}.$$

Since $\varrho(s_1\mathbf{a}, s_2\mathbf{a})\neq 0$, $\sigma(s_1\mathbf{a}, s_2\mathbf{a})\neq 0$ and $t\neq 0$, it follows that $\mathbf{b}=\mathbf{a}$, which contradicts the assumption for a and b. In the case of $\mathbf{d}=\mathbf{b}$ we get the same result.

Let us now suppose that $\mathbf{a}\neq\mathbf{d}\neq\mathbf{b}$. Then from relations (15) we obtain that $\varkappa_1\neq 0$, $\varkappa_2\neq 0$, consequently for all $0\neq s_1\neq s_2\neq 0$

$$\varrho(s_1\mathbf{d}, s_2\mathbf{d}) = \frac{\varkappa_1}{\varkappa_2}\sigma(s_1\mathbf{d}, s_2\mathbf{d}). \tag{16}$$

But, we shall now give a ϱ and σ such that this equation does not hold identically in s_1 and s_2. With relations (12) and (14) equation (16) becomes

$$\|s_1\mathbf{d}\|^2\,\|s_2\mathbf{d}\|^2\,\|(s_1-s_2)\mathbf{d}\|^2 = \frac{\varkappa_1}{\varkappa_2}h(s_1\mathbf{d})\,\|s_2\mathbf{d}\|^2\,\|(s_1-s_2)\mathbf{d}\|^2$$

and consequently

$$\|s_1\mathbf{d}\|^2 = \frac{\varkappa_1}{\varkappa_2}h(s_1\mathbf{d})$$

for all sufficiently small $s_1\neq 0$. Now the function

$$h(\mathbf{x}) = e^{\|\mathbf{x}\|^2}\cdot\|\mathbf{x}\|^2$$

can not satisfy identically the previous equation, which completes the proof.

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