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The heat kernel for p-forms on manifolds of bounded geometry

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1. Introduction

In [3] an approximation result for the eigenvalues below the essential spectrum of the Laplace operator was proved for open manifolds. These eigenvalues were approximated by the eigenvalues of some sequence of semicombinatorical Laplace operators. The essential assumptions were completeness and bounded geometry up to a certain order. Using this, the first author proved, following a paper of DONNELLY [6], the existence of a good fundamental solution of the heat operator acting on functions, or what is the same, the existence of a good heat kernel. For p-forms this existence was presumed. It is widely believed that the existence result holds for p-forms and several authors refer to [4] for example. But in [4] a rigorous proof was given only for functions. Further, in [5] there is a nice existence proof for functions and a uniqueness proof for p-forms. The paper [4] does not contain an existence proof for p-forms. As a matter of fact, we have never seen such a proof until now. This is in some sense understandable because it needs some nontrivial facts that have to be established. In this paper we present an existence proof for a good heat kernel on open manifolds of bounded geometry of infinite order, as expressed by Theorem 4.1. The uniqueness then follows from [5].

The paper is organized as follows. In Section 2 we introduce and summarize some facts on manifolds of bounded geometry. Section 3 is devoted to the main technical lemmas concerning the construction of the heat kernel. Finally, in Section 4 we present the main results of this paper.

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2. Manifolds of bounded geometry

Let (M^N, g) be open and complete. Denote the curvature tensor of (M^N, g) by R and the Levi—Civita derivative by ∇. We consider the following conditions:

$$r_{\mathrm{inj}}(M) = \inf_{x\in M} r_{\mathrm{inj}}(x) > 0, \tag{1}$$

i.e. the injectivity radius possesses a positive lower bound.

(B_m) There exist bounds C_i such that $|\nabla^i R| \leqq C_i$, $0 \leqq i \leqq m$.

Assuming condition (1), we consider further the condition

(BC^m) For every $\varepsilon > 0$, $0 < \varepsilon < r_{\mathrm{inj}}(M)$, and multiindex

$$\alpha = \{\alpha_1, \dots, \alpha_N), \alpha_i \geqq 0, \ |\alpha| = \alpha_1 + \dots + \alpha_N \leqq m$$

and every choice of an orthonormal base in all points $x \in M$ there exist constants C_α independent of x such that $|D^\alpha g_{ij}| \leqq C_\alpha$, $y \in B_\varepsilon(x)$, in normal coordinates x^i defined in the open ball $B_\varepsilon(x)$.

Remarks. 1. (B_0) is equivalent to the boundedness of the sectional curvature. 2. (BC^m) is independent of the choice of the orthonormal base in T_xM. This follows from the chain rule, the triangle inequality and the compactness of the orthogonal group $O(n)$. 3. The boundedness of the $|D^\alpha(g_{ij})|$, $|\alpha| \leqq m$, implies the boundedness of the $|D^\alpha(g^{ij})|$. For $|\alpha| = 0$ this is seen from

$$(g_{ij})(g^{ij}) = E. \tag{2.1}$$

Assuming the validity for $|\alpha| = m-1$, we obtain the validity for m, applying D^α to (2.1), expressing $D^\alpha(g_{ij})$ by the $D^\beta(g_{kl})$, $|\beta| \leqq m$ and $D^\gamma(g^{rs})$, $|\gamma| \leqq m-1$ and applying the induction assumption.

We summarize some relations between the above conditions.

Proposition 2.1. *Let (M^N, g) be open, complete and satisfying* (1).

(a) (BC^m) *implies* (B_{m-2}),

(b) (BC^∞) *and* (B_∞) *are equivalent,*

(c) (B_0) *implies* (BC^0),

(d) (B_i) *implies* (BC^i).

Proof. (a) The curvature tensor can be expressed by derivatives of the g_{ij}, g^{kl} of order $\leqq 2$.

(b) We refer to [4], page 33.

(c) The boundedness of the g_{ij}, assuming (B_0), is just Lemma 1 of [9].

(d) In [10] it was shown that (B_i) implies the boundedness of the Christoffel symbols Γ^j_{ik}. From

$$\Gamma_{j,ik} = g_{rj}\Gamma^r_{ik}, \quad \Gamma_{j,ik}+\Gamma_{i,jk} = (\partial/\partial x^k)g_{ij} \tag{2.2}$$

and (c) we obtain the assertion.

Examples of open manifolds satisfying (1) and (B_∞) are the Riemannian homogeneous spaces, in particular the symmetric spaces of noncompact type.

The existence problem for metrics satisfying (1) and (B_m) is more subtle. The condition (B_∞) does not imply (1), as cusp manifolds of constant curvature $K=-1$ show. Cheeger, Gromov and Taylor presented in Theorem 4.7 of [4] explicit lower bounds for the injectivity radius $r_{\mathrm{inj}}(x)$ by relative volume estimates, assuming additionally curvature bounds. As a trivial conclusion, the injectivity radius of an open manifold in general is governed by the curvature and by additional geometrical entities.

Let us list up some classes of open manifolds admitting a natural construction of metrics of bounded geometry.

Proposition 2.2. *The following classes of smooth open manifolds admit a natural construction of complete metrics satisfying* (1) *and* (B_∞).

(a) *Reductive homogeneous spaces G/H, G beeing a Lie group and H a compact subgroup.*

(b) *Coverings of closed manifolds.*

(c) *Open manifolds which are built up by infinitely often gluing together a finite number of bordisms (manifolds with so called almost periodic ends, cf.* [7]*). In particular, any infinite connected sum of a finite number of closed manifolds or manifolds with a finite number of ends, each of them collared, belong to that class.*

(d) *Leaves of a foliation of a compact manifold.*

(e) *Every finite connected sum of open manifolds, each of them admitting a metric of the above type.*

Proof. (a) Every such manifold admits a metric making it to a Riemannian homogeneous space.

(b) Equip the closed manifold with any metric and take its lift.

(c) If $M_1, \dots, M_r$ are the nondiffeomorphic boundaries, fix a metric g_ϱ at M_ϱ, extend g_ϱ as a product metric to collar neighbohoods and then to the bordisms. For a collared end there is a simpler construction by fixing a product metric at each end and extending the end metrics to the remaining compact part of M.

(d) This item was proved in [8].

(e) The proof is trivial.

Remark. Natural construction here means that the construction of the metric is in a certain sense adapted to the topology of M. Nevertheless, a much more general existence theorem holds true. Namely, we reformulate Theorem 2′ of [8] as

Theorem 2.3. *Every open manifold admits a complete Riemannian metric satisfying* (1) *and* (B_∞).

In the present paper we call an open, complete manifold (M^N, g) satisfying (1) and (B_∞) a manifold of bounded geometry (of infinite order).

In the next Section we need some properties of the Christoffel symbols for such manifolds which we establish now. We recall that (B_∞) implies (BC^∞) according to (2.2). Fixing $x \in M$ and $0 < \varepsilon < r_{\text{inj}}(M)$, we consider geodesic polar coordinates $(r, u) = (r, u^1, \ldots, u^{N-1}) = (x^1, \ldots, x^N)$ around x. Then according to the tensorial transformation rule, we have

$$|D^\alpha g_{ij}(y)| \leqq C_\alpha \tag{2.3}$$

and

$$|D^\alpha g^{kl}(y)| \leqq C'_\alpha \tag{2.4}$$

for all $y \in U_\varepsilon(x)$, C_α, C'_α independent of x. From the definitions of the Christoffel symbols, (2.3) and (2.4) we immediately obtain

Proposition 2.4. *Let* (M^N, g) *be of bounded geometry,* $x \in M$, $0 < \varepsilon < r_{\text{inj}}(x)$, $(r, u^1, \ldots, u^{N-1})$ *geodesic polar coordinates. Then there exist constants* K_α *independent of* x *such that*

$$|D^\alpha \Gamma^k_{ij}(y)| \leqq K_\alpha \tag{2.5}$$

for all $y \in U_\varepsilon(x)$.

3. The main estimates for the heat kernel construction

Let (M^N, g) be open, complete, oriented and of bounded geometry. We denote by Ω resp. Ω^p_0 the vector space of all smooth p-forms with compact support, by ${}^2\Omega^p$ the vector space of all measurable square integrable p-forms and by $D(\bar{\Delta}) \subset {}^2\Omega^p$ the domain of the closure of the Laplace operator

$$\Delta = d\delta + \delta d \colon \Omega^p_0 \to \Omega^p_0 .$$

Since $\bar{\Delta}$ is nonnegative and selfadjoint, the spectral theorem implies representations

$$\bar{\Delta} = \int_0^\infty \lambda \, dE_\lambda, \quad e^{-t\bar{\Delta}} = \int_0^\infty e^{-t\lambda} \, dE_\lambda .$$

If $e^{-t\bar{\Delta}}$ can be written as an integral operator, the kernel of the latter is called the heat kernel of (M^N, g) for p-forms. One asks then for the properties of the kernel.

An integral kernel always exists according to the Schwartz kernel theorem, but this kernel has on open manifolds no importance since it has no mapping properties between L^q-spaces, $1 \leqq q \leqq \infty$.

Let us now make the definitions precise. A two-point form E^p with values $E^p(t, x, y) \in \Lambda^p T_x M \otimes \Lambda^p T_y M$ is called a good global heat kernel, if it satisfies the following conditions:

(H1) $E^p(t, x, y)$ is smooth for $t > 0$.

(H2) $(\partial/\partial t + \Delta) E^p(t, x, y) = 0$, where we apply Δ acts on E^p as a section depending on y.

(H3) $\lim_{t \to 0^+} \int_M E^p(t, x, y) \wedge * \omega_0(y) = \omega_0(x)$ for all $x \in M$ and

$$\omega_0 \in \Omega_0^p, \quad \text{i.e.} \quad E^p(t, x, y) \to \delta_{x,y}.$$

(H4) There exist constants $C_1, C_2 > 0$, depending on l, m, n, such that for all $x, y \in M$, $0 < t < \infty$

$$|(\partial/\partial t)^l \nabla^m \nabla^n E^p(t, x, y)| \leqq C_1 t^{-N/2-(m+n)/2-1} \exp\left(-C_2 r^2(x, y)/t\right).$$

(H5) The heat kernels $E^p(t, x, y)$ and $E^{p+1}(t, x, y)$ are related by $\bar{d}_x(E^p(t, x, y) = \bar{\delta}_y E^{p+1}(t, x, y)$.

The main aim of this paper is an existence proof for a good heat kernel, assuming (M^N, g) to be of bounded geometry. The method of proof consists in summing up iterated convolutions of a certain initial expression, where the convergence is guaranteed by some majorization.

Let us start with preparatory lemmas. Assume $0 < \varepsilon < r_{\text{inj}}$, $\Phi \in C_0^\infty(\mathbf{R})$, $\varphi(\alpha) = 1$ for $|\alpha| < \varepsilon/2$ and $\varphi(\alpha) = 0$ for $|\alpha| > 1$. Then we define $\eta \colon M \times M \to \mathbf{R}$ by means of $\eta(x, y) := \eta(r(x, y))$. We define a smooth two-point form ${}^{(1)}E(t, x, y)$ as follows:

$${}^{(1)}E^p(t, x, y) := (4\pi t)^{-N/2} \exp\left(-r^2(x, y)/4t\right) \sum_{i=0}^{k} t^i U_i(x, y) \eta(x, y) =: S^{p_k}(t, x, y) \eta(x, y),$$

where $U_i(x, y)$, $0 \leqq i \leqq k$ are some smooth two-point p-forms, k fixed.

Lemma 3.1. *The two-point forms $U_i(x, y)$, $0 \leqq i \leqq k$, can be choosen such that*

(i) $(\partial/\partial t + \Delta_y) S^{p_k}(t, x, y) = (4\pi t)^{-N/2} t^{k-N/2} \exp\left(-r^2(x, y)/4t\right) \Delta_y U_k(x, y)$.

(ii) *There exists some constant $D_l > 0$ such that for all $0 \leqq i \leqq k$, $0 \leqq l \leqq k$, we have $|\Delta_y^l U_i(x, y)| \leqq D_l$.*

Proof. The two-point forms $U_i(x, y)$, $0 \leqq i \leqq k$, are the classical Hadamard coefficients. Existence, uniqueness, and recursion formulae for the $U_i(x, y)$ are shown in the literature [2, 11, 12]. The calculation of these Hadamard coefficients leads to a system of differential equations of the following form:

$$(r(\partial/\partial r)+G+k)U_k(x,\cdot)=\Delta U_{k-1}(x,\cdot),\quad r(x,\cdot)<\varepsilon/2,$$
$$U_0(x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}=g_{i_1j_1}(x)\dots g_{i_pj_p}(x),\quad U_{-1}(x,\cdot):=0,$$

where G means some matrix function.

An integral recursion formula for the $U_k(x,y)$ is given by

$$U_k(x,y)=-U_0(x,y)\varrho^{-k}\int_0 r^{k-1}U_0(x,z)^{-1}\Delta_y U_{k-1}(x,z)\,dr \tag{3.1}$$

where $\varrho:=r(x,y)$, $r:=r(x,z)$ and z lies on the geodesic connecting x and y.

Assertion (ii) follows from an analoguous integral recursion formula obtained by covariant differentiation of (3.1) and from the assumption of bounded geometry.

Further we set

$${}^{(1)}R^p(t,x,y):=(\partial/\partial t+\Delta_y)\,{}^{(1)}E^p(t,x,y).$$

Now we will estimate $|{}^{(1)}R^p(t,x,y)|$.

Lemma 3.2. *There exist constants $A_1(T)>0$, $A_2(T)>0$ depending on $T>0$ such that for all $0<t\leqq T$, $x,y\in M$*

$$|{}^{(1)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq A_1(T)t^{k-N/2}\exp\left(-A_2(T)r^2(x,y)/t\right). \tag{3.2}$$

Proof. We use the following well-known formula $\Delta(f\Phi)=(\Delta f)\Phi+f\Delta\Phi-$ $-2\nabla_{\mathrm{grad}\,f}\Phi$, for $f\in C^\infty(M)$, $\Phi\in\Omega^p$. With that we obtain

$$(\partial/\partial t+\Delta_y)S^{p_k}(t,x,y)\eta(x,y)=(4\pi)^{-N/2}t^{k-N/2}\exp\left(-r^2(x,y)/4t\right)\Delta_y U_k(x,y)+$$
$$+S^{p_k}(t,x,y)\Delta_y\eta(x,y)-2\nabla_{\mathrm{grad}\,\eta}S^{p_k}(t,x,y).$$

The estimation of the first term follows from Lemma 3.1 and our assumption of bounded geometry. In the expressions of the second and third ones there occur factors η' and η'' which are zero for $r(x,y)<\varepsilon/2$. So they decrease exponentially to zero as $t\to+0$. Furthermoore $|\Delta_y\eta(x,y)|$ is uniformly bounded because of bounded geometry. These arguments yield the desired estimations.

Corollary 3.3. *Let (M^n,g) be open, complete and of bounded geometry. Then there exist constants $A_1(T)>0$, $A_2(T)>0$ depending on $T>0$ such that for all $0<t\leqq T$ and $x,y\in M$*

$$|{}^{(1)}R^p(t,x,y)|_{(x,y)}\leqq A_1'(T)t^{k-N/2}\exp\left(-A_2(T)r^2(x,y)/t\right), \tag{3.3}$$

where $|\cdot|_{(x,y)}$ means the pointwise norm of two-points forms.

Proof. The assertion follows from (3.2) and from the fact that for manifolds of bounded sectional curvature and our choice of coordinate systems the pointwise Riemannian norm and the euclidean norm are equivalent (cf. [9]).

Now we define for two-point forms A^p and B^p their convolution according to

$$A^p * B^p := \int_0^t \int_M A^p(s, x, z) \wedge * B^p(t-s, z, y)\, d_{\mathrm{vol}_z}\, ds,$$

i.e.:

$$A^p * B^p = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_p}} (A^p * B^p)_{i_1, \ldots, i_p, j_1, \ldots, j_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \otimes dy^{j_1} \wedge \ldots \wedge dy^{j_p}$$

with

$$(A^p * B^p)_{i_1, \ldots, i_p, j_1, \ldots, j_p} :=$$

$$:= \sum_{k_1<\ldots<k_p} \int_0^t \int_M A^p_{i_1, \ldots, i_p, k_1, \ldots, k_p}(s, x, z) B^{p, k_1, \ldots, k_p}_{j_1, \ldots, j_p}(t-s, z, y)\, d_{\mathrm{vol}_z}\, ds.$$

We set ${}^{(i)}R^p := {}^{(1)}R^p * \ldots * {}^{(1)}R^p$ i-times, and assume $k > N/2$.

Lemma 3.4. *Let be $T>0$ and I a positive integer. Then there exist constants $A_3, A_4>0$ such that for $0<t\leqq T$, $1\leqq i\leqq I$, $x, y\in M$ and all $i_1<\ldots<i_p$, $j_1<\ldots<j_p$ we have*

$$|{}^{(i)}R^p(t, x, y)_{i_1, \ldots, i_p, j_1, \ldots, j_p}| \leqq A_3 t^{k-N/2+i-1} \exp\left(-A_4 r^2(x, y)/t\right). \tag{3.4}$$

Proof. We perform mathematical induction. For fixed x by definition of ${}^{(1)}R^p$ the y-support of ${}^{(i)}R^p_{i_1 \ldots, i_p, j_1, \ldots, j_p}$ is contained in $B_{i\varepsilon}(x)$. We consider $i=2$. Then, denoting $\sum\limits_{(k)} := \sum\limits_{k_1<\ldots<k_p}$ we have

$$|{}^{(2)}R^p(t, x, y)_{i_1, \ldots, i_p, j_1, \ldots, j_p}| =$$

$$\begin{aligned} &= \Big|\sum_{(k)} \int_0^t \int_M {}^{(i)}R^p(s, x, z)_{i_1, \ldots, i_p, k_1, \ldots, k_p}\, {}^{(i)}R^p(t-s, z, y)^{k_1, \ldots, k_p}_{j_1, \ldots, j_p}\, d_{\mathrm{vol}_x}\, ds\Big| = \\ &= \Big|\int_0^t \int_M \Big[\sum_{(k)} {}^{(i)}R^p(s, x, z)_{i_1, \ldots, i_p, k_1, \ldots, k_p}\, {}^{(i)}R^p(t-s, z, y)_{i_1, \ldots, i_p, j_1, \ldots, j_p} \times \\ &\qquad \times g^{k_1 i_1} \ldots g^{k_p i_p}\Big]\, d_{\mathrm{vol}_x}\, ds\Big| \leqq A_1^2 D_2\, {}^{(2)}\tilde{R}(t, x, y), \end{aligned} \tag{3.5}$$

with some constant $D_2>0$ and

$$\begin{aligned} {}^{(2)}\tilde{R}(t, x, y) := \int_0^t \int_{B_\varepsilon(x)\cap B_\varepsilon(y)} & s^{k-(N/2)}(t-s)^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times \\ & \times \exp\left(-A_2 r^2(z, y)/(t-s)\right) d_{\mathrm{vol}_z}\, ds. \end{aligned}$$

For the estimation of ${}^{(2)}\tilde{R}(t, x, y)$ we need

Lemma 3.5.

$$r^2(x, z)/s + r^2(z, y)/(t-s) \geqq r^2(x, y)/t + \left(t/s(t-s)\right)[r(x, z) - sr(x, y)/t]^2.$$

For the simple proof which immediately follows from the triangle inequality, we refer to [3].

Lemma 3.5 now implies

$$|^{(2)}\tilde{R}(t,x,y)| \leqq \exp\left(-A_2 r^2(x,y)/t\right) \int_0^t \int_{B_\varepsilon(x)\cap B_\varepsilon(y)} s^{k-(N/2)}(t-s)^{k-(N/2)} \times$$
$$\times \exp\left[-A_2\left(r(x,z)-r(x,y)(s/t)\right)^2\left(t/s(t-s)\right)\right] d_{\mathrm{vol}_x}\, ds \leqq t^{k-(N/2)+1}[k-(N/2)+1]^{-1}\times$$
$$\times \exp\left(-A_2 r^2(x,y)/t\right) \int_0^t \int_{S^{N-1}} \int_0^\varepsilon (t-s)^{k-(N/2)}\times$$
$$\times \exp\left[-A_2\left(r(x,z)-r(x,y)(s/t)\right)^2\left(t/s(t-s)\right)\right] \Theta_x(z)\, dr_x\, du_x\, ds,$$

where (r_x, u_x) are the geodesic polar coordinates of $z\in B_\varepsilon(x)$ and $\Theta_x(z):=$ $:=(\det g_{ij})^{1/2}(z)$. According to the Rauch comparison theorem and our assumption of bounded geometry there exists a constant $D_3>0$ independent of x such that $|\Theta_x(z)|\leqq D_3$ for all $z\in B_\varepsilon(x)$.

We set $\varrho:=r(x,y)$, $r:=r(x,z)$. Then there remains the integral $I(s):=$ $:=\int_0^\varepsilon \exp\left[-A_2 t\left(r-(s/t)\varrho\right)^2 s(t-s)\right] dr$ to be estimated. But $I(s)$ decreases at least like $s^{N/2}$ resp. $(t-s)^{N/2}$ for $s\to+0$ resp. $s\to -t$. For $|^{(2)}\tilde{R}(t,x,y)|$ this implies the estimate

$$|^{(2)}\tilde{R}(t,x,y)| \leqq t^{k-(N/2)+1}[k-(N/2)+1]^{-1}\exp\left(-A_2 r^2(x,y)/t\right) D_3\left(\int_{S^{N-1}} du\right)\times$$
$$\times \int_0^t s^{k-(N/2)} I(s)\, ds \leqq D_3 D_4 D_5\, t^{k-(N/2)+1}[k-(N/2)+1]^{-1}\exp\left(-A_2 r^2(x,y)/t\right),$$

where

$$\int_{S^{N-1}} du \leqq D_4$$

and

$$\int_0^t (t-s)^{k-(N/2)} I(s)\, ds \leqq D_5.$$

Using these estimates and (3.5), we obtain for $0<t\leqq T$ and $x, y, \in M$

$$(3.6)\qquad |^{(2)}R^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq$$
$$\leqq A_1^2 D_2 D_3 D_4 D_5\, t^{k-(N/2)+1}[k-(N/2)+1]^{-1}\exp\left(-A_2 r^2(x,y)/t\right).$$

Using the estimate of Corollary 3.3, and its iteration, we obtain

$$(3.7)\qquad |^{(i)}R(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_1^i D_2^{i-1}\, {}^{(i)}\tilde{R}(t,x,y),$$

where ${}^{(i)}\tilde{R}(t,x,y)$ is the i-fold convolution of $t^{k-(N/2)}\exp\left(-A_2 r^2(x,y)/t\right)$.

In the sequel we need also an estimate for ${}^{(i)}\tilde{R}(t, x, y)$, which we establish again by mathematical induction, namely

$$(3.8)\qquad |{}^{(i)}\tilde{R}(t, x, y)| \leqq (D_3 D_4 D_5)^{i-1} t^{k-(N/2)+i-1} \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right)[k-(N/2)+1]^{-1} \dots [k-(N/2)+i-1]^{-1}.$$

For $i=2$ this is already proved. The induction step $i \to i+1$ shall be done below. Assuming (3.8) for a moment, we obtain from the induction assumption and the estimation of the first i convolution factors of ${}^{(i+1)}\tilde{R}(t, x, y)$

$$|{}^{(i+1)}\tilde{R}(t, x, y)| \leqq (D_3 D_4 D_5)^{i-1}[k-(N/2)+1]^{-1} \dots [k-(N/2)+i-1]^{-1} \times$$
$$\times \int_0^t \int_{B_{i\varepsilon}(x)\cap B_\varepsilon(y)} (t-s)^{k-(N/2)+i-1} s^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times$$
$$\times \exp\left(-A_2 r^2(z, y)/(t-s)\right) d_{\mathrm{vol}_z}\, ds.$$

Denoting the last integral by J, we get

$$|J| \leqq \int_0^t \int_{S^{N-1}} \int_0^\varepsilon (t-s)^{k-(N/2)+i-1} s^{k-(N/2)} \exp\left(-A_2 r^2(x, z)/s\right) \times$$
$$\times \exp\left(-A_2 r^2(z, y)/(t-s)\right) \Theta_x(z)\, dr\, du\, ds \leqq D_3 D_4 t^{k-(N/2)+i}[k-(N/2)+i]^{-1} \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right) \int_0^t \int_0^\varepsilon s^{k-(N/2)} \exp\left(-A_2 t\left(r-(s/t)\varrho\right)^2/s(t-s)\right) dr\, ds \leqq$$
$$\leqq D_3 D_4 D_5 t^{k-(N/2)+1}[k-(N/2)+i]^{-1} \exp\left(-A_2 r^2(x, y)/t\right).$$

This finishes the induction for ${}^{(i+1)}\tilde{R}(t, x, y)$ and shows

$$|{}^{(i+1)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_1^{i+1}(D_2 D_3 D_4 D_5)^i \times$$
$$\times \exp\left(-A_2 r^2(x, y)/t\right) t^{k-(N/2)+i}[k-(N/2)+1]^{-1} \dots [k-(N/2)+i]^{-1}.$$

Furthermore, there exist constants $A_3, A_4 > 0$ such that for $0 < t \leqq T$, $2 \leqq i \leqq I$

$$(3.9)\qquad A_1^i (D_2 D_3 D_4 D_5)^{i-1} \exp\left(-A_2 r^2(x, y)/t\right)[k-(N/2)+1]^{-1} \dots$$
$$\dots [k-(N/2)+i-1]^{-1} \leqq A_3 \exp\left(-A_4 r^2(x, y)/t\right)$$

which finishes the proof of the Lemma.

Lemma 3.6. *For every* $T > 0$ *there exist contstants* $A_5, A_6 > 0$ *such tath*

$$(3.10)\quad |{}^{(2m)}R^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_5 A_6^m t^{k-(N/2)+2m-1}(m!)^{-1} \exp\left(-A_2 r^2(x, y)/t\right)$$

for all $0 < t \leqq T$ *and all positive integers* m.

Proof. The calculations in the proof of Lemma 3.4 give for $i=2m$

$$|^{(2m)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_1^{2m}(D_2 D_3 D_4 D_5)^{2m-1}\times$$
$$\times t^{k-(N/2)+2m-1}[k-(N/2)+1]^{-1}\dots[k-(N/2)+2m-1]^{-1}\exp\big(-A_2 r^2(x, y)/t\big).$$

Using

$$[k-(N/2)+1]\dots[k-(N/2)+2m-1] > m!(m+1)\dots(2m-1)$$

we obtain

$$A_5 A_6^m (m!)^{-1} t^{k-(N/2)+2m-1}\exp\big(-A_2 r^2(x, y)/t\big),$$

as an upper bound for the right-hand side of (3.10) (where $A_5, A_6 > 0$).

Lemma 3.7. *For every* $T>0$ *there exist constants* $A_7, A_8>0$ *such that*

(3.11) $$|^{(2m+1)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_7 A_8^m (m!)^{-1} t^{k-(N/2)+2m}\exp\big(-A_2 r^2(x, y)/t\big)$$

for all $0<t\leqq T$ *and all positive integers* m.

Proof.

$$|^{(2m+1)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_1^{2m+1}(D_2 D_3 D_4 D_5)^{2m}(m!)^{-1} t^{k-(N/2)+2m}\times$$
$$\times\exp\big(-A_2 r^2(x, y)/t\big) \leqq A_7 A_8^m (m!)^{-1} t^{k-(N/2)+2m}\exp\big(-A_2 r^2(x, y)/t\big).$$

Let us define $Q^p := \sum_{i=1} (-1)^{i\,(i)}R^p$, i.e.

$$Q^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p} = \sum_{i=1} (-1)^{i\,(i)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}.$$

Lemma 3.8. *For all* $T>0$ *the series for* $Q^p_{i_1,\dots,i_p,j_1,\dots,j_p}$ *converges absolutely and uniformly. There exist constants* $A_9, A_{10}>0$, *depending on* T, *such that*

(3.12) $$|Q^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_9 t^{k-(N/2)}\exp\big(-A_{10} r^2(x, y)/t\big)$$

for all $0<t\leqq T$, $x, y\in M$, $i_1<\dots<i_p$, $j_1<\dots<j_p$.

Proof. The convergence follows from the preceding two lemmas since $Q^p_{i_1,\dots,i_p,j_1,\dots,j_p}$ can be estimated from above by an exponential series. Furthermore, we obtain from (3.2), (3.4), (3.10), (3.11):

$$|^{(1)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_1 t^{k-(N/2)}\exp\big(-A_2 r^2(x, y)/t\big),$$
$$\sum_{m=1} |^{(2m)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_5\exp\big(-A_2 r^2(x, y)/t\big)\times$$
$$\times\Big[\sum_{m=1} (A_6^m/m!)\, t^{k-(N/2)+2m-1}\Big] \leqq D_5\exp\big(-D_6 r^2(x, y)/t\big),$$
$$\sum_{m=1} |^{(2m+1)}R^p(t, x, y)_{i_1,\dots,i_p, j_1,\dots,j_p}| \leqq A_7\exp\big(-A_2 r^2(x, y)/t\big)\times$$
$$\times\Big[\sum_{m=1} (A_8^m/m!)\, t^{k-(N/2)+2m}\Big] \leqq D_7\exp\big(-D_8 r^2(x, y)/t\big),$$

where D_5, D_6, D_7, D_8 are positive constants. This provides the asserted estimate.

4. The main results

We set $E^p:={}^{(1)}E^p-Q^p*{}^{(1)}E^p$ and show that E^p is the asserted heat kernel.

Main Theorem 4.1. *Let* (M^N, g) *be open, complete, and of bounded geometry. Then there exists a good global heat kernel* $E^p(t, x, y)$ *satisfying the conditions* (H1)–(H4), *and*

(H5) $E^p(t, x, y)=E^p(t, y, x)$ *for all* $x, y\in M$ *(symmetry),*

(H6) $E^p(t+s, x, y)= \int_M E^p(t, x, z)\wedge *E^p(s, z, y)$ *(semigroup property).*

Moreover, E^p *is uniquely determined.*

Proof. (H1) Smoothness is a local property and it is sufficient to establish it for all compact subsets. The kernels ${}^{(1)}E^p$ and ${}^{(1)}R^p$ are smooth by construction. On compact subsets one can differentiate ${}^{(1)}R^p$ under the integral sign, thus establishing the smoothness of ${}^{(1)}R^p$. Also on compact subsets the series for Q^p and its derivatives converge uniformly according to the estimates of Section 3.

(H2) Using, once again, the argument of uniform convergence, we obtain for $0<t\leqq T$ and $k>N/2+2$ (c.f. [2])

$$(\partial/\partial t+\Delta)E^p = (\partial/\partial t+\Delta)({}^{(1)}E^p-Q^p*{}^{(1)}E^p) = {}^{(1)}R^p-Q^p-Q^p*{}^{(1)}R^p =$$
$$= {}^{(1)}R^p-\sum_{i=1}(-1)^i\,{}^{(1)}R^p-\sum_{i=2}(-1)^i\,{}^{(1)}R^p = 0.$$

(H3) For $\omega_0\in\Omega_0^p$ there holds

$$\int_M E^p(t, x, y)\wedge *\omega_0(y) = \sum_{i_1<\ldots<i_p}\Big[\int_M E^p(t, x, y)\wedge *\omega_0(y)\Big]_{i_1,\ldots,i_p,\,j_1,\ldots,j_p} dx^{i_1}\wedge\ldots\wedge dx^{i_p},$$

where

$$\Big[\int_M E^p(t, x, y)\wedge *\omega_0(y)\Big]_{i_1,\ldots,i_p,\,j_1,\ldots,j_p} =$$
$$= \sum_{(k)}\int_M E^p(t, x, y)_{i_1,\ldots,i_p,\,k_1,\ldots,k_p}\,\omega_0(y)^{k_1,\ldots,k_p}\,d_{\mathrm{vol}_y} =$$
$$= \sum_{(k)}\int_M {}^{(1)}E^p(t, x, y)_{i_1,\ldots,i_p,\,k_1,\ldots,k_p}\,\omega_0(y)^{k_1,\ldots,k_p}\,d_{\mathrm{vol}_y} -$$
$$-\sum_{(k)}\int_M (Q^p*{}^{(1)}E^p)(t, x, y)_{i_1,\ldots,i_p,\,k_1,\ldots,k_p}\,\omega_0(y)^{k_1,\ldots,k_p}\,d_{\mathrm{vol}_y}.$$

Introducing normal coordinates centered at x, the formula

$$\lim_{t\to 0^+}\int_M\sum_{(k)}{}^{(1)}E^p(t, x, y)_{i_1,\ldots,i_p,\,k_1,\ldots,k_p}\,\omega_{0,\,l_1,\ldots,l_p}(y)\times$$
$$\times g^{k_1l_1}(y)\ldots g^{k_pl_p}(y)\,d_{\mathrm{vol}_y} = \omega_{0,\,i_1,\ldots,i_p}(x)$$

follows just like in the Euclidean case. The estimate (3.12) for Q^p gives

(4.1) $$|t^{-1}Q^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_9 t^{k-(N/2)+1} \exp\left(-A_{10} r^2(x, y)/t\right).$$

The calculation $Q^p *^{(1)} E^p = t(t^{-1} Q^p *^{(1)} E^p)$, use of (4.1) and a version of Lemma 3.5 lead to an estimation from above by

$$A_9 t \int_0^t \int_M s^{k-(N/2)+1}(t-s)^{k-(N/2)} \exp\left(-A_{10} r^2(x, z)/s\right) \exp\left(-A_2 r^2(z, y)/(t-s)\right) d_{\mathrm{vol}_z}\, ds \leqq$$
$$\leqq t \int_0^t \int_M \exp\left(-A_{11} r^2(x, y)/t\right) s^{k-(N/2)+1}(t-s)^{k-(N/2)} \times$$
$$\times \exp\left[-A_{12}\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right] d_{\mathrm{vol}_z}\, ds$$

where A_{11}, A_{12} are positive constants. Therefore we have for $\omega_0 \in \Omega_0^p$

$$\left|t \sum_{(k)} \int_M (t^{-1} Q^p *^{(1)} E^p)_{i_1, \dots, i_p, k_1, \dots, k_p} \omega_0(y)^{k_1, \dots, k_p} d_{\mathrm{vol}_y}\right| \leqq$$
$$\leqq t\left\{A_9 \int_M \exp\left(-A_{11} r^2(x, y)/t\right) \int_0^t \int_M s^{k-(N/2)+1}(t-s)^{k-(N/2)} \times\right.$$
$$\left.\times \exp\left[-A_{12}\left(r(x, z)-r(x, y)(s/t)\right)^2\left(t/s(t-s)\right)\right] \max_{k_1<\dots<k_p} |\omega_0(y)^{k_1, \dots, k_p}|\, d_{\mathrm{vol}_z}\, ds\, dy\right\}.$$

Since $\operatorname{supp} \omega_0$ is compact, we can cover it by a finite number of ε-balls, $\varepsilon < r_{\mathrm{inj}}$, and apply for the estimation of all three integrals the estimates of Section 3. Thus we prove that the expression $\{\dots\}$ remains bounded as $t \to 0^+$ and $\lim_{t \to 0^+} \int_M (Q^p *^{(1)} E^p) \wedge \wedge * \omega_0(y) = 0$. (H3) is proved.

(H6) In order to show the semigroup property

$$E^p(t, x, y) = \int_M E^p(s, x, z) \wedge * E^p(t-s, z, y)$$

we prove that

$$F^p(t, x, y) := \int_M E^p(s, x, z) \wedge * E^p(t-s, z, y)$$

has the properties of a heat kernel. The uniqueness theorem of [5] then ensures $F^p(t, x, y) = E^p(t, x, y)$. In fact, from (H2) for $E^p(t, x, y)$ we obtain

$$(\partial/\partial t + \Delta_y) F^p(t, x, y) = \int_M E^p(s, x, z) \wedge * E^p(t-s, z, y) +$$
$$+ \int_M E^p(s, x, z) \wedge * \Delta_y E^p(t-s, z, y) = \int_M E^p(s, x, z) \wedge * (-\Delta_y) E^p(t-s, z, y) +$$
$$+ E^p(s, z, x) \wedge * \Delta_y E^p(t-s, z, y) = 0.$$

Property (H3) of $E^p(t, x, y)$ implies

$$\lim_{t\to 0^+} F^p(t, x, y) = \lim_{t\to 0^+} \int_M E^p(s, x, z)\wedge * E^p(t-s, z, y) = \delta_{x,z}\delta_{z,y} = \delta_{x,y}$$

since $t\to 0^+$ implies $s\to 0^+$, $t-s\to 0^+$

(H7) (Symmetry). Using property (H3), we get

$$\lim_{s\to 0^+} \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) = E^p(t, y, x),$$

$$\lim_{s\to t^-} \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) = E^p(t, x, y).$$

Then, according to Duhamel's principle for forms,

$$E^p(t, x, y) - E^p(t, y, x) = \int_0^t \partial/\partial s \int_M E^p(s, x, z)\wedge * E^p(t-s, y, z) =$$

$$= \int_0^t \int_M [\Delta_z E^p(s, x, z)\wedge * E^p(t-s, y, z) - E^p(s, x, z)\wedge * \Delta_x E^p(t-s, y, z)] =$$

$$= \int_0^t \int_M \{\bar{d}_z E^p(s, x, z)\wedge * \bar{d}_z E^p(t-s, y, z) + \bar{\delta}_z E^p(s, x, z)\wedge * \bar{\delta}_z E^p(t-s, y, z) -$$

$$-[\bar{d}_z E^p(s, x, z)\wedge * \bar{d}_z E^p(t-s, y, z) + \bar{\delta}_z E^p(s, x, z)\wedge * \bar{\delta}_z E^p(t-s, y, z)]\} = 0.$$

Here we essentially used the completeness of (M^N, g).

(H4). (Estimates for the derivatives.) Assume $T>0$, $0<t\leqq T$ and $k>N/2+ +(m+n)/2+1$, and begin with $l=m=n=0$. Then the proof is done according to the estimates for ${}^{(1)}E$ and Q^p. Next we consider $\nabla_y^n E^p(t, x, y)$. There holds

$$|\nabla_y^n E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq |\nabla_y^n\, {}^{(1)}E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| +$$

$$+ |\nabla_y^n (Q^p * {}^{(1)}E^p)(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}|.$$

We start with the estimation of the first term.

Lemma 4.2. *There exist positive constants A_{13}, $A_{14}(n, T)$ such that for all $0<t\leqq$ $\leqq T$, $x, y\in M$, $i_1<\dots<i_p$, $j_1<\dots<j_p$*

$$|\nabla_y^n\, {}^{(1)}E^p(t, x, y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq A_{13} t^{-N/2-n/2} \exp\left(-A_{14} r^2(x, y)/t\right). \tag{4.2}$$

The proof is by mathematical induction. For $n=0$ it is done. Assume the assertion for $1,\dots,n-1$. Since (M^N, g) has bounded geometry, there exists a uniformly locally finite cover of M^N by geodesic ε-balls, $0<\varepsilon<r_{\mathrm{inj}}$. According to [1] there exists a constant $D_9>0$, independent of x, such that for all $y\in B_\varepsilon(x)$

$$\left|\nabla_y^n \exp\left(-r^2(x, y)/4t\right)\right| \leqq D_9 \left|(\partial^n/\partial r^n) \exp\left(-r^2(x, y)/4t\right)\right|. \tag{4.3}$$

This follows from Lemma 3 of [1] and the boundedness of the Christoffel symbols together with their derivatives. Using the inequality $e^{-\alpha} \leqq (\alpha e)^{-1}$, we obtain

$$|\partial/\partial r \exp(-r^2/4t)| \leqq [\exp(-r^2/4t)(r^2/4t)]^{1/2} \exp(-r^2/8t) \leqq e^{-1} t^{-1/2} \exp(-r^2/8t).$$

Iteration and application of the product rule gives

$$|(\partial^n/\partial r^n) \exp(-r^2/4t)| \leqq D_{10} t^{-N/2-n/2} \exp(-D_{11} r^2/t) \tag{4.4}$$

with positive constants D_{10}, $D_{11}(n, T)$. According to Lemma 4 of [1] there exists a constant $D_{12}>0$ such that

$$\left|\nabla_y^n (r(x, y))\right| \leqq D_{12}. \tag{4.5}$$

Lemma 3.1 (ii), (4.3)—(4.5) and the derivation rules applied to ${}^{(1)}E^p$ provide the asserted estimation.

In order to estimate the second term, we use the uniform convergence of the integrals and can therefore differentiate under the integral sign:

$$|\nabla_y^n (Q^p * {}^{(1)}E^p)(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq \int_0^t \int_M \sum_{(k)} |Q^p(s, x, z)_{i_1, \dots, i_p, k_1, \dots, k_p}| \times$$

$$\times |\nabla_y {}^{(1)}E^p(t-s, z, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \, |g^{k_1 i_1}(z)| \dots |g^{k_p i_p}(z)| \, d_{\mathrm{vol}_z} \, ds \leqq \tag{4.6}$$

$$\leqq D_{13} t^{k-N/2-n/2} \exp(-D_{14} r^2(x, y)/t), \quad D_{13}, D_{14}(n, T) > 0.$$

Now (4.2) and (4.6) provide the asserted estimate. From the symmetry of $E^p(t, x, y)$ in x, y we obtain the analogous estimate

$$|\nabla_x^m E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq A_{15} t^{-N/2-m/2} \exp(-A_{16} r^2(x, y)/t), \quad A_{15}, A_{16}(m, T) > 0.$$

We now turn to t-derivatives of E^p. Clearly,

$$|(\partial^l/\partial t^l) E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq |(\partial^l/\partial t^l) {}^{(1)}E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| +$$

$$+ |(\partial^l/\partial t^l)(Q^p * {}^{(1)}E^p)(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}|.$$

We start with the first term. Since

$${}^{(1)}E^p = f_1(\eta) \sum_{i=0}^{k} t^i U_i(x, y),$$

and the U_i and η are independent of t, it suffices to estimate $(\partial^l/\partial t^l) f_1$:

$$f_1(t, x, y) := (4\pi t)^{-N/2} \exp(-r^2(x, y)/4t),$$

$$(4\pi)^{N/2} (\partial/\partial t)(t^{-N/2} \exp(-r^2(x, y)/4t)) = (-(N/2) t^{-N/2-1} + t^{-N/2}(r^2/4t^2)) \exp(-r^2/4t).$$

Use of the inequality $e^{-\alpha} \leqq (\alpha e)^{-1}$, like in the proof of Lemma 4.2, gives

$$t^{-N/2} \exp(-r^2/4t)(r^2/4t^2) \leqq 2 e^{-1} t^{-N/2-1} \exp(-r^2/8t).$$

Iterating this procedure and applying the product rule, we obtain

$$|(\partial^l/\partial t^l)\,{}^{(1)}E^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq$$
$$\leqq D_{15}t^{-N/2-l}\exp\left(-D_{16}r^2(x,y)/t\right),\quad D_{15},D_{16}(l,T)>0.$$

We estimate the second term as follows:

$$|(\partial^l/\partial t^l)(Q^p*{}^{(1)}E^p)(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq \Big|(\partial^l/\partial t^l)\sum_{(k)}\int_0^t\int_{B_\varepsilon(y)}(4\pi)^{-N/2}(t-s)^{-N/2}\times$$
$$\times\exp\left(-r^2(z,y)/4(t-s)\right)\sum_{i=0}^k t^iU_{i,i_1,\dots,i_p,k_1,\dots,k_p}(z,y)\times$$
$$\times Q^p(s,x,z)_{l_1,\dots,l_p,j_1,\dots,j_p}g^{k_1l_1}(z)\dots g^{k_pl_p}(z)\,d_{\mathrm{vol}_z}\,ds\Big|\leqq$$
$$\leqq\sum_{(k)}\Big|(\partial^{l-1}/\partial t^{l-1})\Big\{\Big[\lim_{s\to t^-}\int_{B_\varepsilon(y)}-\lim_{s\to 0^+}\int_{B_\varepsilon(y)}\Big](4\pi)^{-N/2}(t-s)^{-N/2}\exp\left(-r^2(z,y)/4(t-s)\right)\times$$
$$\times\sum_{i=0}^k t^iU_{i,i_1,\dots,i_p,k_1,\dots,k_p}Q^p(s,x,z)_{l_1,\dots,l_p,j_1,\dots,j_p}g^{k_1l_1}(z)\dots g^{k_pl_p}(z)\,d_{\mathrm{vol}_z}\,ds\Big\}\Big|\leqq$$
$$\leqq D_{17}t^{k-N/2-1}\exp\left(-D_{18}r^2(x,y)/t\right),\quad D_{17},D_{18}(l,T)>0.$$

Gathering the results, we have

$$|\partial^l/\partial t^l\,E^p(t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq A_{17}t^{-N/2-1}\exp\left(-A_{18}r^2(x,y)/t\right),$$
$$A_{17},A_{18}(l,T)>0.$$

Iterating the derivatives ∇_x^m, ∇_y^n, $\partial^l/\partial t^l$, using the above estimates and the fact that $\nabla_y^i(x,y)$ is bounded thanks to the bounded geometry, we finally obtain the asserted estimate for $0<t\leqq T$. In order to establish (H4) completely, we have still to consider the behaviour of the derivatives for $t\to\infty$. To do this, we essentially use the semigroup property (H6). Until now we have proved

$$|(\partial^l/\partial t^l)\nabla_x^mE^p(2t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq$$
$$\leqq\int_M C_{1,m,l}(T)\,t^{-N/2-m/2-1}\exp\left(-C_{2,m,l}(T)\,r^2(x,z)/t\right)C_{1,0,0}(T)\,t^{-N/2}\times$$
$$\times\exp\left(-C_{2,0,0}(T)\,r^2(z,y)/t\right)d_{\mathrm{vol}_z}.$$

Without loss of generality we assume $C_{2,m,1}(T)>C_{2,0,0}(T)$. Lemma 3.5 now gives

$$|(\partial^l/\partial t^l)\nabla_x^mE^p(2t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq C_{1,m,l}(T)C_{1,0,0}(T)\,t^{-N-m/2-1}\times$$
$$\times\exp\left(-C_{2,0,0}(T)\,r^2(x,y)/2t\right)\int_M\exp\left[\left(-C_{2,0,0}(T)/t\right)(2/1)\left(r(x,z)-r(x,y)/2\right)^2\right]d_{\mathrm{vol}_z}.$$

We denote the latter integral by $I_2(t)$ and state as induction assumption

$$|(\partial^l/\partial t^l)\nabla_x^mE^p(\tilde{k}t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}|\leqq C_{1,m,l}(T)C_{1,0,0}(T)^{\tilde{k}-1}\times$$
$$\times t^{-m/2-1}(t^{-N/2})^{\tilde{k}}\exp\left(-C_{2,0,0}(T)\,r^2(x,y)/t\right)I_2(t)\dots I_{\tilde{k}}(t),$$

where

$$I_{\tilde{k}}(t) := \int_M \exp\left[-C_{2,0,0}(T)/t(\tilde{k}/(\tilde{k}-1))\left(r(x,z)-((\tilde{k}-1)/\tilde{k})\,r(x,y)\right)^2\right] d_{\mathrm{vol}_z}.$$

From the semigroup property we obtain

$$\left|(\partial^l/\partial t^l)\nabla_x^m E^p((\tilde{k}+1)\,t,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}\right| \leqq$$
$$\leqq \int_M C_{1,m,l}(T)\,C_{1,0,0}(T)^{\tilde{k}-1}\,t^{-m/2-1}(t^{-N/2})^{\tilde{k}}\times$$
$$\times \exp\left(-C_{2,0,0}(T)\,r^2(x,z)/\tilde{k}t\right) I_2(t)\dots I_{\tilde{k}}(t)\,C_{1,0,0}(T)\times$$
$$\times \exp\left(-C_{2,0,0}(T)\,r^2(z,y)/t\right) t^{-N/2}\,d_{\mathrm{vol}_z} \leqq C_{1,m,l}(T)\left(C_{1,0,0}(T)\right)^{\tilde{k}} t^{-m/2-1}(t^{-N/2})^{\tilde{k}+1}\times$$
$$\times \exp\left(-C_{2,0,0}(T)\,r^2(x,y)/t\right) I_2(t)\dots I_{\tilde{k}}(t)\times$$
$$\times \int_M \exp\left[(-C_{2,0,0}(T)/t)((\tilde{k}+1)/\tilde{k})\left(r(x,z)-(\tilde{k}/(\tilde{k}+1))\,r(x,y)\right)^2\right] d_{\mathrm{vol}_z} \leqq$$
$$\leqq C_{1,m,l}(T)\left(C_{1,0,0}(T)\right)^{\tilde{k}}(t^{-N/2})^{\tilde{k}+1} I_2(t)\dots I_{\tilde{k}}(t)\,I_{\tilde{k}+1}(t)\times$$
$$\times\, t^{-m/2-1}\exp\left(-C_{2,0,0}(T)\,r^2(x,y)/t\right),$$

where we denoted the last integral by $I_{\tilde{k}+1}(t)$. There exist constants $D_{19}>0$, $k_0>0$, such that for all $\tilde{k}>k_0$

$$t^{-m/2-1}(t^{-N/2})^{\tilde{k}-1} \leqq D_{19}(\tilde{k}t)^{-N/2-m/2-1}.$$

Moreover,

$$I_{\tilde{k}}(t)\to 0 \quad \text{for} \quad t\to 0, \quad I_{\tilde{k}}(t)\to I(t) \quad \text{for} \quad \tilde{k}\to\infty,$$

where

$$I(t) := \int_M \exp\left[-C_{2,0,0}(T)/t\left(r(x,z)-r(x,y)\right)^2\right] d_{\mathrm{vol}_z}.$$

This implies the existence of a constant $k_1>0$ and $t_0\in]0,T]$ such that for all $0<t\leqq t_0$ and $\tilde{k}>k_0$, $\tilde{k}>k_1$

$$|I_{\tilde{k}}(t)| \leqq \left(C_{1,0,0}(T)\right)^{-1}.$$

Therefore, there exists a constant $D_{18}>0$, dependent on t_0, k_0, k_1, such that for all $0<t\leqq t_0$, $x,y\in M$, $\tilde{k}>k_0$, $\tilde{k}>k_1$

$$|(\partial^l/\partial t^l)\nabla_x^m E^p(kt,x,y)_{i_1,\dots,i_p,j_1,\dots,j_p}| \leqq$$
$$\leqq D_{18}(\tilde{k}t)^{-N/2-m/2-1}\exp\left(-C_{2,0,0}(T)\,r^2(x,y)/t\right).$$

Since $t\in]0,t_0[$ was arbitrary, we have the estimation for arbitrary large $\tilde{k}t$.

In a similar manner we estimate $|\nabla_y^n E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}|$. We have for $0 < t \leqq T$

$$|\nabla_y^n E^p(2t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq \int_M C_{1,0}(T) t^{-N/2} \times$$

$$\times \exp(-C_{2,0}(T) r^2(x, z)/t) C_{1,n}(T) t^{-N/2-n/2} \exp(-C_{2,n}(T) r^2(z, y)/t)\, d_{\mathrm{vol}_z} \leqq$$

$$\leqq C_{1,n}(T) C_{1,0}(T) (t^{-N/2})^2 t^{-n/2} \exp(-C_{2,0}(T) r^2(x, y)/t) \times$$

$$\times \int_M \exp[-C_{2,0}(T)/t(2/1)(r(x, z) - (1/2) r(x, y))^2]\, d_{\mathrm{vol}_z}.$$

The last integral shall be denoted by $I_2(t)$. Furthermore, we assumed without loss of generality $C_{2,n}(T) > C_{2,0}(T)$. This implies

$$|\nabla_y^n E(t, x, y)_{i_1, \dots, i_p, j_1 \dots, j_p}| \leqq C_{1,0}(T) C_{1,0}(T) I(t) (t^{-N/2})^2 \times$$

$$\times t^{-n/2} \exp(-C_{2,0}(T) r^2(x, y)/2t).$$

By mathematical induction,

$$|\nabla_y^n E^p(\tilde{k}t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq C_{1,n}(T)(C_{1,0}(T))^{\tilde{k}-1} I_2(t) \dots$$

$$\dots I_{\tilde{k}}(t) (t^{-N/2})^{\tilde{k}} t^{-n/2} \exp(-C_{2,0}(T) r^2(x, y)/\tilde{k}t).$$

There exist constants $t_0, k_0, k_1, D_{19} > 0$, D_{19} dependent on t_0, k_0, k_1, such that for all $0 < t \leqq t_0$, $\tilde{k} > k_0$, $\tilde{k} > k_1$

$$|\nabla_y^n E^p(t, x, y)_{i_1, \dots, i_p, j_1, \dots, j_p}| \leqq D_{19} (\tilde{k}t)^{-N/2-n/2} \exp(-C_{2,0}(T) r^2(x, y)/\tilde{k}t).$$

The time value $t \in]0, t_0[$ was arbitrary and we obtain the estimate for arbitrary large $\tilde{k}t$. Iterating both estimates and using once again the boundedness of $\nabla^l r$, we finally obtain (H4).

(H5). $$\bar{d}_x E^p(t, x, y) = \delta_y E^{p+1}(t, x, y).$$

The property (H3), $\lim_{t \to 0^+} E^p(t, x, y) = \delta_{x,y}$, implies

$$\lim_{s \to 0^+} \int_M E^{p+1}(s, x, z) \wedge * \bar{d}_z E^p(t-s, z, y) = \bar{d}_x E^p(t, x, y),$$

$$\lim_{s \to t^-} \int_M \delta_z E^{p+1}(s, x, z) \wedge * E^p(t-s, z, y) = \delta_y E^{p+1}(t, x, y).$$

Using this, Duhamel's principle and the heat equation, we obtain

$$\bar{d}_x E^p(t, x, y) - \bar{\delta}_y E^{p+1}(t, x, y) =$$

$$= \int_0^t (\partial/\partial s) \int_M E^{p+1}(s, x, z) \wedge * \bar{d}_z E^p(t-s, z, y) =$$

$$= \int_0^t \int_M [\bar{\Delta}_z E^{p+1}(s, x, z) \wedge * \bar{d}_z E^p(t-s, z, y) - E^{p+1}(s, x, z) \wedge * \bar{d}_z \bar{\Delta}_z E^p(t-s, z, y)] =$$

$$= \int_0^t \int_M [\bar{\delta}_z E^{p+1}(s, x, z) \wedge * \bar{\delta}_z \bar{d}_z E^p(t-s, z, y) -$$

$$- \bar{d}_z \bar{\delta}_z E^{p+1}(s, x, z) \wedge * \bar{d}_z E^p(t-s, z, y)] = 0.$$

since (M^N, g) is complete.

This finishes the proof of the main Theorem 4.1.

As it is well known, the existence of a good heat kernel has many good consequences in global analysis, for instance in spectral theory, in the theory of semigroups, and for the existence of characteristic numbers. We do not intend to present all this here, but restrict ourselves to a special class of applications. For many purposes one is interested to invert the Laplace operator Δ outside the L_2-harmonic forms. Denote by ${}^2\Omega^p$ the space of square integrable measurable p-forms on M^N, by ${}^2\Omega^p_k$ the set of all smooth p-forms ω such that $\|\omega\|, \|\Delta\omega\|, \dots, \|\Delta^k\omega\| < \infty$ ($\|\cdot\| = L_2$-norm) and by ${}^2\Omega^{p,k}$ the completion of ${}^2\Omega^p_k$ with respect to ${}^2\|\cdot\|_k$,

$${}^2\|\omega\|_k := \|\omega\| + \|\Delta\omega\| + \dots + \|\Delta^k\omega\|.$$

Let H denote the projection onto

$${}^2\mathscr{H}^p = \{\omega \in \Omega^p \cap {}^2\Omega^p | \bar{d}\omega = \bar{\delta}\omega = 0\} = \ker \bar{\Delta}$$

(since (M^N, g) is complete).

Then one is searching for an operator G satisfying

$$\Delta G\omega = \omega - H\omega$$

and, if possible, for a meaningful integral representation of G. This G is called Green's operator.

Theorem 4.3. *Let (M^N, g) be open, complete, and of bounded geometry. Assume further that $\bar{\Delta} = \bar{\Delta}_p$ has positive eigenvalues below the essential spectrum. Then*

$$G\omega(x) = \int_0 \int_M E^p(t, x, y) \wedge * (\omega - H\omega)(y)\, d_{\mathrm{vol}_y}$$

is a Green operator and has the following properties:

(a) $\|G\omega\| \leqq (2\lambda_1)^{-1/2}\|\omega\|$ *for* $\omega \in \Omega^p_0$, *where* λ_1 *is the first nonzero eigenvalue of* $\bar{\Delta}$. *Hence G can be extended to a bounded linear operator* $G\colon {}^2\Omega^p \to {}^2\Omega^p$.

(b) $G\omega \in {}^2\Omega^{p,k}$ *for arbitrary large* k.
(c) $\omega = H\omega + \bar{d}\delta G\omega + \delta\bar{d}G\omega$ *is the Hodge decomposition.*

Proof. A complete proof is given in [3] under the assumption of the existence of a good heat kernel. This existence we have just now established.

Using Theorem 4.3, we can establish the approximation theorem for the eigenvalues below the essential spectrum by the eigenvalues of semicombinatorical Laplace operators associated to sequences of uniform triangulations also for $0 \leq p \leq N$. For $p=0$ this was completely proved in [3], for $p>0$ under the assumption of the existence of a good heat kernel.

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