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Well-quasiordering depends on the labels

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1. Introduction

A well-quasiordering (WQO) is a quasiordered set containing no infinite decreasing chain and no infinite antichain. A considerable part of the results on WQO is of the form that a concrete category Q is WQO, where $a \leqq b$ means that there exists a Q-morphism from a to b.

As an example let us mention the category T of finite trees with tree embeddings (see [2]). The recent solution of the Wagner's conjecture by ROBERTSON and SEYMOUR ([5]) is also of the form that certain category is WQO.

Other categories are trivially WQO, for example the category F of finite sets and injective mappings or the category H of finite linearly ordered sets and strictly increasing mappings. Still, we come to non-trivial questions, if we introduce more involved orderings:

Let A be a WQO and let Q be a concrete category with finite objects and injective morphisms. We consider a class $Q(A)$ of objects of Q "labeled by" elements of A at each point. We put $a \leqq b$ if there is a morphism from a to b which increases the labels (not necessarily strictly). Now the question is: Is it true that

(1) $$Q(A) \text{ is WQO whenever } A \text{ is WQO?}$$

(1) was proved for F, H by HIGMAN [1] and for T by NASH-WILLIAMS [4]. Of course, it would be useful if (1) were implied by a simpler condition, say,

(2) $$Q(\gamma) \text{ is WQO for any } \gamma \in \mathrm{Ord}.$$

Although this is not known in general, it was proved recently by one of the authors ([3]) for a considerably broad class of categories (for all subcategories of H). Let us note that, by an easy cardinality argument, (2) is equivalent to

(3) $$Q(\omega_1) \text{ is WQO.}$$

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So, for subcategories of H, (3)→(1). However, it has not been known if (1) were not implied by a still weaker condition, say (even!),

$$Q(2) \text{ is WQO.} \tag{4}$$

It is the purpose of this paper to present a bunch of counterexamples of this kind. To be exact, we show that the set

$$M = \{\gamma | (\exists Q)(((\forall \beta < \gamma) Q(\beta) \text{ is WQO}) \& (Q(\gamma) \text{ is not WQO}))\} \tag{5}$$

is confinal in ω_1. We also prove that

$$M \supseteqq \omega,$$

showing that (4) does not imply $Q(3)$ to be WQO.

2. Preliminaries

2.1. Conventions and notation. The cardinality of a set X is denoted by $|X|$. For the ordinals, we use that definition where γ is identified with

$$\{\beta | \beta < \gamma\}.$$

In particular, this will apply to natural numbers. A quasiordering is a reflexive and transitive relation. In a quasiordering, a sequence (a_i) (finite or infinite) is called *bad* if

$$i < j \to a_i \nleqq a_j$$

and is called *good* if

$$i < j \to a_i \leqq a_j.$$

Each infinite sequence contains an infinite good subsequence or an infinite bad subsequence (Ramsey theorem). A quasiordering is called WQO, if no infinite sequence is bad. This definition is equivalent to that used in the Introduction by the Ramsey theorem. For a category C and objects a, b the symbol $C(a, b)$ designates the corresponding hom-set and the symbol Id_a designates the identity on a. For a concrete category, let the forgetful functor be denoted by U.

2.2. Definition. In this paper, a QO-category is a concrete category with finite objects and injective morphisms. For a QO-category Q and a quasiordering A, put

$$Q(A) = \{z = (u_z, c_z) | c_z \text{ is an object of } Q \text{ and } u_z \colon c_z \to A\}.$$

The *quasiordering* on $Q(A)$ is given by $z \leqq t$ if there exists a Q-morphism $\varphi\colon c_z \to c_t$ such that $u_t \circ U(\varphi) \geqq u_z$ (pointwise). We also say that $z \leqq t$ via the morphism φ. In the sequel, we shall use the symbol M for the set defined by formula (5) of the Introduction.

3. The results

To warm up, we start with a special theorem which demostrates the basic idea of our construction.

3.1. Theorem. $M \supseteqq \omega$.

Proof. Let a category Q_k consist of finite sets a_n $(n \in \omega)$ where each a_n is a disjoint union of k sets $a_n^0, \ldots, a_n^{k-1}$. Moreover, we shall assume

$$\lim_{n \to \omega} |a_n^i| = \omega \quad \text{for each } i \in k.$$

The hom-set $Q_k(a_n, a_m)$ will be

(a) $\emptyset$ if $n > m$,

(b) $\{\mathrm{Id}_{a_n}\}$ if $n = m$,

(c) $\{\varphi: a_n \to a_m \mid \varphi \text{ injective and } (\forall i \in k)(\varphi(a_n^i) \subseteqq \bigcup_{j \leqq i} a_m^j) \,\&\, (\exists i \in k)(\varphi(a_n^i) \nsubseteqq a_m^i)\}$

if $n < m$.

To see that $Q_k(k)$ is not WQO, let $z_n(u_n, a_n)$ where u_n sends a_n^i to i for each $i \in k$. It is easily seen that $(z_n)_{n \in \omega}$ is a bad sequence. To prove that $Q_k(i)$ is WQO for $i < k$, introduce an auxilliary category $\bar{Q}_k$ with the same objects as Q_k and with the same morphisms from a_n to a_m for $n \geqq m$, while for $n < m$

$$\bar{Q}_k(a_n, a_m) = \{\varphi: a_n \to a_m \mid \varphi \text{ injective and } (\forall i \in k)(\varphi(a_n^i) \subseteqq \bigcup_{j \leqq i} a_m^j)\}.$$

Let $(z_t) = ((u_t, a_{n(t)}))$ be a bad sequence in $Q_k(i)$, $i < k$. Of course, we have

$$\lim_{t \to \omega} n(t) = \omega,$$

since (z_t) is bad. By HIGMAN's theorem [1], we may assume that (z_t) is good with respect to $\bar{Q}_k(i)$. Let, in $\bar{Q}_k(i)$, $z_s \leqq z_t$ via $\varphi_{s,t}: a_{n(s)} \to a_{n(t)}$. Now, since $i < k$, there is a $j \in i$ such that, for each $K \in \omega$, there exist $p, r \in k, p > r$, and $t(K) \in \omega$ such that

$$|\{x \in a^p_{n(t(K))} \mid u_{t(K)}(x) = j\}| \geqq K,$$

$$|\{x \in a^r_{n(t(K))} \mid u_{t(K)}(x) = j\}| \geqq K.$$

Without loss of generality, we may assume p, r fixed and $t(0) = 0$. Put $t = t(|a^r_{n(0)}| + 1)$. Thus, there exist $x \in a_0^p$, $y \in a_0^r$ such that

$$u_0(x) = u_t(y) = j$$

$$y \notin \operatorname{Im} \varphi_{0,t}.$$

We conclude that, in $Q_k(i)$, $z_0 \leqq z_t$ via a mapping φ given by

$$\varphi(z) = \varphi_{0,t}(z) \quad \text{if} \quad z \neq x$$
$$\varphi(x) = y,$$

contradicting our assumption.

3.2. Theorem. *M is confinal in ω_1.*

Proof below in 3.8.

3.3. The Constructions. Let $\omega \leqq \gamma \leqq \omega_1$. The there exists a bijection s_γ: $\omega \to \gamma$. Let the objects of C_γ be the sets

$$a_\gamma(n) = \{s_\gamma(0), \ldots, s_\gamma(n-1)\}.$$

The hom-set $C_\gamma(a_\gamma(k), a_\gamma(n))$ will be

(1) $\emptyset$ for $k > n$,

(2) $\{\mathrm{Id}_{a_\gamma(n)}\}$ for $k = n$,

(3) the set of all injective mappings $\varphi\colon a_\gamma(k) \to a_\gamma(n)$ such that, for some $j < k$,

(a) $\varphi(s_\gamma(j)) < s_\gamma(j)$,

(b) for $i < j$, $\varphi(s_\gamma(i)) = s_\gamma(i)$,

(c) for $i \leqq j$, $\beta < s_\gamma(i) \to \varphi(\beta) < s_\gamma(i)$.

3.4. Lemma. *C_γ is a QO-category.*

Proof. What remains to show is that, for $k < m < n$,

$$\varphi \in C_\gamma(a_\gamma(k), a_\gamma(n)), \quad \psi \in C_\gamma(a_\gamma(m), a_\gamma(n))$$
$$\psi \circ \varphi \in C_\gamma(a_\gamma(k), a_\gamma(n)).$$

To this end, let φ, ψ satisfy the statement of (3) with constants $j < k, \bar{j} < m$, respectively. We will show that $\psi \circ \varphi$ satisfies it with the constant $\min(j, \bar{j})$. We distinguish two cases:

Case 1. $\bar{j} \geqq j$: The proof of (a), (b), (c) for $\psi \circ \varphi$ is contained in the following computations. (By (a) for φ, $j \leqq \bar{j}$ and (c) for ψ)

$$\psi \circ \varphi(s_\gamma(j)) < s_\gamma(j).$$

For $i < j$ (by (b) for φ, ψ and $j \leqq \bar{j}$)

$$\psi \circ \varphi(s_\gamma(i)) = \psi(s_\gamma(i)) = s_\gamma(i).$$

For $i \leqq j$, $\beta < s_\gamma(i)$, (by (c) for φ, ψ and $j \leqq \bar{j}$)

$$\psi \circ \varphi(\beta) < s_\gamma(i).$$

Case 2. $\bar{j}<j$: Compute again. (By (b) for φ and by (a) for ψ)

$$\psi\circ\varphi(s_\gamma(\bar{j})) = \psi(s_\gamma(\bar{j})) = s_\gamma(\bar{j}).$$

For $i<\bar{j}$, (by (b) for φ, ψ)

$$\psi\circ\varphi(s_\gamma(i)) = \psi(s_\gamma(i)) = s_\gamma(i).$$

For $i\leqq\bar{j}$, $\beta<s_\gamma(i)$, (by (c) for φ, ψ)

$$\psi\circ\varphi(\beta) < s_\gamma(i).$$

3.5. Lemma. *$C_\gamma(\gamma)$ is not* WQO.

Proof. Introduce a sequence $z_n=(u_n, c_n)$ in $C_\gamma(\gamma)$ by putting

$$c_n = a_\gamma(n)$$
$$u_n(s_\gamma(i)) = s_\gamma(i).$$

By condition (a) in 3.3 (3) we see easily that $(z_n)_{n\in\omega}$ is bad in $c_\gamma(\gamma)$.

3.6. Auxilliary definition. Let us call a pair (β, α), $\alpha, \beta\in\omega_1$ *admissible,* if there exist a $\gamma\in\omega_1$, $\gamma\geqq\alpha$, an increasing sequence $(n(i))_{i\in\omega}$, a number $K\in\omega$ and a bad sequence $z_i=(u_i, a_\gamma(n(i)))$ in $C_\gamma(\omega_1)$ such that for each $i\in\omega$

$$|\{\delta < \alpha|\delta\in a_\gamma(n(i))\,\&\,u_i(\delta) \geqq \beta\}| < K.$$

3.7. Lemma.

(1) *If $C_\gamma(\beta)$ is not* WQO *then (β, γ) is admissible.*

(2) *If (β, α) is admissible and $\bar{\alpha}<\alpha$ then $(\beta, \bar{\alpha})$ is admissible.*

(3) *$(0, \omega)$ is not admissible.*

(4) *If $(\beta, \alpha+\omega)$ is admissible then there exists a $\bar{\beta}<\beta$ such that $(\bar{\beta}, \alpha)$ is admissible.*

Proof. (2) and (3) are obvious. Note that in (1) we may use $K=1$. We shall prove (4).

Consider the entities $\gamma, n(i), K, z_i$ constituting the admissibility of $(\beta, \alpha+\omega)$. Let

$$p = \max\{i|\alpha \leqq s_\gamma(i) < \alpha+K\}.$$

Further, let $\alpha_{p+1}=\gamma$,

$$\{s_\gamma(0), \ldots, s_\gamma(p)\} = \{\alpha_0 < \alpha_1 < \ldots < \alpha_p\}.$$

For $i\in p+1$, $t\in\omega$, define $b_i(t)\in F(\omega_1)$ (recall that F is the category of finite sets and injective mappings) in the following way:

$$c_i(t) = \{s_\gamma(j)|j < n(t),\quad \alpha_i < s_\gamma(j) < \alpha_{i+1}\},$$
$$b_i(t) = (u_i|c_i(t),\ c_i(t)).$$

By Ramsey and Higman's theorems there exists an increasing sequence $(t_x)_{x\in\omega}$

such that $n(t_x)>p$ and for $x<y$, $i\in p+1$

$$(*)\qquad u_{t_x}(\alpha_i) \leqq u_{t_y}(\alpha_i)$$

$$(**)\qquad b_i(t_x) \leqq b_i(t_y) \text{ in } F(\omega_1) \text{ via some mapping } \varphi_i(t_x, t_y).$$

Without loss of generality, we may assume $t_x=x$. Now, by $(*)$ and by the definition of K there exists an $\alpha\leqq\bar{\alpha}<\alpha+K$ such that for all t

$$u_t(\bar{\alpha}) < \beta.$$

We will show that $(u_0(\bar{\alpha}), \bar{\alpha})$ is admissible, concluding the proof of (4) by (2).

In fact, for the new K we may take $n(0)$: If, for some $t>0$, there are more than $n(0)$, of $j<n(t)$ with $u_t(n_\gamma(j))\geqq u_0(\bar{\alpha})$ then there exists at least one such j that neither $j\leqq p$ nor $s_\gamma(j)$ lies in the image of $\varphi_i(0, t)$ for any i. Now define $\varphi\colon a_\gamma(n(0))\to\to a_\gamma(n(t))$ by

$$\varphi(s_\gamma(i)) = s_\gamma(i) \quad \text{whenever} \quad i\in p+1, s(i) \neq \bar{\alpha}$$

$$\varphi(\bar{\alpha}) = s_\gamma(j)$$

$$\varphi(\delta) = \varphi_i(0, t)(\delta) \quad \text{if} \quad \delta\in c_i(0).$$

We see easily that $\varphi\in C_\gamma(a_\gamma(n(0)), a_\gamma(n(t)))$ and

$$z_0 \leqq z_t \quad \text{via } \varphi,$$

contradicting the assumption that (z_i) is bad.

3.8. Proof of Theorem 3.2. Define $\gamma(\beta)$ inductively by

$$\gamma(0) = \omega$$

$$\gamma(\beta+1) = \gamma(\beta)+\omega$$

$$\gamma(\beta) = (\lim_{i\to\omega}\gamma(\beta_i))+\omega \quad \text{for} \quad \beta_i\nearrow\beta.$$

It follows from 3.7. (2), (4) that $(\beta, \gamma(\beta))$ is not admissible for any β. Thus, by 3.7. (1), $c_{\gamma(\beta)}(\beta)$ is WQO. This, together with Lemma 3.5, concludes the proof of Theorem 3.2.

References

[1] G. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.*, (3) **2** (1952), 326—336.

[2] J. Kruskal, Well-quasi-ordering, the three theorem, and Vazsony's conjecture, *Trans. Amer. Math. Soc.*, **95** (1960), 210—225.

[3] I. Kříž, and R. Thomas, On well-quasi-ordering finite structures with labels, *Graphs and Combinatorics*, **6** (1990), 41—44.

[4] C. St. J. A. Nash-Williams, On well-quasi-ordering finite trees, *Proc. Camb. Phil. Soc.*, **59** (1963), 833—835.

[5] N. Robertson and P. D. Seymour; Graph minors I—XIII, submitted (1986—1988).

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