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Note on Fourier series with nonnegative coefficients

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1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be its Fourier series. Denote $s_n = s_n(x)$ the n-th partial sum of (1). If $\omega(\delta)$ is a non-decreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 2\pi$, then it will be called modulus of continuity.

Define the following classes of functions

$$H^{\omega} = \{f\colon \|f(x+h) - f(x)\| = O(\omega(h))\}, \tag{2}$$

$$(H^{\omega})^* = \{f\colon \|f(x+h) + f(x-h) - 2f(x)\| = O(\omega(h))\}, \tag{3}$$

where $\|\cdot\|$ denotes the usual maximum norm. If $\omega(\delta) = \delta^{\alpha}$ then we write $\mathrm{Lip}\,\alpha$ instead of $H^{\delta^{\alpha}}$.

In 1948 G. G. LORENTZ [7] proved a theorem containing a coefficient-condition for $f \in \mathrm{Lip}\,\alpha$ in the case if the sequence of the Fourier coefficients is monotonic. Namely he proved the following result.

Theorem A ([7]). *Let $\lambda_n \downarrow 0$ and let λ_n be the Fourier sine or cosine coefficients of φ. Then $\varphi \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ if and only if $\lambda_n = O(n^{-1-\alpha})$.*

Later this result was generalized by R. P. BOAS [1] in 1967 as follows:

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*) This result was partly obtained while the author visited to the Ohio State University, Columbus, U.S.A. in the academic years 1985—86 and 1986—87.

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Theorem B ([1]). *Let* $\lambda_n \geqq 0$ *and let* λ_n *be the sine or cosine coefficients of* φ. *Then* $\varphi \in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ *if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O(n^{-\alpha}), \tag{4}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n^{1-\alpha}). \tag{5}$$

In 1980 L. Leindler in connection with certain investigations in the theory of strong approximation by Fourier series, defined some function classes which are more general than Lip α but narrower than H^ω. Namely he gave the following definition.

Let $\omega_\alpha(\delta)$ $(0 \leqq \alpha \leqq 1)$ denote a modulus of continuity having the following properties:

(i) for any $\alpha' > \alpha$ there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'} \omega_\alpha(2^{-n-\mu}) > 2\omega_\alpha(2^{-n}) \quad \text{holds for all} \quad n \geqq 1; \tag{6}$$

(ii) for every natural number ν there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha} \omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n > N(\nu). \tag{7}$$

Using $\omega_\alpha(\delta)$ L. Leindler defined the function class Lip ω_α in the following way

$$\mathrm{Lip}\,\omega_\alpha = \{f\colon \|f(x+h)-f(x)\| = O(\omega_\alpha(h))\}.$$

Recently the author of the present paper generalized the result of R. P. Boas formulated in Theorem B and some other ones for Lip ω_α instead of Lip α.

For example we proved the following

Theorem C ([8]). *Let* $\lambda_n \geqq 0$ *and* λ_n *be the Fourier sine or cosine coefficients of* φ. *Then*

$$\varphi \in \mathrm{Lip}\,\omega_\alpha \quad (0<\alpha<1)$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega_\alpha\left(\frac{1}{n}\right)\right), \tag{8}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega_\alpha\left(\frac{1}{n}\right)\right). \tag{9}$$

The question of further generalizations for arbitrary $\omega(\delta)$ and H^ω can naturally be arisen.

The first results in this direction were already given by A. I. Rubinstein ([9]) for cosine series, furthermore V. G. Krotov and L. Leindler ([3], see also in [6]) for the sine case. Their results read as follows

Theorem D ([9]). *Let f be an even function belonging to H^ω and let a_n be its Fourier coefficients with $a_n \geqq 0$ $(n=1, 2, \dots)$. Then*

$$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right) \tag{10}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} k a_k = O\left(\frac{1}{n}\int_{1/n}^{\delta_0} \frac{\omega(t)}{t^2}\,dt\right) \tag{11}$$

hold for some fix $\delta_0 > 0$.

If ω satisfies the condition

$$\delta \int_{\delta}^{\delta_0} \frac{\omega(t)}{t^2}\,dt = O(\omega(\delta)) \tag{12}$$

then conditions (10) *and* (11) *are sufficient for*

$$f \in H^\omega.$$

It should be noted that (10) implies (11) for any ω, namely

$$\sum_{k=1}^{n} k a_k = \sum_{k=1}^{n}\sum_{i=k}^{n} a_i = O(1)\sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) = O(1)\int_{1/n}^{\delta_0} \frac{\omega(t)}{t^2}\,dt,$$

and thus for the special moduli of continuity ω satisfying relation (12) the condition (10) itself is a sufficient condition.

Theorem E ([3] Lemma 3, see also in [6]). *If $\lambda_n \geqq 0$ and*

$$g(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx$$

belongs to the class H^ω then

$$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega\left(\frac{1}{n}\right)\right). \tag{13}$$

The aim of this paper is to show that neither (10) nor (13) is sufficient for the corresponding function to be in H^ω; furthermore to give sufficient condition for $f \in H^\omega$ in both cases. We also show that (10) is a necessary and sufficient condition for f to belong to the class $(H^\omega)^*$ (which is broader than H^ω, so this result in this sense is a little sharper than that of Rubinstein). Finally it will be proved that (10) and (13), respectively, is not only a necessary but also sufficient condition for $f \in H^\omega$ and $g \in H^\omega$, if the coefficients a_k and b_k form monotonically decreasing sequences.

2. Now we formulate our results.

Theorem 1. *If* $\lambda_n \geqq 0$ *and* λ_n *are the Fourier sine or cosine coefficients of* φ, *then the conditions*

(14) $$\sum_{k=1}^{n} k\lambda_k = O\left(n\omega\left(\frac{1}{n}\right)\right)$$

and

(15) $$\sum_{k=n}^{\infty} \lambda_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

imply

(16) $$\varphi \in H^{\omega}.$$

Remark. The well-known Weierstrass function

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos 2^n x}{2^n}$$

shows that (15) itself is not sufficient for $\varphi \in H^{\omega}$ (since $f \notin H^{\omega}$ if $\omega(\delta)=\delta$ and (15) is obviously satisfied).

The example

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{4k^2} \sin 3^{4k^2} x$$

proves that from (14) alone (16) does not follow. This function was constructed by A. I. Rubinstein ([9]) in connection with lacunary series. He proved that $g \notin H^{\omega}$, for $\omega(\delta) = \dfrac{1}{|\log_3 \delta|}$. At the same time it can easily be checked that (14) holds.

Theorem 2. *If* $a_k \geqq 0$ *and* a_k *are the Fourier cosine coefficients of* f *then*

(17a) $$f \in (H^{\omega})^*$$

if and only if

(17b) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right).$$

Remark. Notice that (17b) implies

(18) $$\sum_{k=1}^{n} k a_k \leqq K \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right)$$

and using the standard estimation we have that (18) implies

$$f \in H^{\omega_*},$$

where

$$\omega_*(t) := t \sum_{k=1}^{[1/t]} \omega\left(\frac{1}{k}\right).$$

In fact, since from (17a) (18) follows we have that

(19) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

implies $f \in H^{\omega_*}$ and Theorem 2 gives that the same condition (17a) implies

$$f \in (H^{\omega})^*,$$

too. Thus the following question can be arisen: whether

(20) $$f \in H^{\omega_*} \Leftrightarrow f \in (H^{\omega})^*$$

or not.

We can prove that

(21) $$f \in (H^{\omega})^* \Rightarrow f \in H^{\omega_*}$$

but the converse is false. Really, from Theorem 2 we have that

$$f \in (H^{\omega})^* \Rightarrow \sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

which assures that $f \in H^{\omega_*}$, so (21) is proved. In order to prove that

(22) $$f \in H^{\omega_*} \not\Rightarrow f \in (H^{\omega})^*$$

we consider the following function

(23) $$f(x) = \sum_{k=1}^{\infty} \frac{\log n}{n^2} \cos nx$$

and let $\omega(t) = t$, that is, $\omega_*(t) = t \log t$. From Theorem 4 of [8] it follows that

$$f \in H^{\delta \log \delta} \quad (= H^{\omega_*})$$

because both

(24) $$\sum_{h=n}^{\infty} \frac{\log k}{k^2} = O\left(\frac{\log n}{n}\right)$$

and

(25) $$\left\| \sum_{k=1}^{n} \frac{\log k}{k} \sin kx \right\| = O(\log n)$$

hold. And at the same time

$$f \notin (H^{\delta})^* = (H^{\omega})^*,$$

because

$$\frac{1}{2} |f(0+2h) + f(0-2h) - 2f(0)| = \sum_{n=1}^{\infty} \frac{\log n}{n^2} (1 - \cos 2nh) =$$

$$= 2 \sum_{n=1}^{\infty} \frac{\log n}{n^2} \sin^2 nh \geqq 2h^2 \sum_{n=1}^{[1/h]} \log n \frac{\sin^2 nh}{n^2 h^2} \geqq 2h |\log h|,$$

which gives that

$$\|f(x+h)+f(x-h)-2f(x)\| \neq O(h),$$

that is, $f\notin(H^{\delta})^*$ and so (22) is proved.

Theorem 3. *If* $b_k\downarrow 0$ *and* $g(x)=\sum_{k=1}^{\infty} b_k \sin kx$ *then*

(26a) $$g\in H^{\omega}$$

if and only if

(26b) $$\sum_{k=1}^{n} kb_k = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Theorem 4. *If* $a_k\downarrow 0$ *and* $f(x)=\sum_{k=1}^{\infty} a_k \cos kx$ *then*

(27a) $$f\in H^{\omega}$$

if and only if

(27b) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right).$$

3. We require the following lemmas.

Lemma 1. *Let* $\{a_n\}$ *be a sequence of nonnegative numbers and* ω *be a modulus of continuity. Then*

(28) $$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right)$$

implies

(29) $$\sum_{k=1}^{n} k^2 a_k = O\left(n^2\omega\left(\frac{1}{n}\right)\right).$$

Proof.*) Using (28) we have

(30) $$\sum_{k=1}^{n} k^2 a_k = \sum_{k=1}^{n}(2k-1)\sum_{i=k}^{n} a_i \leqq 2\sum_{k=1}^{n} k\omega\left(\frac{1}{k}\right) = I.$$

Since for any ω the inequality

(31) $$\frac{\omega(x_1)}{x_1} \leqq 2\,\frac{\omega(x_2)}{x_2} \quad (0<x_2\leqq x_1)$$

*) This very elegant proof is due to L. Leindler; the author's original one was much more complicated.

(see for example [11] p. 103) holds I can be estimated as follows

$$I \leqq 2n \cdot 2n\omega\left(\frac{1}{n}\right) = 4n^2\omega\left(\frac{1}{n}\right). \tag{32}$$

Thus (30) and (32) give (29).

Lemma 2. *Let* $a_k \geqq 0$ *and* a_k *be the Fourier cosine coefficients of* f. *Then*

$$\sum_{k=n}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right) \quad \textit{and} \quad \left\|\sum_{k=1}^{n} k a_k \sin kx\right\| = O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{33}$$

imply

$$f \in H^{\omega}. \tag{34}$$

This lemma can be proved in the same way as Theorem 4 in [8] for $\omega_1(\delta)$.

4. Proofs.

Proof of Theorem 1. We detail the proof just for cosine series. Set

$$|f(x+2h)-f(x)| = 2\left|\sum_{k=1}^{\infty} \lambda_k \sin k(x+h) \sin kh\right| \leqq \tag{35}$$

$$\leqq 2\Big(\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k\Big) = I + II.$$

Since

$$I \leqq Kh \sum_{k=1}^{[1/h]} k\lambda_k \frac{\sin kh}{kh} \leqq K_1 h \sum_{k=1}^{[1/h]} k\lambda_k, \tag{36}$$

from (14) it follows that

$$I = O(\omega(h)). \tag{37}$$

By using (15) we have that

$$II = O(\omega(h)). \tag{38}$$

So (35), (36), (37) and (38) give that

$$f \in H^{\omega}.$$

Theorem 1 is completed.

Proof of Theorem 2. Suppose that (17b) holds. Then

$$|f(x+2h)+f(x-2h)-2f(x)| = 4\left|\sum_{k=1}^{\infty} a_k \sin^2 kh \cos kx\right| \leqq \tag{39}$$

$$\leqq 4\sum_{k=1}^{\infty} a_k \sin^2 kh = 4h^2 \sum_{k=1}^{[1/h]} k^2 a_k \frac{\sin^2 kh}{k^2 h^2} + \sum_{k=[1/h]}^{\infty} a_k.$$

The first item of the last formula does not exceed $O(\omega(h))$ because of Lemma 1; and from (17b) we get that the second one is also $O(\omega(h))$. So (17b)$\Rightarrow$(17a) is proved.

Turning to prove (17a)$\Rightarrow$(17b) first we note that the proof will be led by the same way as A. I. Rubinstein did in [9]. Let $I_n(x, g)$ be the Jackson polynomial defined by

$$(40)\qquad I_n(x, g) = \frac{3}{2n\pi(2n^2+1)} \int_{-\pi}^{\pi} g(t) \left(\frac{\sin n \frac{t-x}{2}}{\sin \frac{t-x}{2}} \right)^4 dt.$$

This polynomial can be written in the following form

$$(41)\qquad I_n(x, g) = \frac{a_0}{2} + \sum_{k=1}^{2n-2} \varrho_k^{(n)} (a_k \cos kx + b_k \sin kx),$$

where a_k, b_k are the Fourier coefficients of g and $\varrho_k^{(n)}$ are defined as follows

$$(42)\qquad \varrho_k^{(n)} = \frac{1}{2n(2n^2+1)} \left[\frac{(2n-k+1)!}{(2n-k-2)!} - 4 \frac{(n-k+1)!}{(n-k-2)!} \right] \quad \text{for} \quad 1 \leqq k \leqq n-2,$$

$$\varrho_k^{(n)} = \frac{1}{2n(2n^2+1)} \frac{(2n-k+1)!}{(2n-k-2)!} \quad \text{for} \quad n-2 < k \leqq 2n-2.$$

Formula (42) was given by G. P. Safranova ([10]).

Consider the following difference for

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

$$(43)\qquad f(0) - I_n(0; f) = \sum_{k=1}^{2n-2} (1 - \varrho_k^{(n)}) a_k + \sum_{k=2n-1}^{\infty} a_k.$$

It can be proved that the order of approximation by polynomial (40) is $O\left(\omega\left(\frac{1}{n}\right)\right)$ for

$$g \in (H^\omega)^*$$

(see for example [2] pp. 496—497).

Using this fact and that $1 - \varrho_k^{(n)} \geqq 0$ we have from (43)

$$\sum_{k=2n-1}^{\infty} a_k = O\left(\omega\left(\frac{1}{n}\right)\right),$$

which was to be proved.

Theorem 2 is completed.

Proof of Theorem 3. The statement (26a)$\Rightarrow$(26b) was proved by V. G. Krotov and L. Leindler (see Theorem E). Now we suppose (26b). It is obvious that to prove (26a) it is sufficient to show

(44) $$|g(h)-g(0)| \leqq K_1 \cdot \omega(h)$$

and

(45) $$|g(x)-g(x+h)| \leqq K_2 \omega(h), \quad \text{for} \quad 0 < h \leqq x \leqq \pi.$$

First we prove (44).

Set

(46) $$|g(h)-g(0)| \leqq \Big|\sum_{k=1}^{[1/h]} b_k \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kh\Big| = \mathrm{I} + \mathrm{II}.$$

Using (26b) we can estimate I as follows

(47) $$\mathrm{I} \leqq h \sum_{k=1}^{[1/h]} kb_k \frac{\sin kh}{kh} \leqq Kh \sum_{k=1}^{[1/h]} kb_k = O(\omega(h)).$$

From the well-known inequality

(48) $$\Big|\sum_{k=n}^{m} a_k \sin kx\Big| \leqq \frac{4}{x} a_n \quad (a_n \downarrow,\ x \in (0, \pi))$$

it follows that

(49) $$\mathrm{II} \leqq 4 \frac{1}{h} b_{[1/h]}.$$

But taking into account that $b_k \downarrow$, from (26b) we have

(50) $$b_n = O\left(\frac{\omega\left(\frac{1}{n}\right)}{n}\right).$$

From (49) and (50) we get

(51) $$\mathrm{II} = O(\omega(h)).$$

Using (46), (47) and (51) we obtain (44).

Now we verify (45). Consider

(52) $$|g(x+h)-g(x)| = \Big|\sum_{k=1}^{\infty} b_k (\sin kx - \sin k(x+h))\Big| \leqq$$

$$\leqq \Big|\sum_{k=1}^{[1/h]} b_k \cos k(x+h) \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kx - \sin k(x+h)\Big| \leqq$$

$$\leqq \Big|\sum_{k=1}^{[1/h]} b_k \sin kh\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin kx\Big| + \Big|\sum_{k=[1/h]}^{\infty} b_k \sin k(x+h)\Big| = \mathrm{I} + \mathrm{II}' + \mathrm{II}''.$$

By (47) we have

(53) $$\mathrm{I} = O(\omega(h)).$$

Taking into account again (48), (49) and the condition $0<h\leqq x$ we have that the magnitude of either II′ or II″ is $O(\omega(h))$, that is,

$$\text{II}'+\text{II}'' = O(\omega(h)). \tag{54}$$

Thus (52), (53) and (54) give (45) which is the desired statement.

Theorem 3 is completed.

Proof of Theorem 4. Using Theorem 2 and the fact that $H^\omega\subset(H^\omega)^*$ the statement (27a)⇒(27b) can immediately be obtained. Concerning the opposite direction, by Lemma 2, it is enough to show that

$$\Big\|\sum_{k=1}^{n} ka_k \sin kx\Big\| \leqq K\cdot n\omega\left(\frac{1}{n}\right). \tag{55}$$

Let $x\in(0,\pi)$ be fixed and let ν denote $\left[\frac{1}{x}\right]$; if $n>\frac{1}{x}$, then split up the left hand side of (55) into two parts as follows

$$\Big\|\sum_{k=1}^{n} ka_k \sin kx\Big\| \leqq \Big\|\sum_{k=1}^{\nu} ka_k \sin kx\Big\|+\Big\|\sum_{k=\nu=1}^{n} ka_k \sin kx\Big\| = \text{I}+\text{II}. \tag{56}$$

Estimating I we get

$$\text{I} \leqq K_1 x\sum_{k=1}^{\nu} k^2 a_k. \tag{57}$$

Taking into account the monotonity of a_k and (27b) we have

$$ka_k = O\left(\omega\left(\frac{1}{k}\right)\right). \tag{58}$$

From (58) it follows that

$$x\sum_{k=1}^{\nu} k^2 a_k \leqq K_2 x\sum_{k=1}^{\nu} k\omega\left(\frac{1}{k}\right) \leqq K_3 n\omega\left(\frac{1}{n}\right). \tag{59}$$

In the last step we used again inequality (31) and $n>\nu$.

Thus from (57) and (59)

$$\text{I} \doteq O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{60}$$

can be obtained.

Now we estimate the second item in (56). Since

$$\text{II} = \Big\|\sum_{k=\nu}^{n} ka_k \sin kx\Big\| \leqq \Big\|\sum_{k=\nu}^{n}\sum_{i=k}^{n} a_i \sin ix\Big\|+\nu\sum_{i=\nu}^{n} a_i = \text{II}'+\text{II}'' \tag{61}$$

and using again (48) and (58)

$$\text{(62)} \qquad \mathrm{II}' \leqq K \sum_{k=\nu}^{n} \frac{a_k}{x} \leqq K_1 \nu \sum_{k=\nu}^{n} k\omega\left(\frac{1}{k}\right)\frac{1}{k^2} \leqq K_2 n\omega\left(\frac{1}{n}\right).$$

And for II″ using (58) we immediately obtain that

$$\text{(63)} \qquad \mathrm{II}'' \leqq K_3 n\omega\left(\frac{1}{n}\right),$$

and (61), (62), (63) give that

$$\text{(64)} \qquad \mathrm{II} = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Thus (60) and (64) together give (56) which was to be proved.

Theorem 4 is completed.

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