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Pointwise limits of nets of multilinear maps

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Introduction. Motivated by the fact that most of the standard integrals are pointwise limits of the nets of their approximating sums which are either linear or bilinear maps (see [7] and [8]), we establish the most important algebraic and topological properties of the pointwise limit of a net of multilinear maps.

More concretely, using our former results on bounded nets [14] and multipreseminorms [15], we show that the pointwise limit of a net of multilinear maps being equicontinuous at the origin is a selectionally boundedly uniformly continuous multilinear relation whose domain is a closed set whenever the range space is complete.

Having had the necessary definitions, it becomes clear that particular cases of this assertion greatly improve a useful continuity criterion for multilinear maps [3, (18.2) Theorem], a general convergence theorem for net integrals [7, Theorem 3.8] and a part of a generalized Banach—Steinhaus theorem [1, 7. (5)].

1. Prerequisites. Instead of topological vector spaces, it is often more convenient to use preseminormed spaces [9]. A preseminormed space over $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$ is an ordered pair $X(\mathscr{P})=(X, \mathscr{P})$ consisting of a vector space X over $\mathbf{K}$ and a nonvoid family $\mathscr{P}$ of preseminorms on X. A preseminorm on X is a subadditive real-valued function p on X such that $p(\lambda x)\leqq p(x)$ for all $|\lambda|\leqq 1$ and $x\in X$, and $\lim\limits_{\lambda\to 0} p(\lambda x)=0$ for all $x\in X$. Note that these latter properties imply, in particular, that $p(0)=0$ and $p(\lambda x)\leqq p(\mu x)$ for all $|\lambda|\leqq|\mu|$ and $x\in X$.

If $X(\mathscr{P})$ is a preseminormed space, then because of [4, Theorem 6.3], the family of all surroundings

$$B_p^r = \{(x, y)\colon\ p(x-y) < r\},$$

where $p\in\mathscr{P}$ and $r>0$, is a subbase for a uniformly $\mathscr{U}_{\mathscr{P}}$ on X. However, this fact is only of minor importance for us now since among $\mathscr{U}_{\mathscr{P}}$ and the various structures on X induced by $\mathscr{U}_{\mathscr{P}}$ we shall actually need only the induced net convergence $\lim_{\mathscr{P}}=\lim_{\mathscr{U}_{\mathscr{P}}}$ which can also be naturally derived directly from $\mathscr{P}$.

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If $X(\mathscr{P})$ is a preseminormed space, then $\lim_{\mathscr{P}}$ is a relation between nets (x_α) and points x in X such that, after a customary convention in the notation, we have $x \in \lim_{\alpha} {}_{\mathscr{P}}\, x_\alpha$ if and only if $\lim_{\alpha} p(x_\alpha - x) = 0$ for all $p \in \mathscr{P}$. As usual a net (x_α) in $X(\mathscr{P})$ is called a convergent net (a null net) if $\lim_{\alpha} {}_{\mathscr{P}}\, x_\alpha \neq \emptyset$ $(0 \in \lim_{\alpha} {}_{\mathscr{P}}\, x_\alpha)$. Moreover, two nets (x_α) and (y_α) in $X(\mathscr{P})$ are called coherent [12] if $(x_\alpha - y_\alpha)$ is a null net. Note that several useful properties of the convergence $\lim_{\mathscr{P}}$ can be easily derived from the usual properties of the convergence of nets of real numbers by using the above properties of preseminorms.

On the other hand, a net (x_α) in $X(\mathscr{P})$ is called a bounded net (a Cauchy net) if

$$\lim_{\lambda \to 0} \overline{\lim_{\alpha}}\, p(\lambda x_\alpha) = 0 \quad \Big(\lim_{(\alpha, \beta)} p(x_\alpha - x_\beta) = 0\Big)$$

for all $p \in \mathscr{P}$. In [14], we have proved that all Cauchy nets in $X(\mathscr{P})$ are bounded. And a net (x_α) in $X(\mathscr{P})$ is a bounded net (a Cauchy net) if and only if for any subnet (y_β) of (x_α) and any null net (λ_β) of scalars $(\lambda_\beta y_\beta)$ is a null net (any two subnets (z_ν) and (w_ν) of (x_α) are coherent).

Another remarkable feature of this new definition of bounded nets is that a nonvoid subset A of $X(\mathscr{P})$ may henceforth be called bounded if the identity function $(x)_{x \in A}$ of A is bounded as a net whenever A is considered to be directed such that $x \leqq y$ for all $x, y \in A$. Note that A is therefore bounded if and only if

$$\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$$

for all $p \in \mathscr{P}$. And thus nets contained in bounded subsets of $X(\mathscr{P})$ are necessarily bounded.

Having the above definition of bounded nets, we may also define a function f from a subset D of $X(\mathscr{P})$ into another preseminormed space $Y(Q)$ to be boundedly uniformly continuous if $(f(x_\alpha))$ and $(f(y_\alpha))$ are coherent nets in $Y(Q)$ whenever (x_α) and (y_α) are bounded coherent nets in D. Note that f may be called uniformly continuous if it maps coherent nets into coherent nets. Thus, if f is uniformly continuous, then f is also boundedly uniformly continuous. On the other hand, if f is boundedly uniformly continuous, then f is necessarily continuous and the restrictions of f to bounded subsets of D are uniformly continuous.

In the sequel, we shall also need a straightforward notion of a product preseminormed space from [10]. If $X_i(\mathscr{P}_i)$ is a preseminormed space for each i in a nonvoid set I, and moreover

$$X = \mathop{\times}_{i \in I} X_i \quad \text{and} \quad \mathscr{P} = \bigcup_{i \in I} \mathscr{P}_i \circ \pi_i,$$

where π_i is the projection of X onto X_i and $\mathscr{P}_i \circ \pi_i = \{p \circ \pi_i \colon p \in \mathscr{P}_i\}$, then the preseminormed space $X(\mathscr{P})$ is called the Cartesian product of the spaces $X_i(\mathscr{P}_i)$ and the notation

$$X(\mathscr{P}) = \bigtimes_{i \in I} X_i(\mathscr{P}_i)$$

is used. An important consequence of this definition is that a net (x_α) in $X(\mathscr{P})$ is convergent, Cauchy, resp. bounded if and only if each of its coordinate nets $(x_{\alpha i})$ has the corresponding property.

Finally, a real-valued function p on a product vector space $X = \bigtimes_{i=1}^{n} X_i$ is called a multi-preseminorm [15] if it is a preseminorm in each of its variables separately, and moreover

$$p(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) = p(x_1, \ldots, x_{k-1}, \lambda x_k, x_{k+1}, \ldots, x_n)$$

for all $x=(x_i)\in X$, scalar λ and $i, k=1,2,\ldots,n$. The importance of this notion lies mainly in the fact that a multilinear map f from a product preseminormed space

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into an arbitrary preseminormed space $Y(\mathscr{Q})$ is boundedly uniformly continuous if and only if the multi-preseminorm $q\circ f$ is continuous at the origin of $X(\mathscr{P})$ for all $q\in\mathscr{Q}$.

2. Multilinear relations. Since the pointwise limit of a net of multilinear maps is, in general, only a relation which need not be defined on the whole product space, the usual concept of a multilinear map [3, p. 72] has to be subtantially extended.

For this, we need a straightforward notion of a linear relation from [17] which is mainly motivated by the fact that the inverse of a linear function is a linear relation.

Definition 2.1. A relation f from a vector space X over $\mathbf{K}$ into another Y is called *linear* if

$$f(x)+f(y) \subset f(x+y) \quad \text{and} \quad \lambda f(x) \subset f(\lambda x)$$

for all $x, y\in X$ and $\lambda\in\mathbf{K}$.

Remark 1.2. Note that, in other words, this means only that f is a linear subspace of the product space $X\times Y$ such that the set $f(x)=\{y\colon (x, y)\in f\}$ is not empty for all $x\in X$.

After having this self-evident definition now we can easily define a sufficiently general notion of a multilinear relation whose insufficient particular case has already been considered in [18].

Definition 2.3. Let X_i be a vector space over $\mathbf{K}$ for all $i=1, 2, \dots, n$, and

$$X = \bigtimes_{i=1}^{n} X_i.$$

For each $x=(x_i)\in X$ and $i=1, 2, \dots, n$, denote by φ_{xi} the function defined on X_i by

$$\varphi_{xi}(t) = (x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n).$$

A subset D of X will be called *multilinear* if the set

$$D_{xi} = \overset{-1}{\varphi_{xi}}(D)$$

is a linear subspace of X_i for all $x\in X$ and $i=1, 2, \dots, n$.

A relation f from a multilinear subset D of X into a vector space Y over $\mathbf{K}$ will be called *multilinear* if the partial relation

$$f_{xi} = f\circ\varphi_{xi}$$

is a linear relation from D_{xi} into Y for all $x\in X$ and $i=1, 2, \dots, n$.

Remark 2.4. Instead of "$x\in X$" we might only write "$x\in D$" in the above definition. However, this would lead to a further generalization which we do not need here.

Moreover, instead of "multilinear" we may also say "n-linear". Thus, "linear" and "bilinear" can be identified as "1-linear" and "2-linear", respectively.

Concerning multilinear sets and relations, we will only list here a few basic theorems without proofs.

Theorem 2.5. *If X is as in Definition 2.3, then*

$$X_0 = \{x\in X\colon\ x_i = 0 \quad \textit{for some} \quad i = 1, 2, \dots, n\}$$

is the smallest multilinear subset of X.

Theorem 2.6. *If D is a multilinear subset of $X = \bigtimes_{i=1}^{n} X_i$, then*

$$D = \bigcup_{x\in X} \bigcup_{i=1}^{n} (\mathbf{K}x_1)\times\dots\times(\mathbf{K}x_{i-1})\times D_{xi}\times(\mathbf{K}x_{i+1})\times\dots\times(\mathbf{K}x_n).$$

Remark 2.7. This latter theorem, which is also true under a more general definition of multilinear sets, has been pointed out to me by György Szabó.

Theorem 2.8. *If f is a multilinear relation from a multilinear subset D of X into Y, then $f(0)$ is a linear subspace of Y and $f(x)=f(0)$ for all $x\in X_0$.*

Theorem 2.9. *If f is a multilinear relation from X into Y, then there exists a multilinear function φ from X into Y such that*

$$f(x) = \varphi(x)+f(0)$$

for all $x\in X$.

Remark 2.10. Note that if φ is a multilinear function from a multilinear subset D of X into Y and M is a linear subspace of Y, then the relation f defined on D by $f(x)=\varphi(x)+M$ is also multilinear.

By an immediate application of the above assertions, we can at once state the next simple

Example 2.11. A subset D of $\mathbf{K}^n$ is multilinear if and only if either $D=(\mathbf{K}^n)_0$ or $D=\mathbf{K}^n$.

A relation f from $D=\mathbf{K}^n$ or $(\mathbf{K}^n)_0$ into Y is multilinear if and only if there exist a vector $y\in Y$ and a linear subspace M of Y such that

$$f(x) = \Big(\prod_{i=1}^{n} x_i\Big)y+M \quad \text{for all} \quad x\in D.$$

More difficult examples for multilinear sets and relations can be easily obtained from the following obvious, but important theorem which needs only a few properties of convergent nets in preseminormed spaces.

Theorem 1.12. *If (f_α) is a net of multilinear maps from $X=\bigtimes_{i=1}^{n} X_i$ into a preseminormed space $Y(\mathcal{Q})$, then the set*

$$D = \{x\in X\colon\, (f_\alpha(x)) \text{ converges in } Y(\mathcal{Q})\}$$

is a multilinear subset of X and the relation f defined on D by

$$f(x) = \lim_{\alpha}{}_{\mathcal{Q}}\, f_\alpha(x)$$

is a multilinear relation from D into Y.

Remark 2.13. Note that f is a function if and only if $Y(\mathcal{Q})$ is separated in the sense that for each $y\in Y$ with $y\neq 0$ there exists $q\in\mathcal{Q}$ such that $q(y)\neq 0$.

Therefore, in separated preseminormed spaces we may usually restrict ourselves to multilinear functions. But, unfortunately separated preseminormed spaces are often insufficient.

3. Equicontinuity. Before defining a suitable new notion of equicontinuity, which is necessary to rightly state our main results about the pointwise limit of a net of multilinear maps, we shall briefly deal with a corresponding concept of pointwise boundedness.

Definition 3.1. A net (f_α) of functions from a set X into a preseminormed space $Y(\mathcal{Q})$ will be called *pointwise bounded* if $(f_\alpha(x))$ is a bounded net in $Y(\mathcal{Q})$ for all $x\in X$.

Remark 3.2. A nonvoid set $\{f_\alpha\}_{\alpha\in\Gamma}$ of functions from X into $Y(\mathcal{Q})$ may henceforth be called pointwise bounded if the family $(f_\alpha)_{\alpha\in\Gamma}$ is pointwise bounded as a net whenever Γ is considered to be directed such that $\alpha\leqq\beta$ for all $\alpha,\beta\in\Gamma$.

For a preliminary illustration of the appropriateness of these unusual definitions, we can now easily prove a useful characterization of pointwise boundedness in terms of multi-preseminorms.

Theorem 3.3. *If (f_α) is a net of multilinear maps from $X=\mathop{\times}\limits_{i=1}^{n} X_i$ into $Y(\mathcal{Q})$, then the following assertions are equivalent:*

(i) *(f_α) is pointwise bounded;*

(ii) *$M_q=\overline{\lim\limits_{\alpha}}\, q\circ f_\alpha$ is a multi-preseminorm on X for all $q\in\mathcal{Q}$.*

Proof. Because of the fact that $q\circ f_\alpha$ is a multi-preseminorm on X for all α and some of the basic properties of upper limit, it is clear that M_q is always multi-subadditive and

$$M_q(\varphi_{xi}(\lambda x_i))\leqq M_q(x)\quad\text{and}\quad M_q(\varphi_{xi}(\mu x_i))=M_q(\varphi_{xk}(\mu x_k))$$

for all $|\lambda|\leqq 1$, $\mu\in\mathbf{K}$, $x\in X$ and $i,k=1,2,\dots,n$.
Moreover, since

$$M_q(\varphi_{xi}(\lambda x_i))=\overline{\lim_{\alpha}}\, q(\lambda f_\alpha(x))$$

for all $q\in\mathcal{Q}$, $\lambda\in\mathbf{K}$, $x\in X$ and $i=1,2,\dots,n$, it is also clear that

$$\lim_{\lambda\to 0} M_q(\varphi_{xi}(\lambda x_i))=0$$

for all $q\in\mathcal{Q}$, $x\in X$ and $i=1,2,\dots,n$ if and only if (i) holds. Thus, it remains only to show that M_q is necessarily real-valued for all $q\in\mathcal{Q}$ if (i) holds. For this, note that if $x\in X$ and $p=M_q\circ\varphi_{x1}$, then

$$M_q(x)=p(x_1)=p(m(m^{-1}x))\leqq mp(m^{-1}x_1)$$

for all $m\in\mathbf{N}$, whence because of

$$\lim_{m\to\infty} p(m^{-1}x)=0,$$

it is evident that $M_q(x)<\infty$.

Remark 3.4. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from X into $Y(\mathcal{Q})$ is pointwise bounded if and only if $M_q=\sup\limits_{\alpha} q\circ f_\alpha$ is a multi-preseminorm on X for all $q\in\mathcal{Q}$.

Having in mind a particular case of the last statement of Section 1 and our basic concept of boundedness of a net, it seems now quite reasonable to introduce a suitable new notion of equicontinuity.

Definition 3.5. A net (f_α) of multilinear maps from a product preseminormed space

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into another preseminormed space $Y(\mathscr{Q})$ will be called *equicontinuous at the origin of* $X(\mathscr{P})$ if the function

$$M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$$

is continuous at the origin of $X(\mathscr{P})$ for all $q \in \mathscr{Q}$.

Remark 3.6. A nonvoid set $\{f_\alpha\}_{\alpha \in \Gamma}$ of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ may henceforth be called equicontinuous at the origin of $X(\mathscr{P})$ if the family $(f_\alpha)_{\alpha \in \Gamma}$ is equicontinuous at the origin of $X(\mathscr{P})$ as a net whenever Γ is considered to be directed such that $\alpha \leqq \beta$ for all $\alpha, \beta \in \Gamma$.

To let the reader feel the appropriateness of these apparently very strange definitions, we first show that this particular equicontinuity does already imply pointwise boundedness.

Theorem 3.7. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \bigtimes_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ such that (f_α) is equicontinuous at the origin of $X(\mathscr{P})$, then (f_α) is pointwise bounded.

Proof. If $x \in X$ and $q \in \mathscr{Q}$, then we clearly have

$$q(\lambda f_\alpha(x)) \leqq q(|\lambda| f_\alpha(x)) = q\big(f_\alpha(\sqrt[n]{|\lambda|}\, x_1, \ldots, \sqrt[n]{|\lambda|}\, x_n)\big)$$

and hence

$$\overline{\lim_\alpha}\, q(\lambda f_\alpha(x)) \leqq M_q(\sqrt[n]{|\lambda|}\, x_1, \ldots, \sqrt[n]{|\lambda|}\, x_n)$$

for all $\lambda \in \mathbf{K}$. Hence, because of the continuity of M_q at 0,

$$\lim_{\lambda \to 0} \overline{\lim_\alpha}\, q(\lambda f_\alpha(x)) = 0$$

follows. And this shows that (f_α) is pointwise bounded.

Remark 3.8. Hence, it is clear that if $\{f_\alpha\}$ is a set of multilinear maps from $X(\mathcal{P})$ into $Y(\mathcal{Q})$ such that $\{f_\alpha\}$ is equicontinuous at the origin of $X(\mathcal{P})$ then $\{f_\alpha\}$ is pointwise bounded.

Next, we prove a useful characterization of equicontinuity which, together with Theorem 3.3, provides subtantial motivation for introducing and studying multi-preseminorms.

Theorem 3.9. *If (f_α) is a net of multilinear maps from*

$$X(\mathcal{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathcal{P}_i)$$

into $Y(\mathcal{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathcal{P})$;*

(ii) *$M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$ is a boundedly uniformly continuous multi-preseminorm on $X(\mathcal{P})$ for all $q \in \mathcal{Q}$.*

Proof. If (i) holds, then by Theorem 3.7, (f_α) is pointwise bounded. Thus, by Theorem 3.3, $M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$ is a multi-preseminorm on X for all $q \in \mathcal{Q}$. On the other hand, by [15, Theorem 2.7] a multi-preseminorm which is continuous at the origin is necessarily boundedly uniformly continuous. Therefore, (ii) also holds.

The converse implication (ii)$\Rightarrow$(i) is trivial since bounded uniform continuity always implies continuity.

Remark 3.10. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathcal{P})$ into $Y(\mathcal{Q})$ is equicontinuous at the origin of $X(\mathcal{P})$ if and only if $M_q = \sup_\alpha q \circ f_\alpha$ is a boundedly uniformly continuous multi-preseminorm on $X(\mathcal{P})$ for all $q \in \mathcal{Q}$.

4. Main results. To easily prove our main results about the topological properties of the pointwise limit of an equicontinuous net of multilinear maps, we also neeed a somewhat deeper characterization of equicontinuity.

Theorem 4.1. *If (f_α) is a net of multilinear maps from*

$$X(\mathcal{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathcal{P}_i)$$

into $Y(\mathcal{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathcal{P})$;*

(ii) *$\lim_\nu \overline{\lim_\alpha}\, q\big(f_\alpha(x_\nu) - f_\alpha(y_\nu)\big) = 0$ for all $q \in \mathcal{Q}$ whenever (x_ν) and (y_ν) are bounded coherent nets in $X(\mathcal{P})$.*

Proof. Assume that (i) is true, and moreover (x_ν) and (y_ν) are bounded coherent nets in $X(\mathcal{P})$ and $q \in \mathcal{Q}$. If $I = \{1, 2, \ldots, n\}$ and χ_A is the characteristic function of

$A \subset I$, then according to [3, (18.3) Lemma], we have

$$f_\alpha(x_\nu)-f_\alpha(y_\nu) = \sum_{\emptyset \neq A \subset I} f_\alpha(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu)$$

for all α and ν, where the multiplication is taken in the usual pointwise sense. Hence, because of the subadditivity of q and $\overline{\lim}$, it follows that

$$\overline{\lim_\alpha}\, q(f_\alpha(x_\nu)-f_\alpha(y_\nu)) \leqq \sum_{\emptyset \neq A \subset I} M_q(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu)$$

for all ν, where again $M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$.

On the other hand, if $\emptyset \neq A \subset I$, then by our former results mentioned in Section 1, it is clear that

$$(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu) \quad \text{and} \quad ((\chi_I-\chi_A)y_\nu)$$

are bounded coherent nets in $X(\mathscr{P})$. Moreover, since $f_\alpha((\chi_I-\chi_A)y_\nu)=0$ for all α and ν, it is also clear that

$$M_q((\chi_I-\chi_A)y_\nu) = 0$$

for all ν. Thus, by a particular case of Theorem 3.9, we also have

$$\lim_\nu M_q(\chi_A(x_\nu-y_\nu)+(\chi_I-\chi_A)y_\nu) = 0$$

for all $\emptyset \neq A \subset I$. Using these latter equalities, from our previous inequality, we can immediately infer that

$$\lim_\nu \overline{\lim_\alpha}\, q(f_\alpha(x_\nu)-f_\alpha(y_\nu)) = 0,$$

which shows that (ii) is also true.

To prove the converse implication (ii)$\Rightarrow$(i), note that if (x_ν) is a null net in $X(\mathscr{P})$, then by defining $y_\nu=0$ for all ν, we can at once state that (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$ such that $f_\alpha(y_\nu)=0$ for all α and ν. Therefore, if (ii) holds, then we also have

$$\lim_\nu \overline{\lim_\alpha}\, q(f_\alpha(x_\nu)) = 0$$

for all $q \in \mathscr{Q}$. Consequently, the function $M_q = \overline{\lim_\alpha}\, q \circ f_\alpha$ is continuous at the origin of $X(\mathscr{P})$ for all $q \in \mathscr{Q}$, and thus (i) also holds.

Remark 4.2. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ is equicontinuous at the origin of $X(\mathscr{P})$ if and oly if $\lim_\nu \sup_\alpha q(f_\alpha(x_\nu)-f_\alpha(y_\nu))=0$ for all $q \in \mathscr{Q}$ whenever (x_ν) and (y_ν) are bounded coherent nets in $X(\mathscr{P})$.

To partly express a very strong continuity property of the pointwise limit of an equicontinuous net of multilinear maps, we also need the next straightforward

Definition 4.3. A relation f from a subset D of a preseminormed space $X(\mathscr{P})$ into another preseminormed space $Y(\mathscr{Q})$ will be called *selectionally boundedly uniformly continuous* if each selection function φ for f is boundedly uniformly continuous.

Remark 4.4. Note that a selectionally boundedly uniformly continuous relation is, in particular, lower semicontinuous in the usual topological sense [6, p. 32].

Now, having all the necessary preparations, we can easily state and prove the following important addition to Theorem 2.12 which greatly improve the second assertion of [16].

Theorem 4.5. *If* (f_α) *is a net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ *which is equicontinuous at the origin of* $Y(\mathscr{P})$, *then the relation* f *defined by*

$$f(x) = \lim_{\alpha}{}_{\mathscr{Q}} f_\alpha(x)$$

is a selectionally boundedly uniformly continuous relation from its domain D *into* $Y(\mathscr{Q})$.

Proof. Assume that φ is a selection function for f and (x_ν) and (y_ν) are bounded coherent nets in D. If $q \in \mathscr{Q}$, then because of the subadditivity of q and the assumption that $\varphi(x) \in \lim_{\alpha}{}_{\mathscr{Q}} f_\alpha(x)$ for all $x \in D$, we clearly have

$$q(\varphi(x_\nu)-\varphi(y_\nu)) \leqq q(\varphi(x_\nu)-f_\alpha(x_\nu))+q(f_\alpha(x_\nu)-f_\alpha(y_\nu))+q(f_\alpha(y_\nu)-\varphi(y_\nu))$$

and

$$\lim_{\alpha} q(\varphi(x_\nu)-f_\alpha(x_\nu)) = 0 \quad \text{and} \quad \lim_{\alpha} q(f_\alpha(y_\nu)-\varphi(y_\nu)) = 0$$

for all α and ν, respectively. Hence, it follows that

$$q(\varphi(x_\nu)-\varphi(y_\nu)) \leqq \lim_{\alpha} q(f_\alpha(x_\nu)-f_\alpha(y_\nu))$$

for all ν. Hence, by Theorem 4.1, it is clear that

$$\lim_{\nu} q(\varphi(x_\nu)-\varphi(y_\nu)) = 0.$$

Consequently, φ is a boundedly uniformly continuous function of D into $Y(\mathscr{Q})$, and thus the selectional bounded uniform continuity of f is proved.

Since each preseminormed space can be naturally embedded into a complete one, we may usually assume that $Y(\mathscr{Q})$ is complete. In this particular case, the above theorem can be supplemented by the next important

Theorem 4.6. *If (f_α) is a net of multilinear maps from*

$$X(\mathcal{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathcal{P}_i)$$

into $Y(\mathcal{Q})$ which is equicontinuous at the origin of $X(\mathcal{P})$, and $Y(\mathcal{Q})$ is, in addition, complete, then the set

$$D = \{x \in X\colon (f_\alpha(x)) \text{ converges in } Y(Q)\}$$

is a closed subset of $X(\mathcal{P})$.

Proof. Assume that $x \in X$ and (x_ν) is a net in D such that

$$x \in \lim_{\nu}{}_{\mathcal{P}}\, x_\nu .$$

If $q \in \mathcal{Q}$, then because of the subadditivity of q and $\overline{\lim}$, we clearly have

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x) - f_\beta(x)\big) \leqq \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x) - f_\alpha(x_\nu)\big) +$$

$$+ \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x_\nu) - f_\beta(x_\nu)\big) + \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\beta(x_\nu) - f_\beta(x)\big)$$

for all ν, where (α, β) runs in the corresponding product directed set. Moreover, since convergent nets are Cauchy nets, we also have

$$\lim_{(\alpha,\beta)} q\big(f_\alpha(x_\nu) - f_\beta(x_\nu)\big) = 0$$

for all ν. On the other hand, because of $q(-y) = q(y)$ and the definition of upper limit, it is also clear that

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x) - f(x_\nu)\big) = \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\beta(x_\nu) - f_\beta(x)\big) = \overline{\lim_{\alpha}}\, q\big(f_\alpha(x_\nu) - f_\alpha(x)\big)$$

for all ν. Consequently, we have

$$\overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x) - f_\beta(x)\big) \leqq 2\, \overline{\lim_{\alpha}}\, q\big(f_\alpha(x_\nu) - f_\alpha(x)\big)$$

for all ν. Hence, by noticing that (x_ν) and (x) are bounded coherent nets in $X(\mathcal{P})$ and thus by Theorem 4.1

$$\lim_{\nu} \overline{\lim_{(\alpha,\beta)}}\, q\big(f_\alpha(x_\nu) - f_\alpha(x)\big) = 0,$$

we can infer that

$$\lim_{(\alpha,\beta)} q\big(f_\alpha(x) - f_\beta(x)\big) = 0.$$

This shows that $(f_\alpha(x))$ is a Cauchy net in $Y(\mathcal{Q})$. Hence, by the completeness of $Y(\mathcal{Q})$, it is clear that $x \in D$. And thus, we have proved that D is closed in $X(\mathcal{P})$.

Remark 4.7. Particular cases of Theorems 4.5 and 4.6 can be used to derive some essential extensions of a general convergence theorem for net integrals [7, Theorem 3.8].

However, to realize the usefulness of Theorems 4.5 and 4.6 in integration, the reader is rather advised to derive first a uniform convergence theorem for the classical Reimann—Stieltjes integral.

5. Supplements. By using Theorem 4.1, we can also easily prove a remarkable characterization of equicontinuity of a net (f_α) of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ in terms of the induced uniformities $\mathscr{U}_{\mathscr{P}}$ and $\mathscr{U}_{\mathscr{Q}}$.

Theorem 5.1. *If (f_α) is a net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\mathsf{X}}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *$\underline{\lim}_\alpha (f_\alpha^{-1} \circ V \circ f_\alpha)(x)$ is a neighbourhood of x in $X(\mathscr{P})$ for all $V \in \mathscr{U}_{\mathscr{Q}}$ and $x \in X$.*

Proof. If (ii) does not hold, then because of the definition of the induced uniformities and [9, Remark 3.9], there exist $x \in X$, $q \in \mathscr{Q}$ and $\varepsilon > 0$ such that the ball $B_p^{m^{-1}}(x)$ is not contained in the set

$$\underline{\lim}_\alpha (f_\alpha^{-1} \circ B_q^\varepsilon \circ f_\alpha)(x) = \bigcup_\alpha \bigcap_{\beta \geqq \alpha} f_\alpha^{-1}(B_q^\varepsilon(f_\alpha(x)))$$

for all $p \in \mathscr{P}^*$ and $m \in \mathrm{N}$. Thus, for each $v = (p, m) \in \Delta = \mathscr{P}^* \times \mathrm{N}$ there exists $x_v \in B_p^{m^{-1}}(x)$ such that

$$\overline{\lim_\alpha}\, q(f_\alpha(x_v) - f_\alpha(x)) \geqq \varepsilon.$$

Hence, it is clear that $(x_v)_{v \in \Delta}$ is a net in $X(\mathscr{P})$ such that $x \in \lim_v{}_{\mathscr{P}} x_v$, but

$$\overline{\lim_v}\, \overline{\lim_\alpha}\, q(f_\alpha(x_v) - f_\alpha(x)) \geqq \varepsilon,$$

and thus (i) cannot hold because of Theorem 4.1.

Thus, we have proved that (i) implies (ii). To prove the converse implication, note that even the particular case of (ii) when $x = 0$ does already imply (i).

Remark 5.2. Hence, it is clear that a nonvoid set $\{f_\alpha\}$ of multilinear maps from $X(\mathscr{P}$, into $Y(\mathscr{Q})$ is equicontinuos at the origin of $X(\mathscr{P})$ if and only if $\bigcap_\alpha (f_\alpha^{-1} \circ V \circ f_\alpha)(x)$ is a neighbourhood of x in $X(\mathscr{P})$ for all $V \in \mathscr{U}_{\mathscr{Q}}$ and $x \in X$.

Remark 5.3. By using the topological refinement

$$\hat{\mathscr{U}}_{\mathscr{P}} = \{R \subset XxX \colon \forall x \in X \colon \exists U \in \mathscr{U}_{\mathscr{P}} \colon U(x) \subset R(x)\}$$

of $\mathscr{U}_{\mathscr{P}}$ [13], the assertions of Theorem 5.1 and Remark 5.2 can be rephrased in the more instructive form that the net (f_α) (set $\{f_\alpha\}$) is equicontinuous at the origin of $X(\mathscr{P})$ if and only if

$$\underline{\lim}_{\alpha} f_\alpha^{-1} \circ V \circ f_\alpha \in \hat{\mathscr{U}}_{\mathscr{P}} \quad \left(\bigcap_{\alpha} f_\alpha^{-1} \circ V \circ f_\alpha \in \hat{\mathscr{U}}_{\mathscr{P}}\right)$$

for all $V \in \mathscr{U}_{\mathscr{Q}}$.

Note that the "only if parts" of the above assertions are much weaker then the corresponding parts of Theorems 3.9 and 4.1 and Remarks 3.10 and 4.2. In principle, $\lim_{\mathscr{P}}$ and $\mathscr{U}_{\mathscr{P}}$ should be equivalent tools in $X(\mathscr{P})$. However, actually we do not even know that which subfamily of $\hat{\mathscr{U}}_{\mathscr{P}}$ could be used to express the bounded uniform continuity of a function f from $X(\mathscr{P})$ ino $Y(\mathscr{Q})$.

Whenever the net (f_α) of multilinear maps from $X(\mathscr{P})$ into $Y(\mathscr{Q})$ is pointwise convergent in the usual sense that the net $(f_\alpha(x))$ converges in $Y(\mathscr{Q})$ for all $x \in X$, then the converse of Theorem 4.5 is also true. In fact, in this particular case, we can even prove a little more.

Theorem 5.4. *If (f_α) is a pointwise convergent net of multilinear maps from*

$$X(\mathscr{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$ and f is the relation defined on X by

$$f(x) = \lim_{\alpha}{}_{\mathscr{Q}} f_\alpha(x),$$

then the following assertions are equivalent:

(i) *(f_α) is equicontinuous at the origin of $X(\mathscr{P})$;*

(ii) *f is selectionally boundedly uniformly continuous;*

(iii) *f is lower semicontinuous at the origin of $X(\mathscr{P})$.*

Proof. Because of Theorem 4.5 and Remark 4.4, we need only show that (iii) also implies (i). For this, assume that (iii) holds, and let $q \in \mathscr{Q}$ and $M_q = \overline{\lim}_{\alpha}\, q \circ f_\alpha$. If $\varepsilon > 0$, then by the definition of $\mathscr{U}_{\mathscr{Q}}$, the ball $B_q^\varepsilon(0)$ is a neighbourhood of 0 in $Y(\mathscr{Q})$. Thus, because of $0 \in f(0)$ and (iii), the set $U = f^{-1}(B_q^\varepsilon(0))$ is a neighbourhood of 0 in $X(\mathscr{P})$. If $x \in U$, then by the definition of U, there exists $y \in B_q^\varepsilon(0)$ such that $y \in f(x)$. Hence, it is clear that

$$q(f_\alpha(x)) \leqq q(f_\alpha(x) - y) + q(y) \leqq q(f_\alpha(x) - y) + \varepsilon$$

for all α, and

$$\lim_{\alpha} q(f_\alpha(x) - y) = 0.$$

Consequently, we have

$$M_q(x) = \overline{\lim_{\alpha}}\, q(f_\alpha(x)) \leqq \varepsilon.$$

Hence, it is clear that M_q is continuous at the origin of $X(\mathscr{P})$, and thus by Definition 3.5, (i) also holds.

Remark 5.5. Note that to obtain (i) we have only used a particular case of (iii).

As an immediate consequence of Theorem 5.4, we can easily get the essential improvement of [3, (18.2) Theorem] proved directly in [15].

Corollary 5.6. *If f is a multilinear map from*

$$X(\mathscr{P}) = \mathop{\times}_{i=1}^{n} X_i(\mathscr{P}_i)$$

into $Y(\mathscr{Q})$, then the following assertions are equivalent:

(i) *f is boundedly uniformly continuous;*

(ii) *f is continuous at the origin of $X(\mathscr{P})$;*

(iii) *$q \circ f$ is continuous at the origin of $X(\mathscr{P})$ for all $q \in \mathscr{Q}$.*

Proof. To apply Theorem 5.4, note that f is a selection function for the relation F defined on X by

$$F(x) = \lim_{\alpha} {}_{\mathscr{Q}} f_\alpha(x),$$

where α runs in an arbitrary nonvoid directed set.

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