Acta Sci. Math., 55 (1991), 129—139

# AF-algebras with unique trace

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An AF C*-algebra is, by definition, the norm closure of an increasing sequence of finite dimensional C*-algebras. In some sense, these are the simplest noncommutative infinite dimensional C*-algebras.

Our interest in AF-algebras with unique trace is related to the problem of constructing subfactors with a given index of the hyperfinite type $\mathrm{II}_1$ von Neumann factor $R$. For this, one is led to find a sequence of increasing finite dimensional C*-algebras and to take their weak closure in the GNS representation given by a tracial state. If there is only one tracial state, the finite hyperfinite von Neumann algebra one obtains is a factor, hence it is $R$ if it is infinite dimensional.

One way to guarantee the uniqueness of the trace is to fit the situation described in Remark 3: one can apply then either the quoted theorem of Elliott (stated in K-theoretic language) or the Perron—Frobenius theory on matrices with positive entries.

Our approach gives the desired conclusion for a wider class of AF algebras (the matrix given in Remark 2 is not primitive) and establishes some additional properties.

## Statement of the result

Let $A$ be a unital AF C*-algebra, inductive limit of the finite dimensional algebras $\mathbf{C}\cdot 1\subset A_1\subset A_2\subset A_3\subset\ldots$ (1 is the unit of $A$).

We denote by $m_k=(m_1^k, m_2^k, \ldots, m_{c_k}^k)$ the dimension vector of the algebra $A_k$ and by $R_k=(r_{ij}^k)_{i=1,\ldots,c_k;\ j=1,\ldots,c_{k+1}}$ the inclusion matrix for $A_k\subset A_{k+1}$ $(k\geqq 1)$. In particular, ${}^tR_k m_k=m_{k+1}$.

If $w$ is a real vector, $w\geqq 0$ means that its entries are nonnegative. For $w=(w_1, \ldots, w_n)\in\mathbf{R}^n$, $w\geqq 0$, $w\neq 0$, we define

$$\chi(w):=\Big(\sum_{k=1}^{n} w_k\Big)^{-1}\min\Big\{\sum_{i\in I} w_i\,\Big|\, I\subset\{1, 2, \ldots, n\},\ \mathrm{card}\,(I)\geqq n/2\Big\}.$$

Received July 28, 1988.

We consider the multiplicative group $G = \bigcup_{k=1}^{\infty} \mathcal{U}(A_k)$ and its action on $A$ by inner automorphisms.

$$g \in G \overset{\Theta}{\longmapsto} \operatorname{Ad} g \in \operatorname{Int}(A) \subset \operatorname{Aut}(A)$$

$$g(x) \equiv (\operatorname{Ad} g)(x) = gxg^{-1} \quad (g \in G,\, x \in A).$$

We prove the following

Theorem. *With the notations introduced above, let*

$$\varepsilon_k := \min_{j=i,\dots,c_{k+1}} \chi\big((m_i^k r_{ij}^k)_{i=1,\dots,c_k}\big) \quad (k \geqq 1).$$

*If*

$$(*) \qquad \sum_{k=1}^{\infty} \varepsilon_k = \infty,$$

*then:*

(a) *there is a unique normalized trace, denoted by $\tau$, on $A$;*
(b) *$\tau$ is faithful if and only if $A$ is simple;*
(c) *the action $\Theta$ is mixing with respect to the trace $\tau$, i.e.*

$$(\forall)\, x, y \in A_h, \quad (\exists)\, g_n \in G \quad (n \in \mathrm{N}) \quad \textit{such that} \quad \lim_{n\to\infty} \tau\big(g_n(x)y\big) = \tau(x)\tau(y).$$

There are conditions which imply $(*)$ and depend only on the inclusion matrices $R_k$.

Corollary. *With the $R_k$'s introduced above, let*

$$\delta_k := \min_{i,j} r_{ij}^k / \max_{i,j} r_{ij}^k \quad (i = 1, \dots, c_k;\ j = 1, \dots, c_{k+1})$$

*and*

$$\tilde{\varepsilon}_k := \min_{j=1,\dots,c_{k+1}} \chi\big((r_{ij}^k)_{i=1,\dots,c_k}\big).$$

*If*

$$(1) \qquad \sum_{k=2}^{\infty} \delta_{k-1} \tilde{\varepsilon}_k = \infty,$$

*or*

$$(2) \qquad \sum_{k=2}^{\infty} \delta_{k-1} \delta_k = \infty,$$

*then:*

(i) *the algebra $A$ is simple and has a unique normalized trace $\tau$, which is faithful;*
(ii) *the action $\Theta$ is mixing with respect to the trace $\tau$.*

Namely, we shall prove that $(2) \Rightarrow (1) \Rightarrow (*)$.

Remark 1. Condition $(*)$ depends effectively on the particular sequence of algebras $A_n$ defining $A$. Indeed, let $m_1=(1,1,1,1)$, and for $k\geqq 1$,

$$R_{2k+1}=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix},\quad R_{2k}=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix};\quad \text{hence}\quad R_{2k-1}R_{2k}=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}.$$

Then the sum in $(*)$ is zero for the sequence $A_1\subset A_2\subset A_3\subset\ldots$, but it is infinite for the sequence $A_1\subset A_3\subset A_5\subset\ldots$.

Remark 2. Condition $(*)$ does not imply any of the equivalent conditions in (b): let $m_1=(1,1,1)$, and

$$R_k=\begin{pmatrix}1&1&2\\0&1&0\\2&1&1\end{pmatrix}\quad\text{for all}\quad k\geqq 1.$$

Then $\varepsilon_k=1/2$, but the (unique) trace on $A$ has the weights $((1/2)3^{-k+1},0,(1/2)3^{-k+1})$ on $A_k$, hence it is not faithful. One can also see from the Bratteli diagram that $A$ is not simple.

Remark 3. As a special case of the Corollary (part (i)), we can treat the situation dealt with in a theorem of ELLIOTT (Th. 6.1. in [2]), namely when $R_k=R$ for all $k$, where $R$ is a primitive matrix, i.e. there is a nonzero $p$ such that $R^p$ has positive entries. Indeed, if we consider the sequence

$$A_1\subset A_{p+1}\subset A_{2p+1}\subset A_{3p+1}\subset\ldots$$

(which also defines $A$), the inclusion matrices will be constantly $R^p$; hence, the $\delta_k$'s will be all equal and nonzero (because $R^p$ has no zero entry), and then clearly (2) holds.

The proof of Elliott follows different ideas.

## Notations and steps of the proof

Let

$$A_n=\bigoplus_{l=1}^{c_n}A_n^l,\quad A_n^l\cong \mathrm{Mat}_{m_l^n}(\mathbf{C})$$

be the factor decomposition of the $A_n$'s. For $x\in A_n$, we denote by $[x]_n^l$ its $A_n^l$-component and by $\alpha_n^l(x)$ the normalized trace of $[x]_n^l\in A_n^l$:

$$\alpha_n^l(x)=\mathrm{tr}([x]_n^l)=(1/m_l^n)\,\mathrm{Tr}([x]_n^l)$$

(we denote by Tr the canonical trace on a full matrix algebra — i.e. the sum of all diagonal entries — and by tr the normalized one).

If $v=(v_1 \dots v_k)\in \mathbf{C}^k$ is a vector, we write $\omega(v)$ for the "oscillation" of $v$, i.e.

$$\omega(v) := \max_{i,j=1,\dots,k} |v_i - v_j|.$$

Now for any $x\in A_n$, we introduce the vector $\alpha_n(x) := (\alpha_n^1(x), \alpha_n^2(x), \dots, \alpha_n^{c_n}(x))$ and the value $\omega(\alpha_n(x))$. We denote

$$A_\infty := \bigcup_{n=1}^{\infty} A_n.$$

The proof will be divided in a sequence of lemmas.

The first step is to show that for any $x\in A_\infty$, $\lim_{n\to\infty} \omega(\alpha_n(x))=0$, i.e. the entries of $\alpha_n(x)$ tend to become mutually equal. It is here that we use condition $(*)$. This implies that as $n$ goes to infinity, the entries of $\alpha_n(x)$ converge to some complex number $\tau(x)$. This result is derived in Lemma 3, using the results of the previous two lemmas.

In Lemma 4 we check that the map $x\in A_\infty \mapsto \tau(x)\in \mathbf{C}$ defines a tracial state on $A_\infty$ and we show that this is the unique one. So assertion (a) of the Theorem will be proved.

In Lemma 5, using a characterization of simplicity for AF algebras in terms of the inclusion matrices, we prove that the above defined trace is faithful if and only if the algebra $A$ is simple, i.e. assertion (b) of the Theorem.

Assertion (c) of the Theorem (that the action $\Theta$ is mixing with respect to the trace $\tau$) is proved in Lemma 6, after some remarks on finite dimensional $C^*$-algebras.

Finally, in Lemma 7, we show that $(2)\Rightarrow(1)\Rightarrow(*)$ and that if (1) or (2) hold, then the algebra $A$ is simple. Using these facts, the Corollary follows easily from the Theorem.

We emphasize that the whole proof depends on the fact that $\lim_{n\to\infty} \omega(\alpha_n(x))=0$. This is deduced from condition $(*)$ by the estimate given in Lemma 2. One can look for other estimates in order to obtain the same fact from other conditions. Our estimates is insensitive to the equality of all rows of $Q$, when $\|Q\|_\omega=0$, regardless of $\varepsilon$ (see the notations in Lemma 2). We have chosen it because of its relative simplicity.

## The proofs

First of all we clarify how the inclusion matrices $R_k$ and the dimension vectors $m_k$ allow the computation of $\alpha_{n+1}(x)$ from $\alpha_n(x)$. Let $Q_n=(q^n_{ij})_{i=1,\dots,c_{n+1};\ j=1,\dots,c_n}$ be the matrix given by $q^n_{ij}=m^n_j r^n_{ji}/m^{n+1}_i$, i.e.

$$Q_n = \begin{pmatrix} m_1^{n+1} & & 0 \\ & \ddots & \\ 0 & & m_{c_{n+1}}^{n+1} \end{pmatrix}^{-1} {}^tR_n \begin{pmatrix} m_1^{n} & & 0 \\ & \ddots & \\ 0 & & m_{c_n}^{n} \end{pmatrix},$$

and $1_m=(1,1,\dots,1)\in\mathbf{C}^m$. Note that $Q_n(1_{c_n})=1_{c_{n+1}}$ because $m_l^{n+1}=\sum_{k=1}^{c_n} m_k^n r_{kl}^n$ $(l=1,\dots,c_{n+1})$.

Lemma 1. *For any* $x\in A_n$ *we have*

(a) $\alpha_{n+1}(x)=Q_n\alpha_n(x)$,

(b) $\min_{1\le k\le c_n} \operatorname{Re}\alpha_n^k(x)\le \operatorname{Re}\alpha_{n+1}^l(x)\le \max_{1\le k\le c_n}\operatorname{Re}\alpha_n^k(x)$,

$\min_{1\le k\le c_n} \operatorname{Im}\alpha_n^k(x)\le \operatorname{Im}\alpha_{n+1}^l(x)\le \max_{1\le k\le c_n}\operatorname{Im}\alpha_n^k(x)$ *for all* $l=1,\dots,c_{n+1}$.

Proof. (a) Using the information given by the inclusion matrix, it follows that

$$\alpha_{n+1}^l(x)=\operatorname{Tr}\big([x]_{n+1}^l\big)/m_l^{n+1}=\Big(\sum_{k=1}^{c_n} r_{kl}^n \operatorname{Tr}\big([x]_n^k\big)\Big)/m_l^{n+1}=\Big(\sum_{k=1}^{c_n} m_k^n r_{kl}^n\alpha_n^k(x)\Big)/m_l^{n+1}.$$

(b) This is a consequence of the relation $Q_n(1_{c_n})=1_{c_{n+1}}$ and of the fact that $Q_n$ has real nonnegative entries (hence $\alpha_{n+1}^l(x)$ is a weighted average of the entries of $\alpha_n(x)$).

Let us study the matrices $Q=(q_{ij})_{i=1,\dots,n;\ j=1,\dots,m}$ with real nonnegative entries which satisfy $Q(1_m)=1_n$. Note that if $v\in\mathbf{R}^m$ and $\omega(v)=0$, then $\omega(Q(v))=0$ ($\omega(w)=0 \Leftrightarrow w$ is proportional to the vector $1_m$). Since $\omega$ defines a seminorm on any $\mathbf{R}^p$, from the above remark we see that $Q$ induces a linear map $\tilde{Q}\colon \mathbf{R}^m/\omega\to\mathbf{R}^n/\omega$, where $\mathbf{R}^p/\omega$ denotes the quotient space $\mathbf{R}^p/\{v\in\mathbf{R}^p|\omega(v)=0\}$. Hence,

$$\|Q\|_\omega := \sup\{\omega(Q(v))|v\in\mathbf{R}^n,\ \omega(v)\le 1\}$$

is finite. Clearly

$$\omega(Q(v))\le\|Q\|_\omega\,\omega(v) \quad\text{and}\quad \|Q_1Q_2\|_\omega\le\|Q_1\|_\omega\|Q_2\|_\omega$$

whenever $Q_1Q_2$ is defined.

Lemma 2. *Let* $Q=(q_{ij})_{i=1,\dots,n;\ j=1,\dots,m}$ *be a matrix with real nonnegative entries which satisfies* $Q(1_m)=1_n$. *Then* $\|Q\|_\omega\le 1-\varepsilon$, *where*

$$\varepsilon := \min_{i=1,\dots,n}\chi\big((q_{ij})_{j=1,\dots,m}\big).$$

Proof. It is enough to show that if $v=(v_1, \ldots, v_m)\in\mathbf{R}^m$, $w=(w_1, \ldots, w_m)\in\mathbf{R}^m$ are such that $v\geqq 0$, $w\geqq 0$, $\sum_{k=1}^{m} v_k=1$, $\sum_{k=1}^{m} w_k=1$, $\chi(v)\geqq\varepsilon$, $\chi(w)\geqq\varepsilon$, then

$$|\langle\alpha, v\rangle-\langle\alpha, w\rangle| \leqq (1-\varepsilon)\,\omega(\alpha)$$

for any $\alpha=(\alpha_1, \ldots, \alpha_m)\in\mathbf{R}^m$, where $\langle\cdot, \cdot\rangle$ stands for the canonical scalar product of $\mathbf{R}^m$. The desired result will then follow by considering $v=(q_{ik})_{k=1,\ldots,m}$, $w=(q_{jk})_{k=1,\ldots,m}$ for all $1\leqq i, j\leqq n$.

Let $a=\min_k \alpha_k$, $b=\max_k \alpha_k$, $I=[a, b]^m$. Then $\alpha\in I$, $\omega(\alpha)=b-a$. Since the map $f\colon I\to\mathbf{R}$, $f(u):=\langle u, v\rangle-\langle u, w\rangle$ is an affine map, $f(I)=\mathrm{co}\, f(\mathrm{ext}\, I)$, where $\mathrm{ext}\, I$ denotes the set of extreme points of $I$ and co stands for convex hull.

Let $\beta\in\mathrm{ext}\, I$, $\beta=(\beta_1, \ldots, \beta_m)$. Then $\beta_k\in\{a, b\}$ for any $k=1, \ldots, m$. Denote

$$K_a = \{k|1 \leqq k \leqq m,\ \beta_k = a\},\quad K_b = \{k|1 \leqq k \leqq m,\ \beta_k = b\}.$$

One of the sets $K_a$ and $K_b$ has at least $n/2$ elements. Suppose card $K_a\geqq n/2$. Since $\langle\beta, w\rangle\geqq a$, we have

$$f(\beta) = \Big(a \sum_{k\in K_a} v_k+b \sum_{k\in K_b} v_k\Big)-\langle\beta, w\rangle =$$
$$= \Big[b-(b-a) \sum_{k\in K_a} v_k\Big]-\langle\beta, w\rangle \leqq b-(b-a)\,\chi(v)-a \leqq (1-\varepsilon)(b-a).$$

For $v$ instead of $w$ we also obtain $f(\beta)\geqq-(1-\varepsilon)(b-a)$.

The case card $K_b\geqq n/2$ can be treated similarly and we obtain the same results. Thus for any $\beta\in\mathrm{ext}\, I$ we have

$$-(1-\varepsilon)(b-a) \leqq f(\beta) \leqq (1-\varepsilon)(b-a),$$

hence

$$f(I) \subset [-(1-\varepsilon)(b-a),\ (1-\varepsilon)(b-a)],$$

and therefore

$$|f(\alpha)| \leqq (b-a)(1-\varepsilon) = \omega(\alpha)(1-\varepsilon).$$

Recall that $\prod_{n=1}^{\infty}(1-\eta_n)=0$ whenever $0\leqq\eta_n\leqq 1$ and $\sum_{n=1}^{\infty}\eta_n=\infty$. Therefore, by condition $(*)$ we have

$$(**)\qquad \prod_{n=n_0}^{\infty}(1-\varepsilon_n) = 0 \quad (\forall)\, n_0 \geqq 1.$$

Note that due to Lemma 1(a), the $\varepsilon_n$'s defined in the Theorem have the same meaning for the matrices $Q_n$ as $\varepsilon$ for the matric $Q$ in Lemma 2.

For $v=(v_1, \ldots, v_m)\in\mathbf{C}^m$, define

$$\|v\|_\infty := \max_{k=1,\ldots,m} |v_k|.$$

Now we can prove

Lemma 3. *For any* $x \in A_{n_0}$ *we have*

(a) $\lim_{n\to\infty} \omega(\alpha_n(x)) = 0,$

(b) $\lim_{n\to\infty} \|\alpha_n(x) - \tau(x)\cdot 1_{c_n}\|_\infty = 0$ *for some* $\tau(x) \in \mathbf{C}$.

Proof. Let $n \geqq n_0$. Since both $\omega$ and $\|\cdot\|_\infty$ are seminorms, we can deal separately with the real and imaginary parts of $\alpha_n(x)$. Denote by $\operatorname{Re}\alpha_n(x)$ and $\operatorname{Im}\alpha_n(x)$ the vectors whose entries are the real and the imaginary parts, respectively, of the entries of $\alpha_n(x)$. By Lemma 1(a), we see that

$$\operatorname{Re}\alpha_{n+1}(x) = Q_n(\operatorname{Re}\alpha_n(x)),\ \operatorname{Im}\alpha_{n+1}(x) = Q_n(\operatorname{Im}\alpha_n(x)).$$

Lemma 2 implies that

$$\omega(\operatorname{Re}\alpha_{n+1}(x)) \leqq \|Q_n\|_\omega\, \omega(\operatorname{Re}\alpha_n(x)) \leqq (1-\varepsilon_n)\,\omega(\operatorname{Re}\alpha_n(n)).$$

Iterating we get

$$\omega(\operatorname{Re}\alpha_{n+1}(x)) \leqq \prod_{k=n_0}^{n} (1-\varepsilon_k)\,\omega(\operatorname{Re}\alpha_{n_0}(x))$$

and then, by $(**)$,

$$\lim_{n\to\infty} \omega(\operatorname{Re}\alpha_n(x)) = 0.$$

Since

$$\omega(\operatorname{Re}\alpha_n(x)) = \max_{1\leqq l\leqq c_n} \operatorname{Re}\alpha_n^l(x) - \min_{1\leqq l\leqq c_n} \operatorname{Re}\alpha_n^l(x),$$

Lemma 1(b) implies that

$$\lim_{n\to\infty} \|\operatorname{Re}\alpha_n(x) - a1_{c_n}\|_\infty = 0 \quad \text{for some} \quad a \in \mathbf{R}.$$

The vectors $\operatorname{Im}\alpha_n(x)$ can be treated similarly.

Lemma 4. (a) *The mapping* $x \in A_\infty \xmapsto{\tau} \tau(x) \in \mathbf{C}$ *is a continuous normalized trace on* $A_\infty$ *which can be extended by continuity to the whole* $A$.

(b) *Any normalized trace on* $A_\infty$ *equals* $\tau$.

Proof. (a) Linearity follows from the fact that

$$\alpha_n(ax+by) = a\alpha_n(x) + b\alpha_n(y) \quad \text{for any} \quad x, y \in A_n \quad \text{and} \quad a, b \in \mathbf{C}.$$

It is easy to see that $\alpha_n(1) = 1_{c_n}$, hence $\tau(1) = 1$. Since $|\operatorname{tr}([x]_n^l)| \leqq \|[x]_n^l\| \leqq \|x\|$, we see that $\|\alpha_n(x)\|_\infty \leqq \|x\|$, and hence $|\tau(x)| \leqq \|x\|$. Similarly, $\alpha_n(x^*x) \geqq 0$, hence $\tau(x^*x) \geqq 0$.

That $\tau$ is a trace follows from the relation

$$\alpha_n(xy) = \alpha_n(yx) \quad \text{for any} \quad x, y \in A_n,$$

which is a consequence of the definition of $\alpha_n$.

(b) Let $\mu$ be any normalized trace on $A$. Since the factors $A_n^l$ have unique normalized traces, the restriction of $\mu$ to the algebra $A_n$ is described by a nonnegative vector $t_n=(t_n^1, t_n^2, \ldots, t_n^{c_n})$ with $\sum_{k=1}^{c_n} t_n^k=1$. If $x\in A_n$, then

$$\mu(x) = \sum_{k=1}^{c_n} t_n^k \alpha_n^k(x).$$

Then for all $x\in A_n$.

$$|\mu(x)-\tau(x)| = \Big|\sum_{k=1}^{c_n} t_n^k[\alpha_n^k(x)-\tau(x)]\Big| \leqq \sum_{k=1}^{c_n} t_n^k \|\alpha_n(x)-\tau(x)\,1_{c_n}\|_\infty = \|\alpha_n(x)-\tau(x)\,1_{c_n}\|_\infty .$$

Hence Lemma 3 (a) implies that $\mu(x)=\tau(x)$ for all $x\in A_\infty$.

Lemma 5. *Suppose* $(*)$ *holds and* $\tau$ *is the above defined trace. Then* $\tau$ *is faithful if and only if the algebra* $A$ *is simple.*

Proof. Denote by $e_n^l$ the minimal central projection of $A_n$ corresponding to $A_n^l$. It is known that $A$ is simple if and only if for any $n\geqq 1$ and any $1\leqq l\leqq c_n$, there is a $p>n$ such that the inclusion matrix $R_{n,p}=(r_{ij}^{n,p})_{i=1,\ldots,c_n;\, j=1,\ldots,c_p}$ for $A_n\subset A_p$ has only nonzero entries on the $l$-th row (i.e. $A_n^l$ "enters" in all factor summands of $A_p$)—just look at the description of the ideals in the Bratteli diagram of $A$. Since

$$\alpha_p^i(e_n^l) = (r_{li}^{n,p}\, m_l^n)/m_i^p, \quad i = 1, \ldots, c_p,$$

we see that the above condition on the inclusion matrix is equivalent to the fact that $\alpha_p(e_n^l)$ has only nonzero entries.

Suppose first that $\tau$ is faithful. Choose $n\geqq 1$ and $1\leqq l\leqq c_n$. Since $\tau(e_n^l)\neq 0$, and

$$\lim_{p\to\infty} \|\alpha_p(e_n^l)-\tau(e_n^l)\,1_{c_p}\|_\infty = 0,$$

we infer that for $p$ large enough, all the entries of $\alpha_p(e_n^l)$ are nonzero. Thus by the above remark, $A$ must be simple.

The converse implication is obvious since $J:=\{x\in A|\tau(x^*x)=0\}$ is a bilateral ideal and $1\notin J$.

For proving the mixing property of $\Theta$ we need two elementary and possible well-known results which we record below.

For a finite dimensional C*-algebra $N$, with a fixed system of matrix units and $x\in N$, we denote by Diag $(x)$ the set of values which are on the diagonal of $x$.

Remark 4. Let $x\in \mathrm{Mat}_n(\mathbf{C})\cong\mathscr{B}(\mathbf{C}^n)$, $x=x^*$. Then there is a unitary $u\in \mathrm{Mat}_n(\mathbf{C})$ such that $\mathrm{Diag}\big((\mathrm{Ad}\,u)(x)\big)$ has only one element (namely $\mathrm{tr}(x)$). (This statement also holds for $x\neq x^*$ but its proof would be more intricate.)

To see this, notice first that since $x=x^*$, there is an orthogonal basis of $C^n$ with respect to which $x$ has diagonal form, hence the corresponding matrix has real entries.

If we consider $(\operatorname{Ad} u)(x)$ instead of $x$, where $u$ is the unitary matrix that describes the change of coordinates, we may assume that $x \in \mathrm{Mat}_n(\mathbf{R})$.

We shall obtain the assertion by induction. Let $n=2, x=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{R})$. We define

$$u_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathscr{U}(\mathrm{Mat}_2(\mathbf{R})), \quad t \in [0, \pi/2].$$

Since $(\operatorname{Ad} u_0)(x)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(\operatorname{Ad} u_{\pi/2})(x)=\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, and $t \mapsto (\operatorname{Ad} u_t)(x)$ is a continuous function with values in $\mathrm{Mat}_2(\mathbf{R})$, the Darboux property of it implies that there is a $t \in [0, \pi/2]$ such that $(\operatorname{Ad} u_t)(x)$ has equal diagonal entries. Moreover, $(\forall)\, \lambda \in \mathbf{R}$, $\min\{a, d\} \leqq \lambda \leqq \max\{a, d\} \Rightarrow (\exists)\, t \in [0, \pi/2]$, such that

$$(\operatorname{Ad} u_t)(x) = \begin{pmatrix} \lambda & * \\ * & * \end{pmatrix}. \tag{3}$$

The statement is proved for $n=2$. Assume we have proved it for $n-1$, $n \geqq 3$. Let $x=(a_{ij}) \in \mathrm{Mat}_n(\mathbf{R})$. If $x$ has different diagonal entries, one of them, say $a_{11}$, differs from $\operatorname{tr}(x)$. We may assume that $a_{11} < \operatorname{tr}(x)$. There must be an $i_0 \neq 1$ such that $a_{i_0 i_0} > \operatorname{tr}(x)$. We may consider $i_0=2$. Due to (3), there is a unitary

$$\tilde{u}_t = \begin{pmatrix} u_t & 0 \\ 0 & I_{n-2} \end{pmatrix} \in \mathrm{Mat}_n(\mathbf{R})$$

such that

$$x' := (\operatorname{Ad} \tilde{u}_t)(x) = \begin{pmatrix} \operatorname{tr}(x) & * \\ * & x'' \end{pmatrix},$$

where $x'' \in \mathrm{Mat}_{n-1}(\mathbf{R})$. By the inductive assumption there is a $u'' \in \mathrm{Mat}_{n-1}(\mathbf{C})$ such that $\operatorname{Diag}((\operatorname{Ad} u'')(x''))$ has only one value, namely $\operatorname{tr}(x'')$. But $\operatorname{tr}(x'')=\operatorname{tr}(x)$, hence if

$$u' = \begin{pmatrix} 1 & 0 \\ 0 & u'' \end{pmatrix}$$

then $\operatorname{Diag}((\operatorname{Ad} u'\tilde{u}_t)(x))$ has only one value.

Remark 5. Let $N$ be a finite dimensional C*-algebra with a fixed system of matrix units, and let $\mu$ be a normalized trace on $N$. If $x, y \in N$ and $y$ has a diagonal form, then

$$|\mu(xy) - \mu(x)\mu(y)| \leqq \|y\|\, \Delta_N(x),$$

where $\Delta_N(x) = \max\{|a-a'| \,|\, a, a' \in \operatorname{Diag}(x)\}$.

This follows by an easy computation. Suppose that $N=\bigoplus_{i=1}^{m}\mathrm{Mat}_{n_i}(\mathbf{C})$, and let $t=(t_1, \ldots, t_m)$ be the vector of the weights of the minimal projections of the factor summands of $N$ in the trace $\mu$ (so that $\sum_{i=1}^{m} n_i t_i=1$). Let the diagonal entries of $x$ and $y$ be $a_1^1, a_2^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2, \ldots, a_1^m, \ldots, a_{n_m}^m$ and $b_1^1, b_2^1, \ldots, b_{n_1}^1, b_1^2, \ldots, b_{n_2}^2, \ldots, \ldots, b_1^m, \ldots, b_{n_m}^m$, respectively (the upper index indicates the factor summand of $N$).

Then

$$\mu(x)=\sum_{l=1}^{m} t_l \sum_{i=1}^{n_l} a_i^l, \quad \mu(y)=\sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} b_j^k, \quad \mu(xy)=\sum_{k=1}^{m} t_k \sum_{j=1}^{n_k} a_j^k b_j^k$$

(because $y$ has a diagonal form). Since $1=\sum_{l=1}^{m}\sum_{i=1}^{n_l} t_l$,

$$|\mu(xy)-\mu(x)\mu(y)| = \Big|\sum_{l=1}^{m}\sum_{k=1}^{m}\sum_{i=1}^{n_l}\sum_{j=1}^{n_k} t_l t_k (a_j^k b_j^k - a_i^l b_j^k)\Big| \leqq$$

$$\leqq \Big(\sum_{l=1}^{m}\sum_{k=1}^{m}\sum_{i=1}^{n_l}\sum_{j=1}^{n_k} t_l t_k\Big) \max_{k,l,i,j} |a_j^k - a_i^l| \max_{k,j} |b_j^k| = \Delta_N(x)\|y\|.$$

Lemma 6. *Suppose $(*)$ holds and $\tau$ is the trace on $A$ given in Lemma 4. Then the action $\Theta$ is mixing with respect to $\tau$.*

Proof. Choose the systems of matrix units in the $A_n$'s such that the matrix units of $A_n$ are sums of matrix units of $A_{n+1}$ for all $n$. Let $x, y \in A_\infty$, $x=x^*$, $y=y^*$. We may assume $x, y \in A_{n_0}$. Since $y$ is selfadjoint, there is a $u_0 \in \mathcal{U}(A_{n_0})$ such that $(\mathrm{Ad}\, u_0)(y)$ is diagonal in the matrix units system of $A_{n_0}$; moreover, this will hold in all $A_n$, $n \geqq n_0$.

From the Remark 4, we infer that for $n \geqq n_0$ there is a $u_n \in \mathcal{U}(A_n)$ such that

$$\mathrm{Diag}\big([(\mathrm{Ad}\, u_n)(x)]_n^l\big) = \big\{\mathrm{tr}\big([x]_n^l\big)\big\} = \{\alpha_n^l(x)\} \quad \text{for all} \quad l=1, \ldots, c_n.$$

Hence $\Delta_{A_n}\big((\mathrm{Ad}\, u_n)(x)\big)=\omega(\alpha_n(x))$. Since $\lim_{n\to\infty}\omega(\alpha_n(x))=0$ and $\tau\big((\mathrm{Ad}\, u)(x)\big)=\tau(x)$, by Remark 5, we see that

$$\big|\tau\big((\mathrm{Ad}\, u_0^* u_n)(x)y\big)-\tau(x)\tau(y)\big| =$$

$$= \big|\tau\big((\mathrm{Ad}\, u_n)(x)(\mathrm{Ad}\, u_0)(y)\big)-\tau\big((\mathrm{Ad}\, u_n)(x)\big)\tau\big((\mathrm{Ad}\, u_0)(y)\big)\big| \leqq$$

$$\leqq \|(\mathrm{Ad}\, u_0)(y)\|\, \Delta_{A_n}\big((\mathrm{Ad}\, u_n)(x)\big) = \|y\|\,\omega\big(\alpha_n(x)\big) \to 0 \quad \text{as} \quad n\to\infty.$$

So we proved the mixing property for $x, y \in (A_\infty)_h$. That it also holds for any $x, y \in A_h$ can be proved using an obvious approximation argument.

Lemma 7. (a) *With the notations of the Corollary,*

$$1/2\,\delta_{k-1}\delta_k \leqq \delta_{k-1}\tilde{\varepsilon}_k \leqq \varepsilon_k \quad (k \geqq 2);$$

*hence* (2)$\Rightarrow$(1)$\Rightarrow$($*$).

(b) *If any of* (1) *or* (2) *holds, then the algebra $A$ is simple; hence, by Theorem, part* (b), *the unique normalized trace on $A$ is faithful.*

Proof. (a) Since $m_k={}^tR_{k-1}m_{k-1}$, we see that

$$\big(\max_{i,j} r_{ij}^{k-1}\big)\sum_{l=1}^{c_{k-1}} m_l^{k-1} \geqq \sum_{l=1}^{c_{k-1}} r_{lj}^{k-1}m_l^{k-1} = m_j^k \geqq \big(\min_{i,j} r_{ij}^{k-1}\big)\sum_{l=1}^{c_{k-1}} m_l^{k-1}$$

*for any fixed* $j=1, \dots, c_k$. Hence

(4) $$\min_j m_k^j/\max_j m_k^j \geqq \min_{i,j} r_{ij}^{k-1}/\max_{i,j} r_{ij}^{k-1} = \delta_{k-1}.$$

The result can now be obtained using the following straightforward inequalities: for any nonnegative nonzero vectors $w=(w_1, \dots, w_n)$, $a=(a_1, \dots, a_n)$ we have

$$(1/2)\min_i w_i/\max_i w_i \leqq \chi(w);$$

$$\big(\min_i a_i/\max_i a_i\big)\chi(w) \leqq \chi\big((a_1w_1, a_2w_2, \dots, a_nw_n)\big).$$

The first one of these inequalities gives $(1/2)\delta_k \leqq \tilde{\varepsilon}_k$, while the second one and (4) give $\delta_{k-1}\tilde{\varepsilon}_k \leqq \varepsilon_k$.

(b) Both (1) and (2) imply that there is an infinity of $R_k$'s with no zero entries. This implies that $A$ is simple, by the same argument as that used in the proof of Lemma 5.

This concludes the proof of the Theorem.

## References

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