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Restrictions of positive self-adjoint operators

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A densely defined positive symmetric operator in a Hilbert space has a positive self-adjoint extension within the same space. This theorem is well known for a long time and forms a solid part of our knowledge of the theory of unbounded operators in Hilbert space. Hence the restrictions of positive self-adjoint operators to a dense linear subspace are completely characterized by the properties of symmetry and positiveness. The same problem for an arbitrary linear subspace has so far remained unsolved.

The main aim of this note is to give a necessary and sufficient condition for the existence of a positive self-adjoint operator whose restriction to a linear subspace of a Hilbert space is given. Our theorem contains, as a special case, the above mentioned classical result as well as its generalisation given in 1970 by ANDO and NISHIO [1, Theorem 1; Corollary 1] for closed initial operators. Our method of proof follows the proof used in 1983 by the first named author [2, Theorem] in the bounded operator case. Further properties of our extension presented here generalise the results of [3], [4], [5].

This work is a result of a visit in April 1988 of the second named author at the Eötvös University, Budapest.

Let A be a (linear) operator defined on a linear subspace $\mathscr{D}$ of a (complex) Hilbert space $\mathscr{H}$ with values in the space $\mathscr{H}$. Here $\mathscr{D}$ is not assumed to be closed or dense, nor A is assumed to have a closed graph. Throughout the paper we assume that A is symmetric and positive, that is, A has the following property:

$$0 \leqq (Ax, x) \quad \text{for each} \quad x \text{ in } \mathscr{D}. \tag{1}$$

Of course, (1) is necessary for the existence of a positive self-adjoint extension.

Starting with assumption (1) we define a semi inner product $\langle . , . \rangle$ on $\mathscr{D}$ by

$$\langle x, y \rangle := (Ax, y) \quad \text{for } x \text{ and } y \text{ in } \mathscr{D}.$$

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A new Hilbert space appears by the usual construction: let $\mathscr{D}_0=\{x\in D\colon (Ax, x)=0\}$ be the kernel of $\langle\cdot,\cdot\rangle$ and let Q be the quotient map of $\mathscr{D}$ with respect to $\mathscr{D}_0$, that is,

$$Qx = x+\mathscr{D}_0 \quad \text{for all } x \text{ in } \mathscr{D},$$

then $Q(\mathscr{D})$ is a pre-Hilbert space with inner product

$$\langle Qx, Qy\rangle := (Ax, y) \quad \text{for } x, y \text{ in } \mathscr{D}. \tag{2}$$

Now $\hat{\mathscr{H}}$ will denote the completion of $Q(\mathscr{D})$.

Assume first for a moment that x belongs to $\mathscr{D}_0$ if and only if $Ax=0$. Then the formula

$$V(Qx) := Ax \quad \text{for } x \text{ in } \mathscr{D} \tag{3}$$

defines a linear map V from $Q(\mathscr{D})$ into $\mathscr{H}$ factoring A through Q. At the same time we observe that V^* extends Q. Indeed, the identity

$$(VQx, y) = (Ax, y) = \langle Qx, Qy\rangle \quad \text{for } x \text{ and } y \text{ in } \mathscr{D} \tag{4}$$

shows that $V^*y=Qy$. If moreover we assure that $\mathscr{D}(V^*)$ is dense in $\mathscr{H}$, in other words that V^{**} exists, then (3) gives us that $V^{**}V^*$ is a self-adjoint positive extension of A. This is because the closure of V is equal to V^{**} and because V^* is a closed operator with adjoint V^{**}.

Theorem 1. *Let A be a positive linear operator defined on a linear subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$. The following two statements are equivalent:*

(i) *A has a positive self-adjoint extension $\tilde{A}$ in $\mathscr{H}$;*

(ii) *$\mathscr{D}_*:=[y\in\mathscr{H}\colon \sup\{|(Ax, y)|^2\colon x\in\mathscr{D},\ (Ax, x)\leq 1\}<\infty]$ is dense in $\mathscr{H}$.*

Proof. Assume first (i). Then the domain $\mathscr{D}(\tilde{A})$ of $\tilde{A}$ is dense in $\mathscr{H}$. Hence the inclusion $\mathscr{D}(\tilde{A})\subset\mathscr{D}_*$ proves (ii); indeed, to prove that an element y from $\mathscr{D}(\tilde{A})$ belongs to $\mathscr{D}_*$ it is enough to see that for each x from $\mathscr{D}$, $Ax=\tilde{A}x$ holds and

$$|(Ax, y)|^2 = |(\tilde{A}x, y)|^2 \leqq (\tilde{A}x, x)(\tilde{A}y, y) = (Ax, x)(\tilde{A}y, y).$$

Assume now that (ii) holds true. The operator V (see (3)) is then well defined. Indeed, if x is a vector from $\mathscr{D}$ such that $(Ax, x)=0$ then one can show that $(Ax, y)=0$ holds true for each y from $\mathscr{D}_*$. Since $\mathscr{D}_*$ is assumed to be dense in $\mathscr{H}$, we obtain $Ax=0$. Moreover the domain $\mathscr{D}(V^*)$ of V^* is just $\mathscr{D}_*$. Hence V^* is densely defined by the assumption (ii). Here we arrive at the situation mentioned before, and $V^{**}V^*$ is a positive self-adjoint extension of A. The proof of Theorem 1 is complete.

Corollary 1. *Let $A\colon \mathscr{D}\to\mathscr{H}$ be a positive linear densely defined operator. Then A has a positive self-adjoint extension in $\mathscr{H}$.*

Proof. Arguing similarly as in the proof of the implication (i)$\Rightarrow$(ii) of Theorem 1, we show that $\mathscr{D}\subset\mathscr{D}_*$. Thus the condition (ii) of Theorem 1 is satisfied. Hence (i) of Theorem 1, which is our present assertion holds true.

Corollary 2. *For the positive linear operator $A\colon \mathscr{D}\to\mathscr{H}$ the following statements are equivalent:*

(i′) *A has a continuous positive extension $\tilde{A}$ on $\mathscr{H}$;*

(ii′) *$\mathscr{D}_*=\mathscr{H}$;*

(iii′) *there exists a constant $m\geqq 0$ such that*

$$\|Ax\|^2 \leqq m(Ax, x) \quad \textit{for each } x \textit{ from } \mathscr{D}.$$

Proof. Since $\mathscr{H}=\mathscr{D}(\tilde{A})\subset\mathscr{D}_*$ holds true for each continuous positive extension $\tilde{A}$ of A, the implication (i′)$\Rightarrow$(ii′) is immediate. Notice also that $\mathscr{D}(V^*)=\mathscr{D}_*$. So if (ii′) holds true then V^* is an everywhere defined closed operator, that is, V^* is continuous indeed. Hence $V^{**}V^*$ is a continuous positive linear extension of A on $\mathscr{H}$. This proves (ii′)$\Rightarrow$(i′).

If (iii′) holds, the operator V defined by (3) is continuous. Consequently $V^{**}V^*$ is a continuous positive extension of A. Conversely, (i′) implies (iii′) with $m:=\|\tilde{A}\|$.

Corollary 3. *Let $A\colon \mathscr{D}\to\mathscr{H}$ be a positive linear operator with a positive self-adjoint extension $\tilde{A}\colon \tilde{\mathscr{D}}\to\mathscr{H}$. Then $\mathbf{A}:=V^{**}V^*$ has the following properties:*

(iv) $\mathscr{D}(\tilde{A}^{1/2})\subseteqq\mathscr{D}(\mathbf{A}^{1/2})$;

(v) $\|\mathbf{A}^{1/2}x\|^2\leqq\|\tilde{\mathbf{A}}^{1/2}x\|^2$ *for each x in* $\mathscr{D}(\tilde{A}^{1/2})$.

Proof. Starting with positive self-adjoint operator $\tilde{A}$, we can construct the subspace $\tilde{\mathscr{D}}_0$, the quotient map $\tilde{Q}$, the completion $\tilde{\mathscr{H}}$ and the operator $\tilde{V}$ factoring $\tilde{A}$ through $\tilde{Q}$ in the same way as we have obtained $\mathscr{D}_0$, Q, $\hat{\mathscr{H}}$ and V, respectively, from A. Then $\tilde{A}=\tilde{V}^{**}\tilde{V}^*$, because both of these operators are self-adjoint. As in [4], we define an isometry T from $\hat{\mathscr{H}}$ into $\tilde{\mathscr{H}}$(=the completion of $\tilde{Q}(\tilde{\mathscr{D}})$) by the following identity:

$$T(Qx) = \tilde{Q}x \quad \text{for all } x \text{ from } \mathscr{D}.$$

That T is an isometry follows from

$$\langle\tilde{Q}x, \tilde{Q}x\rangle = (\tilde{A}x, x) = (Ax, x) = \langle Qx, Qx\rangle \quad \text{for each } x \text{ in } \mathscr{D}.$$

Since, moreover,

$$(\tilde{V}T)(Qx) = \tilde{V}(TQx) = \tilde{V}\tilde{Q}x = \tilde{A}x = Ax = VQx$$

holds true for each x from $\mathscr{D}$, we conclude that

$$\tilde{V}T|_{Q(\mathscr{D})} = V.$$

Hence, using the fact that T^* is a contraction, we have that

$$\|\mathrm{A}^{1/2}x\|^2 = \|V^*x\|^2 = \|T^*\tilde{V}^*x\|^2 \leqq \|\tilde{V}^*x\|^2 = \|\tilde{A}^{1/2}x\|^2$$

holds for each x in $\mathscr{D}(\mathrm{A}^{1/2})\cap\mathscr{D}(\tilde{A}^{1/2})$. Now, since $\tilde{A}$ extends A, it follows that

$$\mathscr{D}(\tilde{V}^*) = \tilde{\mathscr{D}}_* \subset \mathscr{D}_* = \mathscr{D}(V^*),$$

and therefore

$$\mathscr{D}(\tilde{A}^{1/2}) = \mathscr{D}((\tilde{V}^{**}\tilde{V}^*)^{1/2}) = \mathscr{D}(\tilde{V}^*) \subset \mathscr{D}(\tilde{V}^*) = \mathscr{D}((V^{**}V^*)^{1/2}) = \mathscr{D}(\mathrm{A}^{1/2}).$$

This completes the proof.

Corollary 4. *Let $A\colon \mathscr{D}\to\mathscr{H}$ be a linear operator bounded below by m, that is, such that*

$$m\|x\|^2 \leqq (Ax, x) \quad \textit{holds for all } x \textit{ in } \mathscr{D}.$$

A admits a self-adjoint extension with the same bound if and only if the subspace

$$[y\in\mathscr{H}\colon \sup\{|(Ax-mx, y)|^2\colon x\in\mathscr{D}, (Ax, x) \leqq 1+m\|x\|^2\} < \infty]$$

is dense in $\mathscr{H}$.

Proof. Since for each self-adjoint extension $\tilde{A}$ of A with a bound m, $\tilde{A}-mI$ is a positive self-adjoint extension of the positive (symmetric) operator $A-mI$, the conclusion of Corollary 4 follows from Theorem 1.

Corollary 5. *Any densely defined semibounded linear operator in Hilbert space has a self-adjoint extension with the same bound.*

Proof. Corollary 5 follows from Corollary 4 via arguments used in the proof of Corollary 1.

An extension of [5, Theorem] is the following

Theorem 2. *Let $A\colon \mathscr{D}\to\mathscr{H}$ be a positive linear operator with a positive self-adjoint extension $\tilde{A}$. Let B and C be continuous linear operators on $\mathscr{H}$ leaving $\mathscr{D}$ invariant and such that*

(vi) $ABx = C^*Ax$, $ACx = B^*Ax$ *for all x in $\mathscr{D}$.*

Then, with $\mathrm{A}=V^{**}V^*$ in Theorem 1, we have

(vii) $\mathrm{A}Bx = C^*\mathrm{A}x$, $\mathrm{A}Cx = B^*\mathrm{A}x$ *for all x in $\mathscr{D}(\mathrm{A})$.*

Proof. We define, as in the proof of [5], continuous linear operators $\hat{B}$ and $\hat{C}$ on $Q(\mathscr{D})$ as follows

$$\hat{B}(Qx) = Q(Bx),\ \hat{C}(Qx) = Q(Cx) \quad \text{for each } x \text{ in } \mathscr{D}. \tag{5}$$

To show that $\hat{B}$ and $\hat{C}$ are well-defined and continuous we find estimates for the norm of $\hat{B}(Qx)$ and $\hat{C}(Qx)$ step by step. First we have for any x in $\mathcal{D}$ that

$$\langle \hat{B}(Qx), \hat{B}(Qx)\rangle = (ABx, Bx) = (C^*Ax, Bx) = (Ax, CBx) = \langle Qx, Q(CBx)\rangle \leqq$$
$$\leqq \langle Qx, Qx\rangle^{1/2} \langle Q(CBx), Q(CBx)\rangle^{1/2}.$$

Repeating this argument we obtain

$$\langle \hat{B}(Qx), \hat{B}(Qx)\rangle \leqq \langle Qx, Qx\rangle^{1/2+\dots+1/2^n} \langle Q(CB)^{2^{n-1}}x, Q(CB)^{2^{n-1}}x\rangle^{1/2^2} =$$
$$= \langle Qx, Qx\rangle^{1-1/2^n} (Ax, (CB)^{2^n}x)^{1/2^n} \leqq$$
$$\leqq \langle Qx, Qx\rangle^{1-1/2^n} \|Ax\|^{1/2^n} \|(CB)^{2^n}\|^{1/2^n} \|x\|^{1/2^n}.$$

Passing with n to infinity we get

(6) $$\langle \hat{B}(Qx), \hat{B}(Qx)\rangle \leqq r(CB)\langle Qx, Qx\rangle \quad \text{for each } x \text{ from } \mathcal{D},$$

where $r(CB)(\leqq\|CB\|)$ stands for the spectral radius of CB. (6) tells us that $\hat{B}$ is a well-defined continuous linear operator. $\hat{B}$ has norm not exceeding $r(CB)^{1/2}$. A similar argument applies to show that $\hat{C}$ is also continuous and its norm does not exceed the same value $r(BC)^{1/2}=r(CB)^{1/2}$. Thus both $\hat{B}$ and $\hat{C}$ have unique continuous extensions on $\hat{\mathcal{H}}$ which we also denote by $\hat{B}$ and $\hat{C}$, respectively, as this causes no confusion.

Now we see that $\hat{B}$ and $\hat{C}^*$, hence also $\hat{C}$ and $\hat{B}^*$, coincide since on $Q(\mathcal{D})$ they agree:

$$\langle Qx, \hat{C}^*(Qy)\rangle = \langle \hat{C}(Qx), Qy\rangle = \langle Q(Cx), Qy\rangle = (ACx, y) = (Ax, By) =$$
$$= \langle Qx, B(Qy)\rangle$$

holds true for each x and y in $\mathcal{D}$. On the other hand V interwines $\hat{B}$ and C^* (respectively $\hat{C}$ and B^*). Indeed, if x belongs to $\mathcal{D}$ then

$$V\hat{B}(Qx) = VQ(Bx) = A(Bx) = C^*Ax = C^*V(Qx),$$
$$V\hat{C}(Qx) = VQ(Cx) = A(Cx) = B^*Ax = B^*V(Qx).$$

Hence $C^*V\subset V\hat{B}$ and $B^*V\subset V\hat{C}$. Since C^* is bounded, we get

$$\hat{C}V^* = \hat{B}^*V^* \subset (V\hat{B})^* \subset (C^*V)^* = V^*C.$$

Similar argument shows that $\hat{B}V^*\subset V^*B$. Thus

(viii) $V^*By = \hat{B}V^*y$, $V^*Cy=\hat{C}V^*y$ for every y from $\mathcal{D}_*$.

Returning to the proof of (vii) we see that for each $x\in\mathcal{D}(\mathbf{A})$ and for each $y\in\mathcal{D}_*$ the following identities hold true (using (viii))

$$\langle V^*Bx, V^*y\rangle = \langle \hat{B}V^*x, V^*y\rangle = \langle V^*x, \hat{B}^*V^*y\rangle = \langle V^*x, \hat{C}V^*y\rangle =$$
$$= \langle V^*x, V^*(Cy)\rangle = (C^*V^{**}V^*x, y) = (C^*\mathbf{A}x, y).$$

As a consequence we have that, for each x from $\mathscr{D}(\mathrm{A})$, V^*Bx belongs to $\mathscr{D}(V^{**})$ and at the same time

$$C^*\mathrm{A}x = V^{**}V^*Bx = \mathrm{A}Bx.$$

The other equality of (vii) can be shown similarly. This completes the proof.

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