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On the local spectral radius of a nonnegative element with respect to an irreducible operator

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1. Introduction

The local spectral radius of a nonnegative element of a partially ordered Banach space with respect to a general positive linear continuous operator has been studied in [2]. The main results there gave, among others, sufficient conditions that the local spectral radius be a singularity of the local resolvent function, characterized the distinguished eigenvalues outside the essential spectrum, and sought positive solutions u of the equation $(\lambda - T)u = x$ for positive λ and positive x.

If T is a reducible positive operator, then we may, in general, clearly find nonnegative elements x of the space E such that the local spectral radius $r_T(x)$ of x with respect to T is strictly smaller than the (global) spectral radius $r(T)$ of T. The situation is more delicate, if the operator T is irreducible. The first main result of this paper, Theorem 7, lists four groups of fairly natural conditions, each of which is sufficient for any nonzero x in the positive cone E_+ to ensure that $r_T(x) = r(T)$, assuming T is irreducible. The preceding Propositions 1 through 5 and Remark 6 formulate some more general conditions ensuring $r_T(x) = r(T)$ even if T is reducible, whereas Example 8 shows that the irreducibility of T alone is not sufficient.

The second main result, Theorem 12, yields three groups of conditions, each of which guarantees that the equation $(r(T) - T)u = x$ has no solution u in all of E, assuming that T is irreducible and $x \in E_+ \setminus \{0\}$. The preliminary results contain also here more general conditions. Several examples illustrate the irredundancy of some conditions, or that some other group of conditions is not sufficient.

In the third part of the results we show that if T is irreducible and $r(T) > 0$ is a pole of its resolvent, then some conditions ensure that the equation $(r(T) - T)u =$

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$=(1-P)x$, where P denotes the spectral projection corresponding to the set $\{r(T)\}$, has a positive solution u for all x in E_+. It is also shown that some extra conditions are really needed to ensure the existence of a positive solution u. Further, we show that the algebraic eigenspace to the spectral radius of a compact, nonnegative operator need not have a basis of nonnegative elements, and discuss some connections to works of U. G. ROTHBLUM [8], H. D. VICTORY, JR. [11] and J. KÖLSCHE [5].

2. Preliminaries and notations

Let E be a real Banach space and let T be a linear continuous operator from E into E. By $N(T)$ and $R(T)$ we denote the kernel and the range of T, respectively. As usual ([9], p. 261]), we sometimes identify T with its complex extension $\bar{T}$. In this spirit, e.g., for x in E we define

$$r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n},$$

$$\Omega_T(x) = \{\lambda\in \mathbb{C} \mid r_T(x) < |\lambda|\}$$

and

$$x_T\colon \Omega_T(x) \to E \quad \text{with} \quad x_T(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1} T^k x.$$

We call $r_T(x)$ the local spectral radius of the element x with respect to the operator T, x_T the local resolvent function of the element x with respect to the operator T in its main component $\Omega_T(x)$. Of course $(\lambda-T)x_T(\lambda)=x$ for all $\lambda\in\Omega_T(x)$. We recall some results from [2] which will be used several times in this paper.

Unless explicitely stated otherwise, in the following E will always denote a partially ordered real Banach space with positive cone E_+, and T is a nonnegative operator in E. If $x\neq 0$ is a nonnegative element in E, then

(I) *$r_T(x)$ is a singularity of x_T if E_+ is normal or there is a pole μ of x_T with $|\mu|=r_T(x)$*; see [2, Theorems 6 and 10].

(II) *If E_+ is normal and there exist a $u\in E_+$ and a $\mu\geqq 0$ such that $(\mu-T)u=x$, then $r_T(x)\leqq\mu$*; see [2, Theorem 6].

(III) *If $r_T(x)$ is a pole of x_T, then there exist a $u\in E_+$ and a $\mu>0$ with $(\mu-T)u=x$ if and only if $r_T(x)<\mu$*; see [2, Theorem 10].

The proof for the last two assertions depends essentially on the following inequality: If $u\geqq 0$ and $\mu\geqq 0$ such that $(\mu-T)u=x$ then

$$0 \leqq \frac{(-1)^n}{n!} x_T^{(n)}(\lambda) \leqq \frac{u}{(\lambda-\mu)^n}$$

for all $n=0, 1, 2, \ldots$ and all $\lambda>\max\{\mu, r_T(x)\}$. This inequality was proved in [2, Proposition 5] with the help of the iterated local resolvent; we give here a very simple proof. From $u \geqq 0$ it follows that $\frac{(-n)^n}{n!} u_T^{(n)}(\lambda) = \sum_{k=0}^{\infty} \binom{k+n}{n} \lambda^{-k-n-1} T^k u \geqq 0$ for all $n=0, 1, 2, \ldots$ and all $\lambda > r_T(u)$. From $(\mu - T)u = x$ it follows that $r_T(x) \leqq r_T(u) \leqq \max\{\mu, r_T(x)\}$ and

$$u_T(\lambda) = -\frac{x_T(\lambda) - u}{\lambda - \mu} \quad \text{for} \quad \lambda > \max\{\mu, r_T(x)\}.$$

Differentiating this equality n times and multiplying by $\frac{(-1)^n}{n!}$ we get for $\lambda > \max\{\mu, r_T(x)\}$

$$0 \leqq \frac{(-1)^n}{n!} u_T^{(n)}(\lambda) = -\sum_{j=0}^{n} \frac{(-1)^j}{j!} \frac{x_T^{(j)}(\lambda)}{(\lambda-\mu)^{n-j+1}} + \frac{u}{(\lambda-\mu)^{n+1}} \leqq$$

$$\leqq -\frac{(-1)^n}{n!} \frac{x_T^{(n)}(\lambda)}{\lambda - \mu} + \frac{u}{(\lambda-\mu)^{n+1}}$$

since each summand in the sum is nonnegative, because $(-1)^j x_T^{(j)}(\lambda) \geqq 0$ if $x \geqq 0$ and $\lambda > r_T(x)$. The last inequality is equivalent to the wanted inequality.

3. Results and proofs

Proposition 1. *Let the spectral radius $r(T)$ be a pole of the resolvent $R(\cdot, T)$ of T, and let x be a quasi-interior point in the sense of* [9, p. 241] *of the positive cone E_+. Then $r_T(x) = r(T)$.*

Proof. Let p denote the order of the pole $r = r(T)$, and let $\sum_{k=-p}^{\infty} (\lambda - r)^k Q_k$ be the Laurent expansion of $R(\lambda, T)$ around r. It is well-known that $Q_{-p} \geqq 0$. Assume that $r_T(x) < r(T)$. Then $Q_{-p}x = 0$ and, since x is quasi-interior, we obtain that $Q_{-p} = 0$, a contradiction.

A slightly stronger condition on the spectral radius than in the next proposition was used in [10, Lemma 4] for similar purposes.

Proposition 2. *Let E be a Banach lattice. Let $r(T)$ be a limit point of the set $]-\infty, r(T)[\cap \varrho(T)$, and let x be a quasi-interior point of E_+. Then $r_T(x) = r(T)$.*

Proof. Let $\lambda_0 > r_T(x)$ and $z = x_T(\lambda_0)$. Let E_z denote the principal ideal generated by z. It is well-known that E_z with the cone $E_0 = E_+ \cap E_z$ is an (AM)-space with respect to the norm $\|y\|_z = \inf\{\alpha \in \mathbf{R}_+ : |y| \leqq \alpha z\}$.

The restriction T_0 of T to E_z satisfies

$$T_0 z = T x_T(\lambda_0) = \lambda_0 x_T(\lambda_0) - x \leqq \lambda_0 z.$$

Hence the z-norm of T_0 satisfies $\|T_0\|_z \leqq \lambda_0$, and for the corresponding spectral radius we have $r(T_0) \leqq \lambda_0$.

Assume $r_T(x) < r(T)$, and let $r_T(x) < \lambda_0 < r(T)$. Then $z = x_T(\lambda_0) = \sum_{n=0}^{\infty} \lambda_0^{-n-1} T^n x$ is also a quasi-interior point of E_+, hence the ideal E_z above is dense in the topology of E. By assumption, there exists $\mu \in \varrho(T)$ such that $\lambda_0 < \mu < r(T)$. The operator T_0 above is celarly positive with respect to the cone E_0 and $r(T_0) \leqq \lambda_0$, hence the resolvent $(\mu - T_0)^{-1}$, acting in E_z, is also positive with respect to E_0. Since E is a Banach lattice and E_z is dense in E, the closure of E_0 in the topology of E is E_+. Hence the resolvent $(\mu - T)^{-1}$, acting in E, is also positive with respect to E_+. However, this contradicts $\mu < r(T)$ and [9, App. 2.3, p. 263].

The next result is contained in [7, Theorem 9.1], and can be stated in our terminology as follows.

Proposition 3. *If x is an interior point of the normal cone E_+, then $r_T(x) = r(T)$.*

In fact, a bit more is proved in [7]: under the given conditions we have $r(T) = \lim_{n \to \infty} \|T^n x\|^{1/n}$ (which is clearly equal to $r_T(x)$).

The conditions in the next two propositions were used in [6, Theorem 16.2] for other purposes.

Proposition 4. *Let the cone E_+ be normal and generating, and the E_+-positive operator T be bounded from above by the element v in E_+. If x is a quasi-interior point of E_+, then $r_T(x) = r(T)$.*

Proof. Let $\lambda > r_T(x)$. Then $x_T(\lambda)$ is also a quasi-interior point of E_+, further $r_T(x_T(\lambda)) = r_T(x)$. We have $(\lambda - T) x_T(\lambda) = x \geqq 0$, therefore $T x_T(\lambda) \leqq \lambda x_T(\lambda)$. By assumption and [6, Theorem 16.2], $r(T) \leqq \lambda$ for any $\lambda > r_T(x)$. Hence $r(T) \leqq r_T(x)$, whereas the converse inequality always holds.

Proposition 5. *Let the cone E_+ be normal and generating, $x \in E_+ \setminus \{0\}$, and the E_+-positive operator T be bounded from above by the element x. Then $r_T(x) = r(T)$.*

Proof. Let $u \in E_+$. There is a positive number $\beta = \beta(u)$ such that $Tu \leqq \beta x$. Hence $T^n u \leqq \beta T^{n-1} x$ for every $n = 1, 2, \ldots$. Since E_+ is normal, there is $\gamma \in \mathbf{R}_+$ such that $\|T^n u\| \leqq \gamma \beta \|T^{n-1} x\|$ for all n. Therefore $r_T(u) \leqq r_T(x)$ for every u in E_+. Since E_+ is generating, we have by [2, Lemma 4]

$$r(T) = \max \{r_T(u) \colon u \in E_+\} \leqq r_T(x) \leqq r(T).$$

Remark 6. Assume that T is a positive continuous linear operator acting in the partially ordered Banach space E, the continuous operator A, acting in E, commutes with T and the real number λ satisfies $|\lambda|>r_T(x)$ for some x in E (A and x need not be nonnegative). Then $r_T(Ax)\leqq r_T(x)$ and $r_T(x_T(\lambda))=r_T(x)$. Therefore the assertions of Propositions 1 through 5 remain valid (i.e. $r_T(x)=r(T)$) if instead of x the element Ax or the element $Ax_T(\lambda)$ satisfies (together with T and E) the respective assumptions. The proofs are slight modifications of those given above, thus they will be omitted.

Note further that if T is irreducible, $\lambda>r(T)$ and $x\in E_+\setminus\{0\}$, then $R(\lambda, T)$ commutes with T and $TR(\lambda, T)x$ is a quasi-interior point in E_+. Hence the following theorem is a simple corollary to Propositions 1 through 4 and the remarks above (note that the condition in Proposition 5 is of different, i.e. of more individual, nature).

Theorem 7. *Assume that the irreducible positive continuous linear operator T acting in the partially ordered Banach space E and E satisfy one of the following conditions:*

(i) *$r(T)$ is a pole of the resolvent $R(\cdot, T)$,*

(ii) *$r(T)$ is a limit point of the set $]-\infty, r(T)[\cap\varrho(T)$, and E is a Banach lattice,*

(iii) *the cone E_+ is normal and solid* (i.e. *has a nonvoid interior*),

(iv) *T is bounded from above by an element v in the normal and generating cone E_+.*

Then for any x in $E_+\setminus\{0\}$ we have $r_T(x)=r(T)$.

The following example will show that the irreducibility of T alone does not guarantee that $r_T(x)=r(T)$ for every x in $E_+\setminus\{0\}$.

Example 8. Let E be the real sequence space l^p ($1\leqq p<\infty$) or c_0 with the usual cone $E_+=\{x=(x_i)_{i=1}^\infty\in E\colon\ x_i\geqq 0$ for $i=1,2,\dots\}$. Then x is quasi-interior in E_+ if and only if every $x_i>0$. We shall denote this by $x\gg 0$. Let S be the left shift in E defined by $(Sx)_i=x_{i+1}$ ($i=1,2,\dots$). Let $f\in E'$ act as $fx=x_1$, and let $a=(a_i)\in E$, $a\gg 0$. Define $T\colon E\to E$ by $T=f\otimes a+S$, i.e.

$$(Tx)_i = a_i x_1 + x_{i+1} \quad (i = 1, 2, \dots).$$

T is then a positive irreducible operator. Indeed, for each x in $E_+\setminus\{0\}$ take $k=\min\{i\colon x_i>0\}$. Then $(T^j x)_i=x_{i+j}$ ($1\leqq j<k$), $(T^k x)_i=a_i x_k+x_{i+k}\geqq a_i x_k>0$ for every $i=1,2,\dots$. Thus $T^k x\gg 0$, hence T is irreducible.

It is well known that the Fredholm domain of S is the complement of the unit circle. Since T is a one-dimensional perturbation of S, their essential spectra are identical (cf. [4, Theorem IV. 5.35]). Hence $r(T)\geqq 1$.

Now let $0<q<\frac{1}{2}$, and consider the particular case of the operator T when $a=(q^i)_{i=1}^\infty$. For any $x\in E$ and $\lambda\in\mathbf{R}$ the equality $Tx=\lambda x$ is equivalent to

$$q^i x_1 + x_{i+1} = \lambda x_i \quad (i = 1, 2, \dots).$$

This holds for $\lambda \neq q$ if and only if $(x_i) \in E$, where

$$x_i = \left(\lambda^{i-1} - \sum_{k=1}^{i-1} q^k \lambda^{i-1-k}\right) x_1 = \left(\lambda^{i-1} \frac{\lambda - 2q}{\lambda - q} + \frac{q^i}{\lambda - q}\right) x_1 \quad (i = 2, 3, \ldots).$$

Let λ satisfy $2q \leqq \lambda \leqq 1$, and let $x_1 > 0$. Then $x = (x_i) \in E$, and $x \gg 0$ is an eigenvector corresponding to the eigenvalue $\lambda < 1 \leqq r(T)$. Hence $r_T(x) = \lambda < r(T)$ as stated.

Proposition 9. *If the positive cone E_+, the element x in E_+ and the positive operator T satisfy one of the conditions in Propositions* 1, 3, 4 *or* 5, *further in the cases of Propositions* 4 *or* 5 *we have, in addition, $r(T) > 0$, then the equation $(r(T) - T)u = x$ has no solution u in E.*

Proof. (P) will denote that we are considering the case when the conditions of Proposition P $(P = 1, 3, 4, 5)$ are satisfied.

(1) Assume that there is a solution u in E, and that $R(\lambda, T) = \sum_{k=-p}^{\infty} (\lambda - r)^k Q_k$ is the Laurent expansion of the resolvent around the pole $r = r(T)$ of exact order $p \geqq 1$. Then $Q_{-p} \geqq 0$, $Q_{-p} \neq 0$, and $Q_{-k} = (T - r)^{k-1} Q_{-1}$ for $k = 1, 2, \ldots, p$. By assumption, $Q_{-p} x = (r - T) Q_{-p} u = -Q_{-p-1} u = 0$. Since x is quasi-interior, we obtain $Q_{-p} = 0$, a contradiction.

(3) Since the cone E_+ is normal and solid, a result of M. Krein and M. Rutman (cf. [9, p. 267]) shows that $r(T)$ is an eigenvalue of the dual T' with corresponding eigenvector $f \neq 0$ in the dual cone E'_+. Should a solution $u \in E$ exist, then we should have (denoting the dual pairing by $\langle \cdot, \cdot \rangle$) $\langle f, x \rangle = \langle (r(T) - T') f, u \rangle = 0$, and this contradicts the fact that $f \in E'_+$, $f \neq 0$, and x is an interior point in E_+.

(4) Since $r(T) > 0$, our assumptions imply that $r(T)$ is an eigenvalue of the dual T' with eigenvector f in the dual cone (cf. [7, Proof of Theorem 5.5]). The rest as in case (3).

(5) Let E_x denote the linear manifold of x-measurable elements y of E (cf. [6, p. 34], [7, p. 80]), i.e. those satisfying $-\alpha x \leqq y \leqq \alpha x$ for some $\alpha \in \mathbf{R}_+$. If we set $\|y\|_x = \inf \{\alpha \in \mathbf{R}_+ : -\alpha x \leqq y \leqq \alpha x\}$ then, since E_+ is a normal cone, E_x is a Banach space with respect to the norm $\|\cdot\|_x$, and $E_+ \cap E_x$ is a closed solid normal cone in E_x. Now E_+ is generating and T is bounded from above by x, therefore $R(T) \subset E_x$ and E_x is invariant under T. Assume that there is a solution u, then we obtain from $r(T)u = Tu + x$ and $r(T) > 0$ that $u \in E_x$. It is fairly straightforward to show (cf. [5, p. 80]) that the spectral radii of the operator T in E and in E_x are identical, so we come to the situation of case (3) in the space E_x, and we reach a contradiction.

Corollary 10. *If one of the conditions in Proposition* 9 *is fulfilled, $\lambda \in \mathbf{R}$, and the equation $(\lambda - T)u = x$ has a solution $u \geqq 0$, then $\lambda > r(T)$.*

Proof. By the preceding results, we have $r_T(x)=r(T)$, and for $\lambda=r_T(x)$ there is no solution u in all of E. On the other hand, [2, Theorems 6 and 10] show that there is no solution $u\in E_+$ under the given conditions if $0\leqq\lambda<r_T(x)$. It is clear that there is no solution u in E_+ for $\lambda\in\mathbf{R}\setminus\mathbf{R}_+$. Hence $\lambda>r(T)$.

Remark 11. If the operator A commutes with T, and the element $x=Az$ satisfies (together with T and E) the conditions of Proposition 9, then the equation $(r(T)-T)u=z$ has no solution u in all of E. Indeed, assuming the contrary, the element Au would satisfy $(r(T)-T)Au=x$, which is impossible. The case $A=-$ identity operator is of interest in the next theorem.

Theorem 12. *Let the positive operator T in E be irreducible, satisfy together with E one of the conditions* (i), (iii) *or* (iv) *of Theorem* 7, *in the last case let* $r(T)>0$, *and let* $z\in E_+\cup(-E_+)$ *and* $z\neq 0$. *Then the equation* $(r(T)-T)u=z$ *has no solution* u *in all of* E.

Proof. Let $\lambda>r(T)$ and $A=TR(\lambda,T)$. Then $x=Az=TR(\lambda,T)z$ if $z\in E_+\setminus\setminus\{0\}$ and $x=-Az=-TR(\lambda,T)z$ if $z\in(-E_+)\setminus\{0\}$ is a quasi-interior element of the cone E_+, since T is irreducible. Hence x satisfies conditions (1), (3), or (4) in (see the proof!) Proposition 9, and Remark 11 shows that there is no solution u in E to the equation $(r(T)-T)u=z$.

Remark 13. Much stronger conditions on T and E are imposed in [1; Theorem 1.13] to obtain the assertion of Theorem 12.

It is clear that the assertions of Proposition 9 or Theorem 12 are not valid without extra conditions such as (1), (3), (4) or (5) and (i), (iii) or (iv), respectively. This is shown by Example 8, where T is irreducible and there are quasi-interior elements x in E_+ such that $r_T(x)<r(T)$. Then the element $u=x_T(r(T))$ belongs to E_+ by [2; Lemma 4], and satisfies $(r(T)-T)u=x$.

If V is the Volterra operator defined by $(Vx)(t)=\int_0^t x(s)ds$ for $x\in L^2(0,1)$, then V clearly satisfies condition (4) of Proposition 9 except that we have $r(V)=0$. The elements $u(t)\equiv-1$ and $x(t)\equiv t$ satisfy here $(r(V)-V)u=x$, and x is quasi-interior point in the (usual) cone E_+. Hence the requirement of the positivity of the spectral radius in Proposition 9 is not redundant.

The following example shows that the conditions in Proposition 2 are not sufficient to ensure that $(r(T)-T)u=x$ has no solution u in E for any x in E_+.

Example 14. Let $X=\bigcup_{n=0}^{\infty}[2n,2n+1]\subset\mathbf{R}$ and let $E=C_0(X)$ with the usual positive cone E_+. Let T be the operator of multiplication by $f(t)=(1+t)^{-1}t$ in E. Then $r(T)=1$, and $[(1-T)u](t)=(1+t)^{-1}u(t)$. If $x(t)\equiv(1+t)^{-1}e^{-t}$ then x is quasi-interior in E_+, and the studied equation has the solution $u(t)=e^{-t}$. The element u is quasi-interior in E_+, and the spectrum of the operator T, i.e. the set $\overline{f(X)}\subset\mathbf{R}$, clearly satisfies the condition in Proposition 2.

The next example will show that the series for the main component of the local resolvent function can converge at $r=r_T(x)$ for an E_+-positive operator T and a quasi-interior point x in E_+. Its sum $u=\sum_{n=0}^{\infty} r^{-n-1}T^n x$ is then a positive solution of the equation $(r-T)u=x$.

Example 15. Let $E=c_0$ with the usual positive cone E_+, let T be the left shift in E, and let $x=(1/n^2)_{n=1}^{\infty}$. Then $\|T^k x\|=(k+1)^{-2}$, hence $r_T(x)=1$. Further, the sum $u=\sum_{n=0}^{\infty} T^n x$ exists in E and its j-th component u_j is $\sum_{n=j}^{\infty} n^{-2}$. The solution u of $(r-T)u=x$ is a quasi-interior point of E_+.

Let $T\geqq 0$ be irreducible, and let $r=r(T)>0$ be a pole of the resolvent $R(\cdot, T)$. Then r is a pole of order one ([9], App. 3.2]). Therefore the residuum of $R(\cdot, T)$ at r is the projection P of E on $N(r-T)$ along $R(r-T)$, hence the equation $(r-T)v=(1-P)x$ has solutions v for all $x\in E$.

Proposition 16. *Let $T\geqq 0$ be irreducible, let $r=r(T)>0$ be a pole of its resolvent and let P be the residuum of $R(\cdot, T)$ at r. If E_+ contains interior points, or else T is finite dimensional, then the equation $(r-T)u=(1-P)x$ has solutions $u\geqq 0$ for all $x\in E$ in the first case, and for all $x\geqq 0$ in the second one.*

Proof. $N(r-T)$ is one-dimensional and generated by a quasi-interior element u_0 of E_+ ([9, App. 3.2]). Let v be a solution of $(r-T)v=(1-P)x$, then $(r-T)\cdot(v+\lambda u_0)=(1-P)x$ for all λ. If E_+ has interior elements, then u_0 is such. In this case x can be an arbitrary element of E, and we can choose λ such that $v+\lambda u_0$ is an interior point of E_+.

Consider now the second case, and let $x\geqq 0$. There exists a μ with $Px=\mu u_0$. Then we have

$$v+\lambda u_0=\frac{1}{r}[x+Tv+(\lambda r-\mu)u_0] \quad \text{for all } \lambda.$$

Now we prove that there exists a λ such that $Tv+(\lambda r-\mu)u_0\geqq 0$. Then $v+\lambda u_0\geqq 0$, since $x\geqq 0$. Let $R_0=\bigcup_{k\in\mathbf{N}}\{z\in R(T)\colon -ku_0\leqq z\leqq ku_0\}$. Then R_0 is a linear subspace which is dense in $R(T)$; this follows from $ru_0=Tu_0\in R(T)$ and the fact that $E_0=\bigcup_{k\in\mathbf{N}}\{y\in E\colon -ku_0\leqq y\leqq ku_0\}$ is T-invariant, and is dense in E, since u_0 is a quasi-interior element of E_+. Since T is finite dimensional (i.e. $\dim R(T)<\infty$), we have $R_0=R(T)$ and we can find a λ such that $Tv+(\lambda r-\mu)u_0\geqq 0$.

The question naturally arises whether the conditions in Proposition 16 are redundant. We now give an example of a compact, irreducible operator T such that $r=r(T)>0$, and the equation $(r-T)u=(1-P)x$ has solutions $u\geqq 0$ for some $x\geqq 0$, $x\neq 0$, and has no solution $u\geqq 0$ for other $x\geqq 0$, $x\neq 0$. A consequence of this example will be discussed at the end of this paper.

Example 17. Let $E=c_0$ or $E=l^p$ $(1\leqq p<\infty)$ with the cone E_+ of nonnegative sequences in E, $a=(a_i)\in E'$ (here we identify E' with the corresponding sequence space), and $b=(b_i)\in c_0$. We consider the operator

$$T = a\otimes e^1+SM_b,$$

where e^k is the sequence with 1 in the kth position and 0 in the others, S is the right shift and M_b is the operator of multiplication by b. We have for $x=(x_i)\in E$

$$(Tx)_i = \begin{cases} \sum_{j=1}^{\infty} a_j x_j & \text{if } i = 1, \\ b_{i-1}x_{i-1} & \text{if } i > 1. \end{cases}$$

It is well known that M_b is compact and that the weighted shift SM_b is compact and quasinilpotent [3, Problem 80 for $E=l^2$]. Therefore T, being a one-dimensional perturbation of SM_b, is compact.

Clearly T is non-negative if and only if $a\geqq 0$ and $b\geqq 0$. T is irreducible if $a\gg 0$ and $b\gg 0$, i.e. $a_i>0$ and $b_i>0$ for all i; this follows from

$$(Tx)_1 = \sum_{j=1}^{\infty} a_j x_j, \quad (T^n x)_n = b_{n-1}\cdot\ldots\cdot b_1 (Tx)_1 \quad \text{for} \quad n\geqq 2.$$

Let $\lambda\neq 0$ be an eigenvalue of T and $v=(v_i)$ be a corresponding eigenvector $\neq 0$; this is equivalent to

$$a_1\lambda^{-1}+a_2 b_1\lambda^{-2}+\ldots+a_i b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i}+\ldots = 1$$

and

$$v_i = b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i+1}v_1 \quad \text{if} \quad i\geqq 1,$$

here and in what follows we put $b_{i-1}\cdot\ldots\cdot b_1=1$ if $i=1$. Since $b\in c_0$, we have $(b_{i-1}\cdot\ldots\cdot b_1\lambda^{-i+1}v_1)\in E$ for all $\lambda\neq 0$ and all v_1, and the power series

$$f(\mu) = \sum_{i=1}^{\infty} a_i b_{i-1}\cdot\ldots\cdot b_1\mu^i$$

converges for all μ. Therefore $\lambda\neq 0$ is an eigenvalue of T if and only if $f(1/\lambda)=1$.

Let us assume that $a\gg 0$ and $b\gg 0$. Then $f(1/\lambda)$ is strictly decreasing for $\lambda>0$, $\lim_{\lambda\to 0} f(1/\lambda)=\infty$ and $\lim_{\lambda\to\infty} f(1/\lambda)=0$. Thus there exists exactly one $r>0$ with $f(1/r)=1$. This r is the spectral radius of T, by the Krein—Rutman Theorem, and is a pole of multiplicity one of $R(\cdot, T)$, since T is irreducible and compact ([9, App.

3.2]). Let P be, as in Proposition 16, the residuum of $R(\cdot, T)$ at r. Then P is a projection on the subspace spanned by $\hat{v}=(b_{i-1}\cdot\ldots\cdot b_1 r^{-i+1})_{i=1}^{\infty}$. If $x=(x_i)$, $u=(u_i)$ and $(r-T)u=(1-P)x$, then for $i\geqq 2$

$$u_i = b_{i-1}\cdot\ldots\cdot b_i r^{-i+1}u_1-(i-1)b_{i-1}\cdot\ldots\cdot b_1 r^{-i}\hat{b}_0+$$
$$+r^{-1}x_i+b_{i-1}r^{-2}x_{i-1}+\ldots+b_{i-1}\cdot\ldots\cdot b_2 r^{-i+1}x_2$$

where $\hat{b}_0$ is uniquely determined by $Px=\hat{b}_0\hat{v}$. If $x_2>0$, but $x_i=0$ for $i\neq 2$, then $x\geqq 0$, $x\neq 0$. Therefore $\hat{b}_0>0$, and

$$u_i = b_{i-1}\cdot\ldots\cdot b_2 r^{-i}[rb_1u_1+rx_2-(i-1)b_1\hat{b}_0] \quad \text{if} \quad i\geqq 2.$$

Clearly, it is not possible to choose u_1 in such a way that u_i is non-negative for all i. Therefore the equation $(r-T)u=(1-P)e^2$ has no solution $u\geqq 0$. Nearly the same argument proves that $(r-T)u=(1-P)x$ has no solution $u\geqq 0$ if x is a "finite sequence", $x\geqq 0$, $x\neq 0$.

On the other hand, if we take x such that $x_1=0$ and

$$x_i = b_{i-1}\cdot\ldots\cdot b_1 r^{-i+1}x_0 \quad \text{if} \quad i>1$$

where $x_0>0$, then $x\geqq 0$, $x\neq 0$, and

$$u_i = b_{i-1}\cdot\ldots\cdot b_1 r^{-i}[ru_1+(i-1)(x_0-\hat{b}_0)] \quad \text{if} \quad i\geqq 1.$$

We show that $x_0>\hat{b}_0$ in this case. There exist solutions u of $(r-T)u=(1-P)x$; for the first coordinate in this equation we get using $f(1/r)=1$ and u_i as above,

$$-\sum_{i=2}^{\infty} a_i b_{i-1}\cdot\ldots\cdot b_1 r^{-i}(i-1)(x_0-\hat{b}_0) = x_1-\hat{b}_0 = -\hat{b}_0,$$

and this implies $\hat{b}_0<x_0$. Therefore, for these special $x\in E$ we have nonnegative solutions u of the equation $(r-T)u=(1-P)x$, if we choose a solution with $u_1\geqq 0$.

This example can also be used to show that the algebraic (or generalized) eigenspace to the spectral radius of a compact, non-negative operator need not have a basis of non-negative elements.

Example 18. Let $E=l^p\times l^p$ $(1\leqq p<\infty)$ and

$$T = \begin{pmatrix} T_1 & S_1 \\ 0 & T_1 \end{pmatrix}$$

where T_1 is the operator of the last example and S_1 is a compact, non-negative, nonzero operator in l^p. E is an order continuous Banach lattice, T is compact and non-negative, and $r=r(T)=r(T_1)>0$ is a pole of order 2 of $R(\cdot, T)$. Let $x=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\in E$, then $(r-T)^2x=0$ is equivalent to

$$(*) \qquad (r-T_1)^2x_1 = [(r-T_1)S_1+S_1(r-T_1)]x_2 \quad \text{and} \quad (r-T_1)^2x_2 = 0.$$

Since T_1 is irreducible and compact, $r=r(T_1)$ is a pole of order 1 of $R(\cdot, T_1)$, therefore $(*)$ is equivalent to $(r-T_1)x_2=0$, $(r-T_1)^2x_1=(r-T_1)S_1x_2$, and the last equation has a solution $\hat{x}_1$. If $x_2\neq 0$, then x_2 generates $N(r-T_1)$, so we have $(r-T_1)\hat{x}_1=$ $=S_1x_2-\hat{\lambda}x_2=(1-P_1)S_1x_2$ for some $\hat{\lambda}$, where P_1 is the residuum of $R(\cdot, T_1)$ at r. Therefore $(r-T)^2x=0$ is equivalent to

$$(r-T_1)x_2 = 0 \quad\text{and}\quad (r-T_1)x_1 = (1-P_1)S_1x_2.$$

For each $x\geqq 0$ in $N((r-T)^2)$ with $(r-T)x\neq 0$ we have to and may choose $x_2\geqq 0$, $x_2\neq 0$, in $N(r-T_1)$, therefore x_2 is a quasi-interior element in l^p_+. Since $S_1\geqq 0$, $S_1\neq 0$, we have $S_1x_2\geqq 0$, $S_1x_2\neq 0$. Now we have to look for a solution $x_1\geqq 0$ of $(r-T_1)x_1=(1-P_1)S_1x_2$. But such a solution does not exist in general, since T_1 is the operator of the last example and we can obtain each non-negative, non-zero element in l^p as S_1x_2 by an appropriate choice of S_1 (as a one dimensional non-negative operator).

As a final remark we recall that U. G. Rothblum [8, Theorem 3.1] has shown that for a non-negative matrix the algebraic eigenspace to its spectral radius has a basis of non-negative elements. Generalizing a result of H. D. Victory, jr. [11, Theorem 1] on integral operators in L^p-spaces, J. Kölsche [5, Satz IV. 2.2] has proved: Given $\varepsilon>0$ arbitrarily, for a non-negative, eventually compact operator T in an order continuous Banach lattice there exists a basis for the algebraic eigenspace of T to $r(T)$ such that every vector in this basis has norm 1 but its negative part has norm smaller than or equal to ε. The last example shows that, in general, ε has to be positive in this assertion.

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