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Arithmetical functions satisfying some relations

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1. Let $A(A^*)$ be the set of additive (completely additive) functions, $M(M^*)$ be the set of multiplicative (completely multiplicative) functions. $\|x\| = \min_{k\in \mathbf{Z}} |x-k|$.

Let $L_f(n):=f_0(n)+f_1(n+a_1)+\ldots+f_k(n+a_k)$, where $f_j\in A^*$ and $a_1, \ldots, a_k$ are mutually distinct natural numbers. It is probable that $\|L_f(n)\|\to 0$ $(n\to\infty)$ implies that $f_j(n)\equiv\tau_j \log n+u_j(n)$ (mod 1), with some $\tau_j\in\mathbf{R}$ such that $\tau_0+\ldots+\tau_k=0$ and $L_u(n):=u_0(n)+u_1(n+1)+\ldots+u_k(n+a_k)$ satisfies $L_u(n):\equiv 0$ (mod 1) for every $n\geqq 1$. This question was raised by the author and solved by E. Wirsing in the special case $k=1$.

Furthermore we guess that

$$L_u(n) \equiv 0 \pmod 1 \quad (n = 1, 2 \ldots) \tag{1.1}$$

implies that $u_j(n)\equiv 0$ (mod 1) for every $n\in\mathbf{N}$ and for every j. This was proved for $k=3$, $a_1=1$, $a_2=2$, $a_3=3$ in [2]. Marijke van Rossum investigated the solutions of the relation

$$g_0(\alpha)+g_1(\alpha+1)+g_2(\alpha+2)+g_3(\alpha+3) \equiv 0 \pmod 1 \quad (\forall\alpha\in G), \tag{1.2}$$

where $g_0, \ldots, g_3$ are completely additive functions defined on the set of $\mathbf{G}$ of Gaussian integers. She found that (1.2) has only trivial solutions.

The simple idea to prove that a recursion

$$L_f(n) = f_0(n)+f_1(n+1)+\ldots+f_k(n+k), \quad L_f(n) \equiv 0 \pmod 1 \tag{1.3}$$

has only trivial solution, is the following one:

1) *Initial step:* by taking $L_f(n)\equiv 0$ (mod 1) for $n=1, 2, \ldots, N$ with a large N, solving a linear equation system without multiplication and divisions, one conclude that $f_j(n)\equiv 0$ (mod 1) holds true for all n up to N_0.

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2) *Induction step:* If (1.3) holds and $f_j(n)\equiv 0 \pmod 1$ holds for $k=1, 2, \ldots, n$, then it is true for $k=n+1$ as well, assuming that $n\geqq N_1$, where $N_1\leqq N_0$.
The initial step can be handled by using computer for a moderate size of k. The induction could be deduced simply from the following.

Conjecture. For every integer $k\geqq 1$ there exists a constant $C_0(k)$ such that

$$\min_{P(j)<Q}\ \max_{l=1,\ldots,k}\ \max\{P(jQ+l),\ P(jQ-l)\} < Q$$

hold for every prime $Q>C_0(k)$. Here $P(n)$ denotes the largest prime divisor of n.

This is clearly true, if $k=1$, by choosing $j=1$. The conjecture is open for $k\geqq 2$, and even in the case $k=1$ if we exclude $j=1$.

In Section 2 we shall prove the following

Theorem 1. *Let a, δ be positive integers, $f_1, f_2, f_3\in A^*$ such that $L(n):\equiv f_1(n-a)+ +f_2(n)+f_3(n+\delta)$ satisfies the relation*

$$L(n) \equiv 0 \pmod 1, \tag{1.4}$$

for every integer $n\geqq a+1$. Assume furthermore that $f_j(n)\equiv 0 \pmod 1$ for $j=1, 2, 3$ and for all $n\leqq \max(3, a+\delta)$. Then $f_j(n)\equiv 0 \pmod 1$ $(j=1, 2, 3)$ for all $n\in\mathbb{N}$ and $j=1, 2, 3$.

Hence immediately follows

Theorem 2. *If $f_1, f_2, f_3\in A^*$ and*

$$f_1(n-a)+f_2(n)+f_3(n+b) = 0 \tag{1.5}$$

holds for all $n\geqq a+1$, then for every prime $p>\max(3, a+b)$ the values $f_1(p), f_2(p), f_3(p)$ are determined by the collection of the values $f_1(q), f_2(q), f_3(q)$ taken on at primes $q\leqq\max(3, a+b)$. Thus the set of solutions (f_1, f_2, f_3) of (1.5) forms a finite dimensional space.

Let E denote the operator $Ex_n=x_{n+1}$ in the linear space of infinite sequences, and for an arbitrary polynomial $P(z)=a_0+a_1z+\ldots+a_kz^k$ let $P(E)x_n=a_0x_n+ +a_1x_{n+1}+\ldots+a_kx_{n+k}$. A. Sárközy [4] determined all $f\in M$ which satisfy a linear recurrence. From his theorem one can deduce immediately the following

Lemma 1. *Let $B\geqq 1$ be an integer, $f\in M$ for which $f(n+B)=f(n)$ $(n=1, 2, \ldots)$ holds. Then either $f(n)=0$ for all $n\in\mathbb{N}$, or $f(n)=\chi_B(n)$ for all n coprime to B, where $\chi_B(n)$ is a character $\bmod B$. Let $B=B_1B_2$, $(B_1, B_2)=1$, $B_1=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$, where $f(p_j^{\alpha_j})\neq 0$ $(j=1, \ldots, r)$, $B_2=q_1^{\beta_1}\ldots q_s^{\beta_s}$, where $f(q_l^{\beta_l})=0$. The cases $B_1=1$ or $B_2=1$ are included. Let δ_l be the largest exponent $(\delta_l\geqq 0)$ for which $f(q_l^{\delta_l})\neq 0$. Then $0\leqq\delta_l< <\beta_l$ $(l=1, \ldots, S)$. Let $D=q^{\beta_1-\delta_1}\ldots q^{\beta_s-\delta_s}$. Then $\chi_B(n)=\chi_D(n)$ for $(n, B)=1$, χ_D is a character $\bmod D$. Furthermore $f(p^\gamma)=f(p^\alpha)\,\chi_E(p^{\gamma\alpha})$ holds for all $p^\alpha\| B$ and $\gamma>\alpha$.*

All the functions with the above conditions are periodic $\bmod B$.

In Section 3 we give all the solutions of $V(n+k)=U(n)$ $(n=1, 2, \dots)$ for $U, V\in M$ under the condition $U(n)\neq 0$ if $(n, k)=1$. This equation for completely multiplicative functions was solved earlier in [1]. We present it now as

Lemma 2. *Let $G(n+k)=F(n)$ hold for all $n\in N$, $F, G\in M^*$, $F(n)$ be non-identically zero, $F(n)=0$ if $(n, k)>1$. Then*

a) *$F(n)=G(n)=\chi_k(n)$ is a solution for an arbitrary multiplicative character* $\chi_k \pmod K$,

b) *there is no other solution if* $4|K$ *or if* $(2, K)=1$,

c) *if $K=2R$, $(R, 2)=1$, then all further solutions have the form*

$$F(n) = \chi(n, 8)\psi_R(n),\quad G(n) = \chi(n, 4)F(n),$$

where $\psi_R(n)$ is an arbitrary character $\bmod R$, *$\chi(n, 4)$ is the nonprincipal character* $\bmod 4$, *and $\chi(n, 8)$ is the character* $\bmod 8$ *defined by the relations.*

$$\chi_8(n) = \begin{cases} 1 & n \equiv \pm 1 \pmod 8 \\ -1 & n \equiv \pm 3 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv 1 \pmod 4,$$

$$\chi_8(n) = \begin{cases} 1 & n \equiv \pm 3 \pmod 8 \\ -1 & n \equiv 5, 7 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv -1 \pmod 4.$$

The equation $G(n+k)=F(n)$, $F(1)\neq 0$ implies that $F(n)G(n)\neq 0$ for $(n, k)=1$, assuming that F and G are completely multiplicative. This is not true if we assume only that $F, G\in M$.

In Section 4 we solve the equation $G(n+1)=F(n)$ for $F, G\in M$ without any additional conditions.

2. Proof of Theorem 1. The case $a=b=1$ has been proved in [2]. We may assume that $(a, b)=1$. Indeed, by substituting $n\delta$ into the place of n, observing that $f_j(\delta)\equiv 0 \pmod 1$, we have

$$f_1(n-a_1)+f_2(n)+f_3(n+a_1) \equiv 0 \pmod n \quad (\forall n),$$

and $f_j(n)\equiv 0 \pmod 1$ $(j=1, 2, 3)$ for every $n\leqq \max(3, a+b)$, $a=\delta a_1$, $b=\delta b_1$.

Let A_n denote the event that $f_j(n)\not\equiv 0 \pmod 1$ holds for at least one j. We shall prove that under the condition of the theorem there exists no such an integer. If such an n exists, then $n\geqq k+1$, furthermore the smallest n for which A_n is true has to be a prime number P.

Now we distinguish three cases according to the parity of a and b. Let $k=a+b$.

Case I: a and b are odd numbers. Since P is the smallest integer n for which A_n is true, therefore $f_3(P)\equiv 0 \pmod 1$ cannot occur, since $f_2(P-a)\equiv 0 \pmod 1$,

$f_1(P-k)\equiv 0 \pmod 1$. Similarly, $f_2(P)\equiv 0 \pmod 1$, since $2|P+b$, and $\frac{P+b}{2}<P$. Thus $f_1(P)\equiv\alpha\ (\not\equiv 0) \pmod 1$. Since $L(P+a)\equiv 0 \pmod 1$, and $2|P+a$, $\frac{P+a}{2}<P$, $f_2(P+a)\equiv 0 \pmod 1$, therefore $f_3(P+k)\equiv -\alpha \pmod 1$.

Let now $\delta|k$, $\delta>1$. Since $L(P+a)\equiv 0 \pmod 1$, $L\left(P+\frac{k}{\beta}-b\right)\equiv 0 \pmod 1$, therefore

(2.1) $$f_1(\delta P)+f_2(\delta P+a)+f_3(\delta P+k)\equiv 0 \pmod 1$$

(2.2) $$f_1\left(P+\frac{k}{\delta}+k\right)+f_2\left(P+\frac{k}{\delta}-b\right)+f_3\left(P+\frac{k}{\delta}\right)\equiv 0 \text{ mod } 1),$$

$f_1\left(P+\frac{k}{\delta}-k\right)\equiv 0 \pmod 1$. If $f_3(P+k/\delta)\equiv\beta\not\equiv 0 \pmod 1$, then k/δ is an even number, since in the opposite case $2|P+k/\delta$, and from $\frac{1}{2}(P+k/\delta)<P$ it would follow $f_3(\cdot)\equiv 0 \pmod 1$. But then $f_2(P+k/\delta-b)\equiv-\beta\not\equiv 0 \pmod 1$, $P+\frac{k}{\delta}-b$ is an even number and $\frac{1}{2}\left(P+\frac{k}{\delta}-b\right)<P$. This cannot be occur. Thus $f_3(\delta P+k)\equiv$ $\equiv f_3(\delta)+f_3\left(P+\frac{k}{\delta}\right)\equiv 0 \pmod 1$. So we have

(2.3) $$f_2(\delta P+a)\equiv -\alpha \pmod 1 \text{ whenever } \delta|k,\ \delta>1.$$

Assume first that $3|k$. Then, from (2.3) we have $f_2(3P+a)\equiv-\alpha \pmod 1$. Since $2|3P+a$, therefore $3P+a=2Q$, where Q is a prime number, $P<Q<2P$. Since $f_1(Q-a)+f_2(Q)+f_3(Q+b)\equiv 0 \pmod 1$, $2|Q-a$, $2|Q+b$, $Q-a<2P$, $Q+b<$ $<2(P+k)$, therefore $f_1(Q-a)\equiv 0 \pmod 1$, $f_3(Q+b)\equiv 0 \pmod 1$, and so $f_2(Q)\equiv$ $\equiv 0 \pmod 1$, $\alpha\equiv 0 \pmod 1$. It remains the case $3\nmid k$. Since $f_3(P+k)\not\equiv 0 \pmod 1$, and from (2.3), $f_2(2P+a)\not\equiv 0 \pmod 1$, thus P, $P+k$, $2P+a$ are prime numbers.

Assume first that $3\nmid a$. Since $P>3$, therefore either $3|2P+a$ or $3|4P+a$. Since $f_2(2P+a)\not\equiv 0 \pmod 1$, therefore $3\nmid 2P+a$, so $3|4P+a$. Let us consider now

(2.4) $$f_1(4P)+f_2(4P+a)+f_3(4P+k)\equiv 0 \pmod 1.$$

We shall prove that $f_2(4P+a)\equiv 0 \pmod 1$. Since $4P+a=3Q$, it is true, if Q is a composite number. If it is a prime, then we may consider

$$f_1(Q-a)+f_2(Q)+f_3(Q+b)\equiv 0 \pmod 1,$$

which by $2|Q+b$, $2|Q-a$, $Q<2P$ gives that $f_2(Q)\equiv 0 \pmod 1$. So, from (2.4) we infer $f_3(4P+k)\equiv-\alpha \pmod 1$. If $4|k$, then it cannot be occur, since $P+k$ is the smallest integer n for which $f_3(n)\not\equiv 0 \pmod 1$. If $k=2l$, $(l,2)=1$, then

$f_3(2P+l) \equiv -\alpha \pmod 1$. If $k=2l$, $(l,2)=1$, then $f_3(2P+l) \equiv -\alpha \pmod 1$. But

$$f_1(2P-l)+f_2(2P-l+a)+f_3(2P+l) \equiv 0 \pmod 1. \tag{2.5}$$

Since $2|a-l$, $2|2P-l+a<2P+a$, therefore $f_2(2P-l+a) \equiv 0 \pmod 1$, and so $f_1(2P-l) \equiv \alpha \pmod 1$.

Since $2P-l$, $(2P-l)+l=2P$, $2P+l$ cover all the residue classes mod 3, $3 \nmid 2P$, thus $3|2P+l$ or $3|2-l$. Both of these cases imply that $\alpha \equiv 0 \pmod 1$.

It remains the case $3|a$ and $3 \nmid k$. Then $k \equiv b \pmod 3$. Let $Q:=P+k$. Then $f_3(Q) \equiv -\alpha \pmod 1$. Let us consider $f_1(2Q-k)+f_2(2Q-b)+f_3(2Q) \equiv 0 \pmod 1$. Since $2Q-k \equiv 2Q-b \pmod 3$, $3|2Q-b$, and $2Q-b<3(P+a)$, would imply $f_2(2Q-b) \equiv 0 \pmod 1$, $f_1(2Q-k) \equiv 0 \pmod 1$, thus we may assume that $3 \nmid 2Q-b$. But then $P, P+k, 2P+k$, are coprime to 3. Since $3 \nmid k$, $3 \nmid P$, therefore either $P \equiv k \pmod 3$ or $P \equiv -k \pmod 3$. In both cases, at least one of $P, P+k, 2P+k$ is a multiple of 3. This is a contradiction.

By this the proof of Case I is completed.

Case II: a is odd, b is even. Let $n=P$ be the smallest integer for which A_n holds true. Then n is a prime, $P>3$, $P>k$. We can see, similarly as earlier, that $f_2(P) \equiv \alpha \not\equiv 0 \pmod 1$ with some α, $f_1(P) \equiv 0$, $f_3(P) \equiv 0 \pmod 1$. Observe that $f_3(n) \equiv 0 \pmod 1$ if $n<P+b$, and that $f_3(P+b) \equiv -\alpha \pmod 1$, which immediately follows from $L(P) \equiv 0 \pmod 1$. Furthermore, we can get that $f_1(n) \equiv 0 \pmod 1$, if $n<2P-a$. It is enough to prove this for odd, even for prime number integer $n=Q$. Since $L(Q+a) \equiv 0 \pmod 1$, $2|Q+a$, $2|Q+k$, $Q+a<2P$, therefore $f_2(Q+a) \equiv \equiv 0 \pmod 1$, $f_3(Q+k) \equiv 0 \pmod 1$, and so $f_1(Q) \equiv 0 \pmod 1$ as well. Then, for $\delta|b$, $\delta>1$, we get that $f_3(\delta P+b) \equiv 0 \pmod 1$, and by $L(\delta P) \equiv 0 \pmod 1$, that

$$f_1(\delta P-a) \equiv -\alpha \pmod 1 \quad \text{if} \quad \delta|b \quad \text{and} \quad \delta>1. \tag{2.6}$$

Let us consider the equation $L(3P) \equiv 0 \pmod 1$.

Since $2|3P-a$, $3P-a=2Q$, $Q<2P-a$, therefore $f_1(3P-a) \equiv 0 \pmod 1$. This implies that either $\alpha \equiv 0 \pmod 1$, or $3 \nmid b$, furthermore in the second case that $f_3(3P+b) \equiv -\alpha \pmod 1$. Thus $3P+b$ is a prime number since if it would be composite then its prime factors would be smaller than $P+b$. So P, $P+b$, $3P+b$ are prime numbers greater than 3, thus $P \equiv b \pmod 3$.

Since $2|b$, thus from (2.6) it follows that $2P-a$ is a prime, and so that $3 \nmid 2b-a$. If $4|b$, then by (2.6) we get that $4P-a$ is a prime, and $f_1(4P-a) \equiv -\alpha \pmod 1$. Assume that $2\|b$, $b=2b_1$. Since $P \equiv b \pmod 3$, $P \equiv 2b_1 \pmod 3$, from $L(2P+b_1-b) \equiv \equiv 0 \pmod 1$, by $2|2P+b_1-k<P$, $3|2P+b_1-b$ we deduce that $f_1(2P+b_1-k) \equiv \equiv 0 \pmod 1$, $f_2(2P+b_1-b) \equiv 0 \pmod 1$, and so that $f_3(2P+b_1) \equiv 0 \pmod 1$. But then, from $L(4P) \equiv 0 \pmod 1$ we have

$$f_1(4P-a)+f_2(4P)+f_3(2(2P+b_1)) \equiv 0 \pmod 1,$$

and so that $f_1(4P-a)\equiv-\alpha$ (mod 1). Thus $4P-a$ is a prime, since in the case $4P-a=3Q$, $Q<2P-a$ would imply $f_1(4P-a)\equiv 0$ (mod 1). So $P, P+b, 2P-a, 4P-a$ are all prime numbers which can be occur only if $3|a$.

It remained to consider the case $3|a$, $P\equiv b$ (mod 3). Furthermore $f_1(4P-a)\equiv\equiv-\alpha$ (mod 1). Since $3|2(P+b)-b$, $3|2(P+b)-b-a$, and $L(2(P+b)-b)\equiv 0$ (mod 1), therefore $f_1(2(P+b)-b)\equiv 0$ (mod 1), $f_2(2(P+b)-b-a)\equiv 0$ (mod 1), consequently $f_3(2(P+b))\equiv 0$ (mod 1), which implies $\alpha\equiv 0$ (mod 1).

The proof of Case II is completed.

Case III: *a is even, b is odd*. Then we have $f_1(P)\equiv\alpha(\not\equiv 0)$ (mod 1), $f_2(P+a)\equiv\equiv-\alpha$ (mod 1, $P+a$ is a prime number. Furthermore, $f_2(n)\equiv 0$ (mod 1) if $n<P+a$. Now we observe that $f_3(n)\equiv 0$ (mod 1) for all $n<2P+k$. Since $f_3(2)\equiv 0$ (mod 1), therefore enough to prove this for odd prime Q. Let $Q<2P+k$. If $f_3(Q)\not\equiv 0$ (mod 1), then by $L(Q-b)\equiv 0$ (mod 1) we have that $f_1(Q-k)+f_2(Q-b)\not\equiv 0$ (mod 1). But $2|Q-b$, $2|Q-k$, and $Q-k<2P$, $Q-b<2(P+a)$. Consequently $f_3(Q)\equiv 0$ (mod 1).

Let $\delta|a$ and $\delta>1$. By $f_2(P+a/\delta)\equiv 0$ (mod 1), and $L(\delta P+a)\equiv 0$ (mod 1) we deduce that

$$f_3(\delta P+k)\equiv-\alpha \pmod 1 \quad \text{if} \quad \delta>1 \quad \text{and} \quad \delta/a. \tag{2.7}$$

Let $\mu|k$. Since $L(\mu P+a)\equiv 0$ (mod 1) and $f_3\left(\mu P+\mu\cdot\frac{k}{\mu}\right)\equiv 0$ (mod 1), therefore

$$f_2(\mu P+a)\equiv-\alpha \pmod 1 \quad \text{if} \quad \mu|k. \tag{2.8}$$

Assume now that $\mu>1$. Then $L(2\mu P+a)\equiv 0$ (mod 1), $2\mu P+k=(\mu 2P+k/\mu)$, $2P+k/\mu<2P+k$, $f_3(2\mu P+k)\equiv 0$ (mod 1), and so

$$f_2(2\mu P+a)\equiv-\alpha \pmod 1 \quad \text{if} \quad \mu|k \quad \text{and} \quad \mu>1. \tag{2.9}$$

So $P, P+a, 2P+k$ are prime numbers.

Since $2|3P+k$, $\frac{3P+k}{2}<2P+k$, therefore $f_3(3P+k)\equiv 0$ (mod 1), and so, by $L(3P+a)\equiv 0$ (mod 1) we have $f_2(3P+a)\equiv-\alpha$ (mod 1). This implies that either $\alpha\equiv 0$ (mod 1) or $3\nmid a$. Assume that $3\nmid a$. Since $P, P+a$ are primes larger than 3, therefore $P\equiv a$ (mod 3). If $4|a$, then $f_3(4P+k)\equiv-\alpha$ (mod 3) and 3 cannot be a divisor of $4P+k$ if $\alpha\not\equiv 0$ (mod 3), consequently $4P+k$ is a prime number. If $2\|a$, $a=2a_1$, then by

$$f_1(4P)+f_2(2(2P+a_1))+f_3(4P+k)\equiv 0 \pmod 1$$

$$f_1(2P-a_1)+f_2(2P+a_1)+f_3(2P+a_1+b)\equiv 0 \pmod 1$$

and by taking into account that $3|2P-a_1$, $2|a_1+b$, first we deduce that $f_1(2P-a_1)\equiv\equiv 0$ (mod 1), $f_3(2P+a_1+b)\equiv 0$ (mod 1) and so that $f_2(2P+a_1)\equiv 0$ (mod 1), we have $f_3(4P+k)\equiv -\alpha \pmod 1$. This implies that $4P+k$ is a prime number. Since $0,\ 2P,\ 2\cdot 2P$ are incongruent residues $\bmod 3$, therefore so are $k, 2P+k, 4P+k$, consequently one of them is a multiple of 3. Since $2P+k$, $4P+k$ are primes larger than 3, only the case $3|k$ can be occur. Assume that $3|k$. Then $a\equiv -b \pmod 3$. From

$$f_1(2P+a)+f_2(2P+2a)+f_3(2P+2a+b) \equiv 0 \pmod 1$$

we have $3|2P+a$, $3|2P+2a+b$, which implies that $f_1(2P+a)\equiv 0 \pmod 1$, $f_3(2P+2a+b)\equiv 0 \pmod 1$, and so that $f_2(P+a)\equiv 0 \pmod 1$, which can be occur only if $\alpha\equiv 0 \pmod 1$.

This completes the proof of Case III. The theorem is proved.

3. Let us consider now the equation

$$V(n+K) = U(n) \quad (n = 1, 2, \dots), \tag{3.1}$$

where U, V are multiplicative functions, K is a fixed positive integer. We are interested in to give all the solutions under the condition

$$U(n) \neq 0 \quad \text{whenever} \quad (n, K) = 1. \tag{3.2}$$

The same equation for completely multiplicative functions was considered in our earlier paper [1]. We solved (3.1) for $K=1$ assuming (3.2) in [1]. The case $K>1$ is more complicated. Assume that (3.1) and (3.2) hold.

Let

$$H(n) := \frac{V(n)}{U(n)} \tag{3.3}$$

be defined on the set of integers n, coprime to K. Let furthermore

$$\delta_p(m) := H(p)H(m)H(m+k)\dots H\big(m+(p-2)K\big). \tag{3.4}$$

If $(p, n(n+K))=1$, then

$$H(p) = \frac{V\big(p(n+k)\big)}{U(pn)} = \frac{1}{H(pn+K)\dots H\big(pn+(p-1)K\big)}, \tag{3.5}$$

i.e.

$$\delta_p(pn+K) = 1 \quad \text{if} \quad \big(p, n(n+K)\big) = 1. \tag{3.6}$$

Let $p>q$, $r=p-q+1$. Then

$$\begin{aligned}\delta_p(m) &= H(p)\big[H(m)H(m+K)\dots H\big(m+(q-2)K\big)\big]\times\\ &\quad\times\big[H\big(m+(q-1)K\big)\dots H\big(m+(p-2)K\big)\big] =\\ &= H(p)\frac{\delta_q(m)}{H(q)}\cdot\frac{\delta_r\big(m+(q-1)K\big)}{H(r)},\end{aligned}$$

and so

$$\frac{H(p)}{H(q)H(r)} = \frac{\delta_p(m)}{\delta_q(m)\cdot\delta_r(m+(q-1)K)}. \tag{3.7}$$

We should like to give some conditions which imply that the right hand side equals 1. This holds true if all the next relations are satisfied, with a suitable integer m:

$$m \equiv K(\bmod p);\ m \equiv K(\bmod q);\ m+(q-2)K \equiv 0 \quad (\bmod r), \tag{3.8}$$

$$\left(\frac{m-K}{p}\cdot\frac{m+(p-1)K}{p}, p\right) = 1;\ \left(\frac{m-K}{q}\cdot\frac{m+(q-1)K}{q}, q\right) = 1, \tag{3.9}$$

$$\left(\frac{m+(q-2)K}{r}\cdot\frac{m+(q-1)K-K+rK}{r}, r\right) = 1; \quad (pqr, K) = 1. \tag{3.10}$$

Let

$$K^* = \begin{cases} K & \text{if } K \text{ is even,} \\ 2K & \text{if } K \text{ is odd.} \end{cases}$$

Assume that r is given, $(r, K)=1$. Let λ be an integer which will be chosen later, $\eta:=\lambda K^*$. Let p and q by defined by

$$p = (1+\eta)r,\ q = \eta r+1.$$

If (3.8), (3.9), (3.10) hold with some m, then

$$H(p) = H(1+\lambda K^*)H(r) \tag{3.11}$$

is valid.

We shall search m in the form $m=pq\,v+K$. The conditions $m\equiv K$ $(\bmod p)$, $m\equiv K$ $(\bmod q)$, $m+(q-2)K\equiv pq\,v+(q-1)K\equiv 0$ $(\bmod r)$ are satisfied clearly, the condition $(pqr, K)=1$ is equivalent to $(r(1+\eta((\eta r+1), K)=1$ which is true since $(r, K)=1$ was assumed.

We have

$$\frac{m-K}{p}\cdot\frac{m+(p-1)K}{p} = qv(qv+K),\ \frac{m-K}{q}\cdot\frac{m+(q-1)K}{q} = pv(pv+K),$$

$$m+(q-2)K = pqv+(q-1)K = [(1+\eta)qv+\eta K]r,$$

$$m+(q-2)K+rK = [(1+\eta)qv+(\eta+1)K]r = (1+\eta)r(qv+K).$$

So, to satisfy (3.9), (3.10) we have to find such v, for which

$$\big(qv(qv+K), p\big) = 1, \quad \big(pv(pv+K), q\big) = 1 \tag{3.12}$$

$$\big(((1+\eta)qv+\eta K)\cdot(1+\eta)(qv+K), r\big) = 1 \tag{3.13}$$

simultaneously hold.

The condition $(p, q)=1$ will be guaranteed by restricting r to satisfy the relation

$$(r(r-1), 1+\eta) = 1. \tag{3.14}$$

Since η is an even number, there exists such an r. Now we prove that (3.14) implies that $(p, q)=1$. Assume the contrary. Let $\delta|(p, q)$, δ be a prime number. Since $p=(1+\eta)r$, $q=\eta r+1$, therefore $\delta\nmid r$, and so $\delta|1+\eta$. But $q=(\eta+1)r+ +(1-r)$, whence $\delta|1-r$. This case was excluded by (3.14).

Now our conditions can be rewritten in the form

$$(v(pv+K), q) = 1 \tag{1}$$

$$(v(qv+K), p) = 1 \tag{2}$$

$$((1+\eta)qv+\eta K, r) = 1 \tag{3}$$

$$(qv+K, r) = 1. \tag{4}$$

Since (2) implies (4), therefore (4) can be omitted. Since $p=(1+\eta)r$, then we may substitute them with

$$(v(pv+K), q) = 1 \tag{A}$$

$$(v(qv+K), r) = 1 \tag{B}$$

$$(v(qv+K), (1+\eta)) = 1 \tag{C}$$

$$((1+\eta)qv+\eta K, r) = 1. \tag{D}$$

Since $(p, q)=1$, therefore $(q, r)=1$, consequently q, r, $1+\eta$ are pairwise coprime integers. To prove that (A), (B), (C), (D) hold simultaneously with a suitable v, it is enough to show that there is a solution of (B) and (D), furthermore that of (A), and of (C).

Since q and $1+\eta$ are both odd numbers, therefore (A) and (C) can be solved.

Assume that there exist no v for which (B) and (D) would hold simultaneously. Then there exists a prime divisor Q of r such that for every integer v, either $(v(qv+K), Q)=Q$ or $((1+\eta)qv+\eta K, Q)=Q$. Let us observe that it can be occur only if $Q=3$, i.e. if $3|r$.

If $3|r$, then $3\nmid K$, $q\equiv 1 \pmod 3$, thus we have $v(qv+K)\equiv v(v+K) \pmod 3$, $(1+\eta)qv+\eta K\equiv(1+\eta)v+\eta K \pmod 3$. If $3|\eta$, then the last congruence can be reduced to $\equiv v \pmod 3$. In this case (B) and (D) can be solved as well.

We shall exclude the case when $3|r$ and $3\nmid\eta$, i.e. the case: $3|r$ and $\eta\equiv 1$ (mod3.) Since $H(p)=H(q)H(r)$, by (3.9) we have

$$H(1+\lambda K^*) = H(1+\lambda rK^*) \tag{3.15}$$

if

$$(r(r-1), 1+\lambda K^*) = 1 \quad (r, K) = 1 \tag{3.16}$$

and in the case $3|r$, the relation $\eta \not\equiv 1(3)$ holds.

Lemma 3. *If* $(\lambda, K)=1$, $(\mu, K)=1$ *and in the case* $3\nmid K$, $\lambda K^* \not\equiv 1 \pmod 3$, $\mu K^* \not\equiv 1 \pmod 3$, *then*

$$H(1+\lambda K^*) = H(1+\mu K^*) \tag{3.17}$$

Proof. We can find positive integers r and s such that

$$r\lambda = s\mu \tag{3.18}$$

and

$$(r(r-1), 1+\lambda K^*) = 1 \tag{3.19}$$

$$(s(s-1), 1+\mu K^*) = 1. \tag{3.20}$$

Indeed, if $\delta=(\lambda, \mu)$, $\lambda=\delta\lambda_1$, $\mu=\delta\mu_1$, then $r=\mu_1 t$, $s=\lambda_1 t$ is a solution of (3.18) for every positive integer t. Assume that $(t, K)=1$. Then $(r, K)=(s, K)=1$ holds true. Since K is coprime to both of the integers $1+\lambda K^*$, $1+\mu K^*$, we have to consider only the solvability of (3.19) and that of (3.20). Both of them have solutions.

Assume that there exists no t for which (3.19) and (3.20) would be satisfied. Then there would exist a prime divisor Q of $(1+\lambda K^*, 1+\mu K^*)$ such that $\mu_1 t(\mu_1 t-1)\cdot\lambda_1 t\cdot(\lambda_1 t-1)\equiv 0 \pmod Q$ holds for every integer t.

We have $(\lambda_1\mu_1, Q)=1$. Furthermore $Q|(\lambda-\mu)K^*$, $(Q, K^*)=1$, therefore $Q|\delta(\lambda_1-\mu_1)$. $Q|\delta$ cannot be occur, thus $\lambda_1-\mu_1\equiv 0 \pmod Q$. Consequently our congruence can be reduced to the form $t(\lambda_1 t-1)\equiv 0 \pmod Q$. But it has at most two solutions mod Q, consequently there is a t for which both of (3.19), (3.20) holds. By this we proved our Lemma 3.

Lemma 4. *If* $A\equiv B \pmod{K^*K}$ *and* $(A, K^*)=1$, *then*

$$H(A) = H(B). \tag{3.21}$$

Proof. Let $3\nmid K$. Assume first that $3\nmid A$ and $3\nmid B$ or $3|(A, B)$. In the former case let $A_1=3A$, $B_1=3B$, in the second case $A=A_1$, $B=B_1$. In both cases $A_1=B_1 \pmod 3$.

If Θ is such an integer for which $A_1\Theta\equiv 1+K^* \pmod{K^*K}$ holds, then $B_1\Theta\equiv 1+K^* \pmod{K^*K}$ is satisfied as well. Writing $A_1\Theta=1+\lambda K^*$, $B_1\Theta=1+ +\mu K^*$, $\lambda K^*\not\equiv 1$, $\mu K^*\not\equiv 1$ obviously hold. Since the solutions Θ give a whole residue class $(\bmod K^*K)$, which is reduced to the module, we can choose Θ to be a large prime. By Lemma 3 we have $H(A_1\Theta)=H(B_1\Theta)$, which implies that $H(A_1)=H(B_1)$, and so that $H(A)=H(B)$.

If $3|A$, $3\nmid B$, then the general solution of the congruence $B\Theta\equiv 1+K^*\pmod{K^*K}$ can be written as $\Theta=\Theta^*+hK^*K$ $(h=0,1,2,\ldots)$ where Θ^* is a particular solution. Since $B\Theta\equiv B\Theta^*+hBK^*K\pmod 3$, $3\nmid BK^*K$, therefore $B\Theta\equiv 1\pmod 3$ holds if h is falling into the appropriate residue class mod 3. Then $A\Theta\equiv 0\pmod 3$. We may choose Θ to be a large prime, and by Lemma 2, $H(A\Theta)=H(B\Theta)$ we conclude that $H(A)=H(B)$.

In the case $3|K$ we get the lemma similarly, but without taking care of the requirements $\lambda K^*\not\equiv 1$, $\mu K^*\not\equiv 1\pmod 3$.

Let χ_0 be the principal character mod K^*K. Since the conditions of Lemma 1 are satisfied for the function $f(n):=\chi_0(n)H(n)$, $B=K^*K$, therefore there exists a character χ_{K^*K} such that

$$H(n)=\chi_{K^*K}(n)\quad\text{whenever}\quad (n,K^*K)=1. \tag{3.22}$$

We distinguish two cases according to the parity of k.

Case K=even. For every m, n integers coprime to K, let

$$A(m,n):=\frac{U(mn)}{U(m)U(n)},$$

$$S(m,n):=\frac{1}{\chi(n+K)}\prod_{l=1}^{m}\chi(mn+lK),$$

where χ is the character given in (3.22). Since χ is periodic mod K^2, therefore $S(m,n)$ is periodic mod K^2 in both of its variables m and n. Furthermore, $A(m,n)=1$ if m and n are coprimes.

Since

$$U(n)=V(n+K)=H(n+K)U(n+K)=\chi(n+K)U(n+K),$$

consequently

$$U(nm)=\chi(mn+K)\chi(mn+2K)\ldots\chi(mn+mK)U\big(m(n+K)\big)=S(m,n)U\big(m(n+K)\big),$$

i.e.

$$A(m,n)=S(m,n) \tag{3.23}$$

holds under the condition $(mn,K)=1$, $(m,n+K)=1$.

Let p be an arbitrary prime, $(p,K)=1$. Then p is an odd integer. Take $m=p^\alpha$, $n=pv$, where $(v,p)=1$. Then $A(p^\alpha,pv)=\dfrac{U(p^{\alpha+1})}{U(p)U(p^\alpha)}$. Since $(n+K,m)=1$ clearly holds, therefore

$$A(p^\alpha,pv)=S(p^\alpha,pv).$$

Since $S(p^\alpha,pv)=S(p^\alpha,pv+K^2)=A(p^\alpha,pv+K^2)=1$, we deduced that

$$U(p^{\alpha+1})=U(p^\alpha)U(p)$$

valid for all prime pover p^α coprime to K. This shows that U is completely multiplicative on the set $(n, K)=1$. Since $V=H\cdot U$, and H is completely multiplicative on the set $(n, K)=1$, so is V. Therefore, we may apply Lemma 2 for the characterization of the solution (U, V) at least on the set $(n, K)=1$.

Case $K=odd$. Let $n=2^\gamma \nu$, $\gamma\geqq 1$ and $(\nu, K)=1$. Then

$$1 = \frac{V(2^\gamma \nu+K)}{U(2^\gamma \nu)} = \frac{V(2^{\gamma+1}\nu+2K)}{U(\nu(U(2^\gamma)}\cdot\frac{U(2^{\gamma+1})}{V(2)U(2^{\gamma+1})} = \frac{U(2^{\gamma+1})}{V(2)U(2^\gamma)}H(2^{\gamma+1}\gamma+K).$$

Thus we proved that

$$H(2^{\gamma+1}\nu+K) = D_\gamma, \quad \text{for every} \quad (\nu, 2K) = 1, \tag{3.24}$$

where

$$D_\gamma = \frac{U(2^{\gamma+1})}{V(2)U(2^\gamma)}, \quad (\gamma \geqq 1) \tag{3.25}$$

Similarly, we can prove that

$$H(2^{\gamma+1}\nu-K) = E_\gamma \quad \text{for every} \quad (\nu, 2K) = 1, \tag{3.26}$$

$$E_\gamma = \frac{U(2)V(2^\gamma)}{V(2^{\gamma+1})} \quad \gamma \geqq 1 \tag{3.27}$$

From (3.22) we know that $H(n)=\chi(n)$ for $(n, 2K)=1$, where χ is a character mod $2K^2$ For odd K we can prove more, namely that H is periodic mod $2K$. The worst case is the case $3\nmid K$. Assume that $3\nmid K$.

If $K^*\equiv 1$ (mod 3), then, by Lemma 2,

$$H(1+3K^*) = H(1+4K^*); \quad (\lambda = 3, \mu = 4),$$

if $K^*\equiv -1$ (mod 3), then

$$H(1+2K^*) = H(1+3K^*) \quad (\lambda = 2, \mu = 3),$$

consequently, by

$$H(1+\nu K^*) = \chi_{2K^2}(1+\nu K^*) = \chi_{2K^2}(1+K^*)^\nu = H(1+K^*)^\nu,$$

we get that $H(1+K^*)=\chi_{2K^2}(1+K)=1$. If $3|K$, then we have $H(1+K^*)=H(1+2K^*)$, and conclude to the same result. But then $H(1+\nu K^*)=\chi_{2K^2}(1+\nu K^*)=1$ holds for every integer ν. If $A\equiv B$ (mod K^*) such that $(A, K^*)=1$, then one can choose a large prime Θ such that $A\Theta\equiv 1$ (mod K^*), which implies that $B\Theta\equiv 1$ (mod K^*), and $H(A\Theta)=H(B\Theta)$, whence by $(A, \Theta)=(B, \Theta)=1$, $H(\Theta)\neq 0$, we infer $H(A)=H(B)$. So we proved that H is periodic mod $2K$; consequently, by Lemma 1,

$$H(n) = \chi_{2K}(n) \quad \text{if} \quad (n, 2K) = 1. \tag{3.28}$$

Let us consider now (3.24). Observe that if $v_1, v_2, \ldots, v_s$, $S=\varphi(2K)$ is a complete reduced residue system $\bmod 2K$, then so is $2^{\gamma+1}v_j+K$ $(j=1, \ldots, S)$. Indeed, these numbers are coprime to $2K$, and if $2^{\gamma+1}v_i+K\equiv 2^{\gamma+1}v_j+K \pmod{2K}$, for some suitable $i\neq j$, then $K|(v_i-v_j)$. Since v_i, v_j are odd numbers, therefore $2|(v_i-v_j)$, so $v_i\equiv v_j \pmod{2K}$, which cannot be occur. It implies that the left-hand side does not change its value if v run over a reduced residue set, whence we have that $H(n)=1$ for every $(n, 2K)=1$, furthermore that $D_\gamma=1$ and similarly that $E_\gamma=1$ for every $\gamma\geqq 1$. From the relation $D_\gamma E_\gamma=1$ we obtain that

$$H(2^{\gamma+1}) = \frac{H(2^\gamma)}{H(2)} \quad (\gamma \geqq 1),$$

which implies that $H(2^2)=1$. We shall show that there exists such an integer Γ for which $H(2^\Gamma)=H(2^{\Gamma+1})$, which will imply that $H(2)=1$, and so that $H(2^\gamma)=1$ for every $\gamma\geqq 1$.

To do this, let us consider the product

$$\Delta(s, n) = \prod_{l=1}^{s-1} H(sn+lK)$$

defined for positive integers s, n such that $(sn, K)=1$. Observing that for $(s, n+K)=1$ we have

$$U(sn) = H(sn+K) \ldots H(sn+sK)\, U\big(s(n+K)\big) = \Delta(s, n)\, U(s)\, U(n),$$

consequently, if additionally $(s, n)=1$, then

$$\Delta(s, n) = 1.$$

Assume that the conditions

$$(s, n) = 1, \quad (s, n+K) = 1, \quad (s, K) = (n, K) = 1 \tag{3.29}$$

hold for some pairs of integers s, n. They imply that $\Delta(s, n)=1$. Let us change n by $N=n+RsK$, where R is an arbitrary positive integer. Since the conditions (3.29) will be held replacing n by N, therefore $\Delta(s, N)=1$ holds for all $R\geqq 1$. Let $A_l=sn+lK$, then $A_1<A_2<\ldots<A_{s-1}$. Let Γ_0 be so large that $A_{s-1}-A_1<2^{\Gamma_0}$. Let us choose $R=R_1$ such that $2^\Gamma \| A_0+s^2R_1K$. Let $b_2, \ldots, b_{s-1}$ be defined as the exponents of 2, such that $2^{b_j}\| A_j+s^2R_1K$ $(j=2, \ldots, s-1)$. It is clear that $\max b_j<\Gamma_0$. Now we choose an R_2 such that $2^{\Gamma+1}\| A_0+s^2R_2K$. For this choice of R the exponents of 2 in $A_j+s^2R_2k$ $(j=2, \ldots, s-1)$ are unchanged, $2^{b_j}\| A_j+s^2R_2K$. Thus we have

$$1 = \Delta(s, n+R_1sK) = H(2^\Gamma)\prod_{l=2}^{s-1} H(2^{b_l}) = H(2^{\Gamma+1})\prod_{l=2}^{s-1} H(2^{b_l}) = \Delta(s, n+R_2sK).$$

whence we have $H(2^\Gamma)=H(2^{\Gamma+1})$.

So we proved that $U(n)=V(n)$ on the set $(n,K)=1$. By taking $f(n)=$ $=\chi_0(n)U(n)$, where $\chi_0(n)$ is the principal character $\bmod K$, we have $f(n+K)=$ $=f(n)$ for all $(n,K)=1$. From Lemma 1 we get that $U(n)=V(n)=\chi_K(n)$ on the set $(n,K)=1$. Hence, by Lemma 3, after a simple discussion we shall deduce our

Theorem 3. *Let* $K\geqq 1$ *be an integer,* $F,G\in M$ *such that* $G(n+K)=F(n)$ *holds for every* $n\in\mathbf{N}$, *furthermore that* $F(n)\neq 0$ *if* $(n,K)=1$. *Then the following assertions hold:*

(A) $F(n)=G(n)=\chi(n;K)$ *on the set* $n,(n,K)=1$,

or

(B) *in the case* $K=2R$, $(R,2)=1$,

$$G(n)=\chi(n;4)F(n);\quad F(n)=\chi(n;8)\chi(n;R),$$

for every n, $(n,K)=1$, *where* $\chi(n;4)$ *is the nonprincipal character* $\bmod 4$; *by*

$$\chi(n;8)=\begin{cases}-1 & n\equiv\pm 1\,(\bmod 8)\\ 1 & n\equiv\pm 3\,(\bmod 8)\end{cases}\quad \text{if}\quad R\equiv 1\,(\bmod 4),$$

$$\chi(n;8)=\begin{cases}1 & n\equiv 1,3\,(\bmod 8)\\ -1 & n\equiv 5,7\,(\bmod 8)\end{cases}\quad \text{if}\quad R\equiv -1\,(\bmod 4).$$

(C) *If* $\delta\geqq 1$ *and* $p^{\delta+1}|K$, *then* $F(p^\delta)=0$ *holds if and only if* $G(p^\delta)=0$ *is satisfied. In the case* (A), *if* p *is odd and* $F(p^\delta)\neq 0$ *then* $G(p^\delta)=F(p^\delta)$ *and* $\chi(n;K)$ *is periodic with the period* K/p^δ. *In the case* (A), *if* $p=2$ *and* $F(p^\delta)\neq 0$, *then* $\chi(n;K)$ *is periodic with the period* $K/2^{\delta-1}$ *and* $G(p^\delta)=\chi\big(1+(K/2^\delta);K\big)F(p^\delta)$. *In the case* (B), *if* p *is odd, then* $G(p^\delta)=\chi(p^\delta;4)F(p^\delta)$, *and* $F(p^\delta)\neq 0$ *implies that* $\chi(n;R)$ *is periodic* $\bmod R/p^\delta$.

(D) *In the case* (B), $F(2^\gamma)=G(2^\gamma)=0$ *for every* $\gamma\geqq 1$.

(E) *If* $p^\alpha\|K$, *then* $F(p^\alpha)=0$ *is true if and only if* $G(p^\gamma)=0$ *for every* $\gamma>\alpha$ *furthermore* $G(p^\alpha)=0$ *if and only if* $F(p^\gamma)=0$ *is satisfied for every* $\gamma>\alpha$. *If* $p>2$, *then the statement* $G(p)=0$, $F(p)=0$ *are equivalent.*

(F) *If* $p^\alpha\|K$ *and* $F(p^\alpha)\neq 0$ *or* $G(p^\alpha)\neq 0$, *then* $\chi(n;K)$ *is induced by* $\chi(n;K_1)$, $K=p^\alpha K_1$ *in case* (A), *and* $\chi(n;R)$ *is induced by* $\chi(n;R_1)$, $R=p^\alpha R_1$ *in case* (B).

(G) *In case* (A) *let* $K=B_1B_2$, $(B_1,B_2)=1$, *where* B_1 *is the product of those prime powers* p^α, $p^\alpha\|K$, *for which at least one of* $G(p^\alpha)\neq 0$, $F(p^\alpha)\neq 0$ *holds. Then* $\chi(n;K)$ *is induced by some character* $\chi(n;B_2)$, *and*

$$\frac{G(p^\alpha)}{\chi(p^\alpha;B_2)}=\frac{F(p^\gamma)}{\chi(p^\gamma;B_2)}\quad (\textit{for every } \gamma>\alpha)$$

$$\frac{F(p^\alpha)}{\chi(p^\alpha;B_2)}=\frac{G(p^\gamma)}{\chi(p^\gamma;B_2)}\quad (\textit{for every } \gamma>\alpha),$$

moreover for $p \neq 2$,

$$F(p^\alpha) = G(p^\alpha)$$

hold.

(H) *In the case* (B) *let* $R=D_1\cdot D_2$, $(D_1, D_2)=1$, *where* D_1 *is the product of the prime powers* p^α, $p^\alpha \| R$, *for which* $F(p^\alpha)\neq 0$, *then* $\chi(n; R)$ *is induced by a character* $\chi(n; D_2)$. *Then*

$$a(p; \gamma) := \frac{F(p^\gamma)}{\chi(p^\gamma; 8)\chi(p^\gamma; D_2)} = a(p; \alpha)$$

$$b(p; \gamma) := \frac{G(p^\gamma)}{\chi(p^\gamma; 8)\chi(p^\gamma; D_2)} = b(p; \gamma)$$

hold for every $\gamma>\alpha$, *furthermore*

$$G(p^\gamma) = X(p^\alpha; 4)F(p^\gamma)$$

for every $\gamma \geqq \alpha$.

If F and G is such a pair of functions for which the above conditions hold, then the relation $G(n+K)=F(n)$ $(n\in N)$ *is satisfied.*

Proof. We shall prove only the necessity of the conditions, the sufficiency part can be verified easily. (A) and (B) were proved earlier. To prove (E) take $n=p^\gamma v$, where $\gamma>\alpha$, $(v, K)=1$, and consider only the equations $G(p^\gamma v+K)=F(p^\gamma v)$, $F(p^\gamma v-K)=G(p^\gamma v)$. Since $p^\gamma v\pm K=p^\alpha(p^{\gamma-\alpha}v\pm K_1)$, $K=p^\alpha K_1$, and $(p^{\gamma-\alpha}v\pm K_1, K)=$ $=1$, $F(p^{\gamma-\alpha}v-K_1)\neq 0$, $G(p^{\gamma-\alpha}v+K_1)\neq 0$, and since the same is true if $\gamma=\alpha$, $p>2$, for v, $(v(v-K_1), K)=1$, we obtain (E).

Now we prove (C). The assertion that $F(p^\delta)=0$ iff $G(p^\delta)=0$ is clear. Consider first the case (A). Assume that $F(p^\delta)\neq 0$. Let $n=p^\delta v$, $K=p^\alpha K_1$, $p^\alpha\|K$, $\delta<\alpha$. Then $G(p^\delta)G(v+p^{\alpha-\delta}K_1)=F(p^\delta)F(v)$, whence

$$a := \frac{G(p^\delta)}{F(p^\delta)} = \frac{\chi(v; K)}{\chi(v+p^{\alpha-\delta}K_1; K)} \quad \text{if} \quad (v, K) = 1. \tag{3.30}$$

If we write this equation replacing v by $v+s\,p^{\alpha-1}K_1$, and multiply the equations for $s=0, \dots, v-1$, we get that

$$a^v = \frac{\chi(v; K)}{\chi(v+vp^{\alpha-\delta}K_1; K)},$$

whence we obtain, that

$$a^v = \frac{1}{\chi(1+p^{\alpha-\delta}K_1, K)}$$

is true for every v, $(v, K)=1$. The right hand side does not depend on v. If $2\nmid K$ we can choose $v=1$, $v=2$ and conclude that $a=1$. If $2|K$, then we take $v=K-1$,

$v=K+1$, and deduce that $a^2=1$. In both cases we have

$$\chi(v+2p^{\alpha-\delta}K_1, K) = \chi(v; K) \quad \text{if} \quad (v, K) = 1,$$

which implies that $\chi(v, K)$ is periodic with period $2p^{\alpha-\delta}K_1$, and so it is periodic with $(2p^{\alpha-\delta}K_1, K)$. This implies condition (C) for the case (A).

Now we shall consider case (B). Observe that for the characters given in (B), the product

$$T_R(\mu) := \frac{\chi(\mu; 8)}{\chi(\mu+2R; 8)\chi(\mu; 4)} = -1 \tag{3.31}$$

for every odd μ and for $R \equiv \pm 1 \pmod 4$.

Assume that $p \neq 2$, $p^\delta | R$, $R=p^\alpha R_1$, $p^\alpha \| R$, $\delta<\alpha$. By choosing $n=p^\delta \alpha$, starting from the relation $G(p^\delta)G(v+2p^{\alpha-\delta}R_1)=F(p^\delta)F(v)$, substituting the values for $F(v)$ and $G(v+2p^{\alpha-\delta}R_1)$ given in (B), after some calculation we obtain

$$\frac{G(p^\delta)}{F(p^\delta)} = -\chi(p^\delta; 4)T_R(p^\delta v)\frac{\chi(v; R)}{\chi(v+2p^{\alpha-\delta}R_1; R)},$$

whence, by (3.32) we have that

$$b := \frac{G(p^\delta)}{(p^\delta; 4)F(p^\delta)} = \frac{\chi(v; R)}{\chi(v+2p^{\alpha-\delta}R_1; R)},$$

for every v, $(v, 2R)=1$. Arguing as at the former case we deduce that $b^2=1$, and so that $\chi(\cdot, R)$ is periodic $\bmod 4p^{\alpha-\delta}R_1$, and so $\bmod (4p^{\alpha-\delta}R_1, R)=p^{\alpha-\delta}R_1$. But then $b=1$, $G(p^\delta)=\chi(p^\delta; 4)F(p^\delta)$. This proves condition (C).

The next step is to prove (D). Assume that $G(2)\neq 0$, choose $n=2^\gamma v$, $\gamma \geqq 2$. Then

$$G(2)G(2^{\gamma-1}v+R) = F(2^\gamma)F(v)$$

and by using the explicit form of F and G, after some cancellation, we have

$$G(2)\chi(2^{\gamma-1}v+R; 4)\chi(2^{\gamma-1}v+R; 8)\chi(2^{\gamma-1}; R) = F(2^\gamma)\chi(v; 8). \tag{3.32}$$

If $\gamma \geqq 4$, then the left-hand side does not depend on v, while $\chi(v; 8)$ does. It implies that $F(2^\gamma)=0$ for $\gamma \geqq 4$, and so $G(2)=0$. We can prove impossibility of the case $F(2)\neq 0$ similarly. By (C) the proof of (D) is completed.

Let us prove now (G). By choosing $n=p^\gamma v$, $(v, K)=1$, $\gamma>\alpha$, $p^\alpha \| K$, $K=K_1 p^\alpha$, under conditions (A), we have

$$\frac{G(p^\alpha)}{F(p^\gamma)} = \frac{\chi(v; K)}{\chi(p^{\gamma-\alpha}v+K_1; K)}, \tag{3.33}$$

which is valid if $G(p^\alpha)\neq 0$. Assume $G(p^\alpha)\neq 0$. Then $F(p^\gamma)\neq 0$ holds for $\gamma>\alpha$, and the right-hand side does not depend on v. Let $\gamma \geqq 2\alpha$. Then the denominator is periodic $\bmod K_1$, which implies that $\chi(\nu+K_1;K)=\chi(\nu;K)$, consequently $\chi(\nu;K)=\chi(\nu;K_1)$ with some character $\bmod K_1$, and so the right-hand side is $\chi(p^{\gamma-\alpha};K_1)$. This assertion hold for every $\gamma>\alpha$, and in the case $p\neq 2$ even for $\gamma=\alpha$. The case $F(p^\alpha)\neq 0$ is similar. Doing this for all p^α, $p^\alpha\|B_1$, we get that $\chi(n;K)$ is periodic $\bmod B_2$, and this leads to the equations given in (G). We proved the first part of (F), as well.

Let us finally consider (H). Let $G(p^\alpha)\neq 0$. Let $R=p^\alpha R_1, \gamma>\alpha, n=p^\gamma\nu, (\nu,K)=1$. Then $p>2$. From $G(p^\alpha)G(p^{\gamma-\alpha}\nu+2R_1)=F(p^\gamma)F(\nu)$, we deduce that

$$\frac{G(p^\alpha)}{F(p^\gamma)}\cdot\frac{\chi(p^{\gamma-\alpha};8)}{\chi(p^\alpha;4)}=-\frac{\chi(\nu,R)}{\chi(p^{\gamma-\alpha}\nu+2R_1,R)}\cdot T_R(p^\gamma\nu),$$

which by (3.31) and by choosing $\gamma\geqq 2\alpha$, gives that $X(\cdot,R)$ is periodic $\bmod 2R$ and so $\bmod R$. Furthermore the right-hand side equals $\chi(p^{\gamma-\alpha};R_1)$, for every $\gamma\geqq\alpha$. We can deduce a similar formula assuming $F(p^\alpha)\neq 0$. Doing this for every p^α, $p^\alpha\|D_1$, we can finish the proof rapidly.

By this the proof of our theorem is completed.

4. Let $A, G\in M$ be connected by the equation $G(n+1)=F(n)$. This was solved in Section 3 under the additional condition $F(n)\neq 0$ $(n=1,2,\dots)$. It was found that $F(n)=G(n)=1$ identically.

Let now α be such an exponent for which $2^\alpha-1=P$, where P is a prime power, $P=Q^\beta$, allowing the case $\beta=1$. Let $G_\alpha, F_\alpha\in M$ as follow: $F(1)=G(1)=1$, $G_\alpha(2)=1$, $G_\alpha(2^\alpha)=F_\alpha(P)=$arbitrary nonzero value, $F_\alpha(n)=0$ if $n\neq 1, P$; $G_\alpha(n)=0$ if $n\neq 1,2,2^\alpha$. It is clear that F_α and G_α will be multiplicative functions, and the equation $G_\alpha(n+1)=F_\alpha(n)$ $(n=1,2,\dots)$ will be true.

It is an open question, whether $2^\alpha-1$ can be a prime power for infinitely many α or not. The list of $\alpha=2,3,5$ shows that such α values exist.

We shall prove the next

Theorem 4. *If $F, G\in M$ and $G(n+1)=F(n)$ holds for every $n\in N$, then either $F(n)=G(n)$ are identically zero, or identically one, or there exists an integer $\alpha\geqq 2$ such that $2^\alpha-1=$prime power$=P$, such that $G(2)=F(1)=G(1)=1$, $G(2^\alpha)=$ $=F(P)$ and $F(n)=0$, $G(n)=0$ holds for all other $n\in N$.*

Proof. Let $\mathcal{P}$ be the set of those prime powers P for which $F(P)\neq 0$, and $\mathcal{R}$ be the set of those powers Q for which $G(Q)\neq 0$. Let $\overline{\mathcal{P}}$, $\overline{\mathcal{R}}$ denote the complement sets with respect to the whole set of prime powers. If $\mathcal{P}$ or $\overline{\mathcal{P}}$ are empty sets, then so are $\mathcal{R}$ and $\overline{\mathcal{R}}$, and these lead to the equation $F(n)=G(n)$ as it was proved in Section 3. Thus, we may assume that $\mathcal{P}$ and $\mathcal{R}$ are non-empty proper subsets of the whole set of the prime powers.

It is well known that all solutions of the Diophantine equation $3^x-2^y=1$ are $x=y=1$, and $x=2, y=3$ while $2^x-3^y=1$ implies that $x=2, y=1$.

Lemma 5. *Let P be the smallest integer n, for which $F(n)G(n)=0$. Then P=prime power, furthermore $P=2,4$ or 8; $F(P)=0$ and $G(P)\neq 0$.*

Proof. It is clear that the smallest integer n for which $F(n)G(n)=0$ holds, has to be a prime power P, and $G(n)=F(n-1)\neq 0$. Thus $F(P)=0$.

Assume first that P is even, and $P>2$. Then $P=2^a$. We have $G(P+1)=0$, $G(2P+2)=F(2P+1)=0$. From the minimality of P we have that both of $P+1$ and $2P+1$ are prime powers. Since at least one of them is a multiple of 3, therefore either $2^a+1=3^b$ or $2^{a+1}+1=3^b$, which implies that $P=4$ or $P=8$.

Assume that P is an odd number. Then $G(P+1)=0$, and we can get rapidly that $P+1=2^s$. If $3|P$, then $P=3^a$, $2^s-3^a=1$, whence $s=2$, $a=1$, i.e. $P=3$ follows. In this case $F(2)\neq 0$, $F(3)=0$. But $F(2)\neq 0$, $\Rightarrow G(3)\neq 0$, $G(6)=F(5)\neq 0$, $F(10)\neq 0$, $G(11)\neq 0$, $\Rightarrow G(22)\neq 0$, $\Rightarrow F(21)=F(3)F(7)\neq 0$, $\Rightarrow F(3)\neq 0$. This leads to a contradiction. If $3|P$, the $2P+1\equiv 0(3)$, $G(2P+1)=0$, and we deduce that $2P+1=3^b$, whence $2^{s+1}-3^b=1$, and so $s=1$, $P=1$ follows. This cannot occur. We finished the proof of our lemma.

Lemma 6. *In the notations of Lemma 5, $P=4$ or $P=8$ cannot be occur.*

Proof. I. The case $P=8$. Then $\{2,3,2^2,5,7\}\in\mathcal{P}$, $\{2,3,2^2,5,7,2^3\}\in\mathcal{R}$, whence $G(5\cdot 3\cdot 7)=G(105)\neq 0$, $F(104)=F(2^3)\cdot(13)\neq 0$, $F(2^3)\neq 0$, and this is a contradiction.

II. The case $P=4$. Then $\{2,3\}\in\mathcal{P}$, $\{2,3,2^2\}\in\mathcal{R}$, and so

$$7=2\cdot 3+1\in\mathcal{R},\ G(7\cdot 3)=G(21)=F(20)=F(5\cdot 4)\neq 0,\quad \text{i.e.}\quad F(4)\neq 0,$$

contrary to our assumption.

Lemma 7. *If $\mathcal{R}$ contains at least one odd prime powers, then $F(n)$ and $G(n)$ are nowhere zero.*

Proof. Assume that $\varkappa$ is the smallest odd prime power in $\mathcal{R}$. $\varkappa=3$ would imply that $F(2)\neq 0$, and this case was treated earlier. Assume that $\varkappa>3$. Then $\varkappa-1$ is a power of 2, since in the opposite case, $\varkappa-1=2^sA$, $A>1$ would imply that $F(2^s)\neq 0$, $G(2^s+1)\neq 0$, and $2^s+1<\varkappa$. Thus $\varkappa-1=2^s\in\mathcal{P}$. Since $G(2)\neq 0$, therefore $0\neq G(2\varkappa)=F(2\varkappa-1)$. If $3|\varkappa$, then $\varkappa=3^b$, and from the equation $3^b-2^s=1$ we deduce that either $\varkappa=3$ $(b=1)$, or $\varkappa=3^2$ $(b=2)$. If $\varkappa=3$ then $2\in\mathcal{P}$, which was considered earlier. If $\varkappa=3^2$, then

$$\{2,3^2\}\in\mathcal{R},\ 2^3\in\mathcal{P},\ G(18)\neq 0,\ 17\in\mathcal{P},\ F(136)=F(8\cdot 17)\neq 0,$$

$$137\in\mathcal{R},\ G(1233)=G(9\cdot 137)\neq 0,\ F(1232)=F(2^4\cdot 7\cdot 11)\neq 0,\ 11\in\mathcal{R},$$

$G(12)=G(4\cdot 3)\neq 0$, $3\in\mathcal{R}$, which is a contradiction. Assume that $3\nmid\varkappa$. Then $3|2\varkappa-1$, $F(2\varkappa-1)\neq 0$. If $2\varkappa-1$ is not a power of 3, then $2\varkappa-1=3^bB$, where $B>1$, $3\nmid B$, consequently $B\geqq 5$, $3^b\in P$, $F(B)=G(B+1)\neq 0$, and the odd parts of both of 3^b+1, $B+1$ have to be 1, taking into account the minimality of $\varkappa$. But then $3^b+1=2^l$, whence $b=1$, $B=2^d-1$, $d\geqq 3$, and $2(2^s+1)-1=3\cdot(2^d-1)$, i.e. $2^{s+1}-3\cdot 2^d=-4$, which is impossible, since $s+1\geqq d\geqq 3$.

We finished the proof of our lemma.

Lemma 8. *If $\mathcal{P}$ contains at least two distinct odd prime powers, then $\mathcal{R}$ contains at least one odd number.*

Proof. Let $Q_1, Q_2\in\mathcal{P}$ be odd numbers. Assume first that $(Q_1, Q_2)=1$. If the lemma fails to hold, then $G(Q_1+1)\neq 0$, $G(Q_2+1)\neq 0$, $G(Q_1Q_2+1)\neq 0$, and so $Q_1+1=2^a$, $Q_2+1=2^b$, $Q_1Q_2+1=2^c$, $a>b\geqq 2$. Then $(2^c-1)=(2^a-1)(2^b-1)$ and the two sides of this equation are incongruent mod 2^b.

It remains the case when $Q_1=Q^u$, $Q_2=Q^v$ with some odd prime Q. Let $Q_1+1=$ $=2^a$, $Q_2+1=2^b$, $a>b\geqq 2$. Hence we get that $Q_j\equiv -1 \pmod 4$, i.e. that $Q\equiv -$ $-1 \pmod 4$, u, v are both odd numbers. First we observe that $Q^v+1|Q^u+1$. But then $v|u$, which can be proved easily. Assume that $u=kv+r$, where $0\leqq r<v$. If k is an even number, $k=2h$,

$$Q^u+1 = Q^r(Q^{2hv}-1)+Q^r+1,$$

which by $Q^v+1|Q^{2hv}-1$ implies that $Q^v+1|Q^r+1$, and this cannot occur. If k is an odd number, then $Q^u+1=Q^r(Q^{kv}+1)+(1-Q^r)$, and by $Q^v+1|Q^{kv}+1$, $Q^v+1|Q^r-1$, which implies that $r=0$.

So we have, $u=kv$, k is odd. In the same way, starting from $Q^v|Q^u$, we deduce that $b|a$, $a=bt$. So we have

$$Q^v+1 = 2^b,\ Q^{kv}+1 = 2^{bt},\ t\geqq 2.$$

Then

$$2^{bt}-1 = (2^b-1)^k \equiv -1+(1^k)\cdot 2^b \pmod{2^{b+1}}$$

which is impossible for odd k.

The proof of our lemma is completed. By this we proved our theorem.

Remark. The general case $G(n+K)=F(n)$ can be treated similarly, at least for small fixed values of K, but it involves the knowledge of all solutions of Diophantine equations like $a^x-b^y=h$ for some values of a, b, h.

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