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Strong limit theorems for quasi-orthogonal random fields. II

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1. Introduction. Let $\{X_{ik}: i, k \geqq 1\}$ be a random field (in abbreviation: r.f.). We say that $\{X_{ik}\}$ is quasi-orthogonal if

$$EX_{ik}^2 = \sigma_{ik}^2 < \infty \tag{1.1}$$

and there exists a double sequence $\{\varrho(m, n): m, n \geqq 0\}$ of nonnegative numbers such that

$$|EX_{ik}X_{jl}| \leqq \varrho(|i-j|, |k-l|)\sigma_{ik}\sigma_{jl} \quad (i, j, k, l \geqq 1) \tag{1.2}$$

and

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varrho(m, n) < \infty. \tag{1.3}$$

In the special case when $\varrho(m, n)=0$ except $m=n=0$, we say that $\{X_{ik}\}$ is an orthogonal r.f.

2. Main results. We will study the almost sure (in abbreviation: a.s.) behavior of the Cesàro type means

$$\zeta_{mn} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right) X_{ik} \quad (m, n \geqq 1) \tag{2.1}$$

as $m+n \to \infty$.

Theorem 1. *If $\{X_{ik}\}$ is a quasi-orthogonal r.f. and*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sigma_{ik}^2}{i^2k^2} < \infty, \tag{2.2}$$

then

$$\lim_{m+n \to \infty} \zeta_{mn} = 0 \quad a.s. \tag{2.3}$$

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It is instructive to compare Theorem 1 with the corresponding result in [4, Theorem 1] according to which

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(i+1)]^2[\log(k+1)]^2<\infty \tag{2.4}$$

is a sufficient (and in the monotonic case, necessary) condition for the following strong law of large numbers:

$$\lim_{m+n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}X_{ik}=0 \quad \text{a.s.} \tag{2.5}$$

The surprising fact is that the logarithmic factors are missing in condition (2.2). We note that the logarithms are to the base 2 in this paper.

We will prove Theorem 1 in a more general setting which provides information on the rate of convergence in (2.3). In the sequel, p and q denote nonnegative integers.

Proposition 1. *If the conditions of Theorem* 1 *are satisfied and* $\varepsilon>0$, *then*

$$P[\sup_{m\ge 2^p}\sup_{n\ge 2^q}|\zeta_{mn}|>\varepsilon]=O(1)\Big\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}+ \\ +\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\Big\}. \tag{2.6}$$

Applying the well-known Kronecker lemma (see, e.g. [5, p. 35]), Proposition 1 implies Theorem 1.

We note that a result analogous to Proposition 1 was proved in [3, Theorem 4] for sequences of random variables (in abbreviation: r.v.'s).

We also consider other Cesàro type means defined by

$$\tau_{nm}=\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big(1-\frac{i-1}{m}\Big)X_{ik}\quad (m,n\ge 1). \tag{2.7}$$

Clearly, the τ_{mn} are intermediate between the rectangular arithmetic means occurring in (2.5) and the means (2.1).

Theorem 2. *If* $\{X_{ik}\}$ *is a quasi-orthogonal r.f. and*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2<\infty, \tag{2.8}$$

then

$$\lim_{m+n\to\infty}\tau_{mn}=0 \quad a.s. \tag{2.9}$$

A more general statement giving information on the convergence rate in (2.9) reads as follows.

Proposition 2. *If the conditions of Theorem* 2 *are satisfied and* $\varepsilon>0$, *then*

$$P[\sup_{m\geqq 2^p}\sup_{n\geqq 2^q}|\tau_{mn}|>\varepsilon]=O(1)\Big\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+ \tag{2.10}$$

$$+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}[\log(k+1)]^2+\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+$$

$$+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2\Big\}.$$

Condition (2.8) lies between (2.2) and (2.4) (cf. conclusions (2.3), (2.5), and (2.9)).

We guess that the logarithmic factor in condition (2.8) is exact.

Conjecture. If $\{\sigma_{ik}\geqq 0\}$ is a double sequence such that

$$\frac{\sigma_{ik}}{k}\geqq\frac{\sigma_{i,k+1}}{k+1}\quad (i,k\geqq 1)$$

and

$$\sum_{i=r}^{\infty}\sum_{k=r}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}[\log(k+1)]^2=\infty \tag{2.11}$$

with $r=1$, then there exists an orthogonal r.f. $\{X_{ik}\}$ such that

$$EX_{ik}=0,\ EX_{ik}^2\leqq\sigma_{ik}^2\quad (i,k\geqq 1)$$

and

$$\limsup_{m+n\to\infty}|\tau_{mn}|=\infty\quad \text{a.s.}$$

If condition (2.11) is satisfied with any $r\geqq 1$, then we can state

$$\limsup_{m,n\to\infty}|\tau_{mn}|=\infty\quad \text{a.s.}$$

3. Proof of Proposition 1. We begin with a known result [2].

Lemma 1. *If* $\{X_{ik}\}$ *satisfies conditions* (1.1)—(1.3), *and* $\{a_{ik}\}$ *is any sequence of numbers, then*

$$E[\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}a_{ik}X_{ik}]^2=O(1)\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n}a_{ik}^2\sigma_{ik}^2\quad (a,b\geqq 0;\ m,n\geqq 1). \tag{3.1}$$

We emphasize that in the proofs of Propositions 1 and 2 the condition that $\{X_{ik}\}$ is a quasi-orthogonal r.f. is used only to the extent that this implies the moment inequality (3.1).

Now we turn to the proof of Proposition 1. We start with the inequality

$$(3.2)\qquad P[\sup_{m\geqq 2^p}\sup_{n\geqq 2^q}|\zeta_{mn}|>\varepsilon]\leqq\sum_{r=p}^{\infty}\sum_{s=q}^{\infty}P[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\zeta_{mn}|>\varepsilon].$$

Let $2^r\leqq m\leqq 2^{r+1}$ and $2^s\leqq n\leqq 2^{s+1}$. Since

$$(3.3)\qquad \zeta_{mn}=\zeta_{2^r,2^s}+(\zeta_{m,2^s}-\zeta_{2^r,2^s})+(\zeta_{2^r,n}-\zeta_{2^r,2^s})+(\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s})$$

we can estimate as follows

$$(3.4)\qquad P[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\zeta_{mn}|>\varepsilon]\leqq P\left[[|\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right]+\sum_{j=1}^{3}P_{rs}^{(j)},$$

where

$$P_{rs}^{(1)}=P\left[\max_{2^r<m\leqq 2^{r+1}}|\zeta_{m,2^s}-\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right],$$

$$P_{rs}^{(2)}=P\left[\max_{2^s<n\leqq 2^{s+1}}|\zeta_{2^r,n}-\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right],$$

$$P_{rs}^{(3)}=P\left[\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}}|\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right].$$

By the Chebyshev inequality and (3.1),

$$(3.5)\qquad P\left[|\zeta_{2^r,2^s}|>\frac{\varepsilon}{4}\right]\leqq\frac{16}{\varepsilon^2}E\zeta_{2^r,2^s}^2=\frac{O(1)}{\varepsilon^2}\sum_{i=1}^{2^r}\sum_{k=1}^{2^s}\sigma_{ik}^2.$$

By the Cauchy inequality,

$$(3.6)\qquad [\max_{2^r<m\leqq 2^{r+1}}|\zeta_{m,2^s}-\zeta_{2^r,2^s}|]^2\leqq\sum_{m=2^r+1}^{2^{r+1}}m[\zeta_{m,2^s}-\zeta_{m-1,2^s}[^2.$$

An elementary calculation shows that

$$\zeta_{m,2^s}-\zeta_{m-1,2^s}=\sum_{i=1}^{m}\sum_{k=1}^{2^s}a_{ik}(m,s)X_{ik}$$

where

$$a_{ik}(m,s)=\frac{1}{2^s}\left(1-\frac{k-1}{2^s}\right)\left[\frac{(i-1)(2m-1)}{m^2(m-1)^2}-\frac{1}{m(m-1)}\right].$$

Clearly,

$$|a_{ik}(m,s)|\leqq\frac{1}{m(m-1)2^s}.$$

Hence, by the Chebyshev inequality and (3.1),

$$(3.7)\qquad P_{rs}^{(1)}\leqq\frac{16}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}mE[\zeta_{m,2^s}-\zeta_{m-1,2^s}]^2=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=1}^{2^s}\frac{\sigma_{ik}^2}{m(m-1)^2 2^{2s}}.$$

The symmetric counterpart of (3.7) is

$$P_{rs}^{(2)} = \frac{O(1)}{\varepsilon^2} \sum_{n=2^s+1}^{2^{s+1}} \sum_{i=1}^{2^r} \sum_{k=1}^{n} \frac{\sigma_{ik}^2}{n(n-1)^2 2^{2r}}. \tag{3.8}$$

Finally, by the Cauchy inequality,

$$\Big[\max_{2^r<m\leqq 2^{r+1}} \max_{2^s<n\leqq 2^{s+1}} |\zeta_{mn}-\zeta_{m,2^s}-\zeta_{2^r,n}+\zeta_{2^r,2^s}|\Big]^2 \leqq$$

$$\leqq \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} mn[\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1,n-1}]^2$$

and by an elementary calculation,

$$\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1\,n-1} = \sum_{i=1}^{m} \sum_{k=1}^{n} b_{ik}(m,n) X_{ik}$$

where

$$b_{ik}(m,n) = \left[\frac{(i-1)(2m-1)}{m^2(m-1)^2} - \frac{1}{m(m-1)}\right]\left[\frac{(k-1)(2n-1)}{n^2(n-1)^2} - \frac{1}{n(n-1)}\right].$$

Clearly,

$$|b_{ik}(m,n)| \leqq \frac{1}{m(m-1)n(n-1)}.$$

Hence, by the Cauchy inequality and (3.1),

$$P_{rs}^{(3)} \leqq \frac{16}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} mnE[\zeta_{mn}-\zeta_{m-1,n}-\zeta_{m,n-1}+\zeta_{m-1,n-1}]^2 = \tag{3.9}$$

$$= \frac{O(1)}{\varepsilon^2} \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^s+1}^{2^{s+1}} \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\sigma_{ik}^2}{m(m-1)^2 n(n-1)^2}.$$

Next, we combine the above estimates in four parts.

Part 1. By (3.2) and (3.5), while decomposing the inner double sum and interchanging the order of summations, we get that

$$\sum_{r=p}^{\infty} \sum_{s=q}^{\infty} P\left[|\zeta_{2^r,2^s}| > \frac{\varepsilon}{4}\right] = O(1) \sum_{r=p}^{\infty} \sum_{s=q}^{\infty} \frac{1}{2^{2r}2^{2s}} \times \tag{3.10}$$

$$\times \Big\{ \sum_{i=1}^{2^p} \sum_{k=1}^{2^q} + \sum_{i=1}^{2^p} \sum_{k=2^q+1}^{2^s} + \sum_{i=2^p+1}^{2^r} \sum_{k=1}^{2^q} + \sum_{i=2^p+1}^{2^r} \sum_{k=2^q+1}^{2^s} \Big\} \sigma_{ik}^2 =$$

$$= O(1) \left\{ \frac{1}{2^{2p}2^{2q}} \sum_{i=1}^{2^p} \sum_{k=1}^{2^q} \sigma_{ik}^2 + \frac{1}{2^{2p}} \sum_{i=1}^{2^p} \sum_{k=2^q+1}^{\infty} \frac{\sigma_{ik}^2}{k^2} + \right.$$

$$\left. + \frac{1}{2^{2q}} \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{2^q} \frac{\sigma_{ik}^2}{i^2} + \sum_{i=2^p+1}^{\infty} \sum_{k=2^q+1}^{\infty} \frac{\sigma_{ik}^2}{i^2k^2} \right\}.$$

Part 2. By (3.2) and (3.7), we obtain in a similar way that

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(1)} = O(1) \sum_{m=2^p+1}^{\infty} \sum_{s=q}^{\infty}\sum_{i=1}^{m}\Big\{\sum_{k=1}^{2^q}+\sum_{k=2^q+1}^{2^s}\Big\}\frac{\sigma_{ik}^2}{m^3 2^{2s}} = \tag{3.11}$$

$$= O(1)\left\{\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{k^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\right\}.$$

Part 3. By (3.2) and (3.8),

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(2)} = O(1)\left\{\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\right\}. \tag{3.12}$$

Part 4. By (3.2) and (3.9),

$$\sum_{r=p}^{\infty}\sum_{s=q}^{\infty} P_{rs}^{(3)} = O(1) \sum_{m=2^p+1}^{\infty}\sum_{n=2^q+1}^{\infty}\Big\{\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}+\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{n}+\sum_{i=2^p+1}^{m}\sum_{k=1}^{2^q}+ \tag{3.13}$$

$$+\sum_{i=2^p+1}^{m}\sum_{k=2^q+1}^{n}\Big\}\frac{\sigma_{ik}^2}{m^3n^3} = O(1)\left\{\frac{1}{2^{2p}2^{2q}}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\sigma_{ik}^2+\frac{1}{2^{2p}}\sum_{i=1}^{2^p}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{k^2}+\right.$$

$$\left.+\frac{1}{2^{2q}}\sum_{i=2^p+1}^{\infty}\sum_{k=1}^{2^q}\frac{\sigma_{ik}^2}{i^2}+\sum_{i=2^p+1}^{\infty}\sum_{k=2^q+1}^{\infty}\frac{\sigma_{ik}^2}{i^2k^2}\right\}.$$

Collecting (3.2) and (3.10)—(3.13) yields (2.6) to be proved.

4. Proof of Proposition 2. This proof is essentially a combination of the techniques of Section 3 and the proof of [4, Proposition 1]. Therefore, we do not go into full details.

The next lemma is a version of the well-known Rademacher—Menshov inequality (see, e.g. [1, Theorem 2]).

Lemma 2. *If* $\{X_{ik}\}$ *satisfies conditions* (1.1)—(1.3), *and* $\{a_{ik}\}$ *is any sequence of numbers, then*

$$E\Big[\max_{1\le l\le n}\Big|\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+l} a_{ik}X_{ik}\Big|\Big]^2 = O(1)[\log 2n]^2\sum_{i=a+1}^{a+m}\sum_{k=b+1}^{b+n} a_{ik}^2\sigma_{ik}^2 \tag{4.1}$$

$$(a, b \ge 0;\ m, n \ge 1).$$

To start the proof of Proposition 2, assume that $2^r\le m\le 2^{r+1}$ and $2^s\le n\le 2^{s+1}$ with nonnegative integers r and s. Obviously, it is enough to prove (2.10) for the slightly modified means

$$\tau_{mn}^* = \frac{1}{m2^s}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)X_{ik}$$

in the place of τ_{mn}. We use a decomposition analogous to (3.3), according to which we can write

$$P\big[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\tau^*_{mn}|>\varepsilon\big]\leqq P\Big[|\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big]+\sum_{j=1}^{3}Q^{(j)}_{rs} \tag{4.2}$$

(cf. (3.4)), where

$$Q^{(1)}_{rs}=P\Big[\max_{2^r<m\leqq 2^{r+1}}|\tau^*_{m,2^s}-\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big],$$

$$Q^{(2)}_{rs}=P\Big[\max_{2^s<n\leqq 2^{s+1}}|\tau^*_{2^r,n}-\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big],$$

$$Q^{(3)}_{rs}=P\Big[\max_{2^r<m\leqq 2^{r+1}}\max_{2^s<n\leqq 2^{s+1}}|\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big].$$

Imitating the corresponding steps in the proof of Proposition 1, it is easy to verify that

$$P\Big[|\tau^*_{2^r,2^s}|>\frac{\varepsilon}{4}\Big]=\frac{O(1)}{\varepsilon^2}\sum_{i=1}^{2^r}\sum_{k=1}^{2^s}\sigma^2_{ik} \tag{4.3}$$

and

$$Q^{(1)}_{rs}=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=1}^{2^s}\frac{\sigma^2_{ik}}{m(m-1)^2 2^{2s}} \tag{4.4}$$

(cf. (3.5) and (3.7), respectively).

The following two estimates are different from (3.8) and (3.9). By the Chebyshev inequality and (4.1),

$$\begin{aligned}Q^{(2)}_{rs}&=\frac{O(1)}{\varepsilon^2}\frac{[\log 2^{s+1}]^2}{2^{2r}2^{2s}}\sum_{i=1}^{2^r}\sum_{k=2^s+1}^{2^{s+1}}\Big(1-\frac{i-1}{m}\Big)^2\sigma^2_{ik}=\\&=\frac{O(1)}{\varepsilon^2}\frac{1}{2^{2r}}\sum_{i=1}^{2^r}\sum_{k=2^s+1}^{2^{s+1}}\frac{\sigma^2_{ik}}{k^2}[\log 2k]^2.\end{aligned} \tag{4.5}$$

To estimate $Q^{(3)}_{rs}$, we set $\eta_{mn}=\tau^*_{mn}-\tau^*_{m,2^s}$. Then

$$\eta_{mn}-\eta_{2^r,n}=\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}.$$

Similarly to the reasoning in (3.6) we estimate as follows

$$\begin{aligned}&\big[\max_{2^r\leqq m\leqq 2^{r+1}}\max_{2^s\leqq n\leqq 2^{s+1}}|\tau^*_{mn}-\tau^*_{m,2^s}-\tau^*_{2^r,n}+\tau^*_{2^r,2^s}|\big]^2\leqq\\&\leqq\sum_{m=2^r+1}^{2^{r+1}}m\big[\max_{2^s<n\leqq 2^{s+1}}|\eta_{mn}-\eta_{m-1,n}|\big]^2.\end{aligned}$$

A simple computation shows that

$$\eta_{mn}-\eta_{m-1,n}=\sum_{i=1}^{m}\sum_{k=2^s+1}^{n}c_{ik}(m,n)X_{ik},$$

where

$$c_{ik}(m,n)=\frac{1}{2^s}\left[\frac{(i-1)(2m-1)}{m^2(m-1)^2}-\frac{1}{m(m-1)}\right].$$

Clearly,

$$|c_{ik}(m,n)|\leqq\frac{1}{m(m-1)2^s}.$$

Thus, by the Chebyshev inequality and (4.1),

$$Q_{rs}^{(3)}=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}m[\log 2^{s+1}]^2\sum_{i=1}^{m}\sum_{k=2^s+1}^{2^{s+1}}c_{ik}^2(m,n)\sigma_{ik}^2= \tag{4.6}$$

$$=\frac{O(1)}{\varepsilon^2}\sum_{m=2^r+1}^{2^{r+1}}\sum_{i=1}^{m}\sum_{k=2^s+1}^{2^{s+1}}\frac{\sigma_{ik}^2}{m(m-1)^2 2^{2s}}[\log 2k]^2=\frac{O(1)}{\varepsilon^2}\sum_{i=1}^{2^{r+1}}\sum_{k=2^s+1}^{2^{s+1}}\frac{\sigma_{ik}^2}{2^{2r}k^2}[\log 2k]^2.$$

Now to complete the proof on the basis of (4.2)—(4.6) we have to go along the same lines as in the proof of Proposition 1 (cf. Parts 1—4 there).

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