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General results on strong approximation by orthogonal series

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1. Introduction. Let $\{\varphi_n(x)\}$ denote an orthonormal system on a finite interval (a, b). In this paper we shall consider real orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.1}$$

It is well known that the partial sums $s_n(x)$ of any such series converge in the L^2 norm to a function $f(x) \in L^2(a, b)$.

The following theorem, proved in [6], provides a quantitative estimate for the pontwise approximation of $f(x)$ by the arithmetic means of $s_n(x)$:

Let $0<\gamma<1$. *If*

$$\sum_{n=0}^{\infty} c_n^2 n^{2\gamma} < \infty, \tag{1.2}$$

then

$$\frac{1}{n+1} \sum_{k=0}^{n} s_k(s) - f(x) = o_x(n^{-\gamma})$$

almost everywhere (a.e.) in (a, b).

This result was extended by G. Sunouchi [17] to strong approximation. Earlier G. Alexits, who was first to propose the problem of strong approximation, in cooperation with his coauthors established various results pertaining to Fourier series [2], [3]. As far as we know it was Sunouchi's result the first to deal with strong approximation by general orthogonal series. His result reads as follows:

Let $0<\gamma<1$ *and* $\varkappa>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$\left\{ \frac{1}{A_n^{\varkappa}} \sum_{k=0}^{n} A_{n-k}^{\varkappa-1} |s_k(x) - f(x)|^p \right\}^{1/p} = o_x(n^{-\gamma})$$

a.e. in (a, b), *where* $A_n^{\varkappa} = \binom{n+\varkappa}{n}$.

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After several articles of the first author have dealt with strong approximation [9], [10], [11], the following two general results (the first for Cesàro means, the second for Riesz means) were established by the first author and H. SCHWINN [14]:

Theorem A. *Let* $\gamma>0$, $\varkappa>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$C_n(f, \varkappa, p, \mathbf{v}; x) := \left\{\frac{1}{A_n^{\varkappa}} \sum_{k=0}^{n} A_{n-k}^{\varkappa-1} |s_{\nu_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.3}$$

a.e. in (a, b) *for any increasing sequence* $\mathbf{v}:=\{\nu_k\}$ *of positive integers.*

Theorem B. *Let* $\gamma>0$, $\beta>0$. *If* (1.2) *holds and* $0<p\gamma<\beta$, *then*

$$R_n(f, \beta, p, \mathbf{v};\ x) := \left\{(n+1)^{-\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_{\nu_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.4}$$

a.e. in (a, b) *for any increasing sequence* $\mathbf{v}:=\{\nu_k\}$ *of positive integers.*

L. REMPULSKA [15] investigated the approximation properties of generalized Abel means of orthogonal series. One of her results, relevant to our present interest, is as follows:

Let q *be a non-negative integer and* $\gamma>0$. *If* (1.2) *holds, then*

$$(1-t)^{q+1} \sum_{k=0}^{\infty} \binom{q+k}{k} t^k s_k(x) - f(x) = \begin{cases} o_x((1-t)^{\gamma}) & \text{if } q+1>\gamma, \\ o_x((1-t)^{\gamma}|\log(1-t)|) & \text{if } q+1=\gamma, \\ O_x((1-t)^{q+1}) & \text{if } q+1<\gamma, \end{cases}$$

a.e. in (a, b), *as* $t\to 1^-$.

This result was extended to strong Abel means, by the first author, in [8]:

Theorem C. *Let* q *be a non-negative integer and* $\gamma>0$. *If* (1.2) *holds and* $0<p\gamma<1$, *then*

$$Q(f, q, p, \mathbf{v};\ t) := \left\{(1-t)^{q+1} \sum_{k=0}^{\infty} \binom{q+k}{k} t^k |s_{\nu_k}(x)-f(x)|^p\right\}^{1/p} = o_x((1-t)^{\gamma});$$

furthermore if $p\gamma\geqq 1$ *and* $p\leqq 2$, *then*

$$Q(f, q, p, \mathbf{v};\ t) = \begin{cases} o_x((1-t)^{\gamma}) & \text{if } q+1>p\gamma, \\ o_x((1-t)^{\gamma}|\log(1-t)|^{1/p}) & \text{if } q+1=p\gamma, \\ O_x((1-t)^{(q+1)/p}) & \text{if } q+1<p\gamma, \end{cases}$$

hold a.e. in (a, b), *as* $t\to 1^-$, *for any increasing sequence* $\mathbf{v}:=\{\nu_k\}$ *of positive integers.*

An investigation, pertaining to the Riesz means dealing with a question similar to the special case when $q+1=p\gamma$ in Theorem C, was started in [10]. These results are often referred to as "limit case" theorems, since the restrictions, concerning the parameters, $p\gamma<1$ and $p\gamma<\beta$ are replaced by $p\gamma=1$ and $p\gamma=\beta$, respectively.

Theorem D. *Let $\varkappa$ and p be positive numbers. If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2/p} < \infty,$$

then

$$C_n(f, \varkappa, p, \mathbf{v};\ x) = o_x\big(n^{-1/p}(\log n)^{1/p}\big)$$

a.e. in (a, b) for any increasing sequence $\mathbf{v}:=\{v_k\}$ of positive integers.

This corresponds to the case $\gamma=1/p$.

Theorem E. *Let β and p be positive numbers. If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\beta/p} < \infty,$$

then

$$R_n(f, \beta, p, \mathbf{v};\ x) = o_x\big(n^{-\beta/p}(\log n)^{1/p}\big)$$

a.e. in (a, b) for any increasing sequence $\mathbf{v}:=\{v_k\}$ of positive integers.

This corresponds to the case $\gamma=\beta/p$.

The aim of our present paper is to extend these results of strong approximation to certain more general classes of strong summation methods. These methods will include, as we shall show, a large family of Hausdorff transformations and $[J,f]$-transformations. We hope that the forthcoming result will throw additional light on the common kernel of the previously established results.

2. The main result. Let $\boldsymbol{\alpha}:=\{\alpha_k(\omega)\}$, $k=0, 1, \dots$ denote a sequence of non-negative functions defined for $0\leqq\omega<\infty$, satisfying

$$\sum_{k=0}^{\infty} \alpha_k(\omega) \equiv 1.$$

We shall assume that the linear transformation of real sequences $\mathbf{x}:=\{x_k\}$ given by

$$A_\omega(\mathbf{x}) := \sum_{k=0}^{\infty} \alpha_k(\omega)\, x_k, \quad \omega\to\infty$$

is regular [4; p. 49]. Let $\gamma:=\gamma(t)$ and $g(t)$ denote non-decreasing positive functions defined for $0\leqq t<\infty$, furthermore let $\boldsymbol{\mu}:=\{\mu_m\}$, $m=0, 1, \dots$ denote a fixed, increasing sequence of integers with $\mu_0=0$. We shall assume throughout this paper that the following conditions are satisfied:

There exist positive integers N and h so that

$$\mu_{m+1} \leqq N \cdot \mu_m, \quad m = 1, 2, \dots \tag{2.1}$$

$$\gamma(\mu_{m+1}) \leqq N \cdot \gamma(\mu_m), \quad m = 1, 2, \dots \tag{2.2}$$

$$\gamma(\mu_{m+h}) \geqq 2\gamma(\mu_m), \quad m = 1, 2, \dots . \tag{2.3}$$

For $r>1$, $\omega>0$ and $m=0, 1, \dots$ we define

$$\varrho_m(\omega, r) := \left\{\frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\overline{\mu_{m+1}-1}} (\alpha_k(\omega))^r\right\}^{1/r}. \tag{2.4}$$

In terms of the quantities introduced above we are ready to state our main result.

Theorem 1. *Let $p>0$. Suppose that there exist $r>1$ and a constant $K(r, \boldsymbol{\mu}, \gamma)$ such that for any $\omega>0$*

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K(r, \boldsymbol{\mu}, \gamma)\big(g(\omega)/\gamma(\omega)\big)^p. \tag{2.5}$$

If

$$\sum_{n=1}^{\infty} c_n^2 \gamma(n)^2 < \infty, \tag{2.6}$$

then

$$A_\omega(f, p, \mathbf{v};\ x) := \Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)\, |s_{\nu_k}(x) - f(x)|^p\Big\}^{1/p} = O_x\big(g(\omega)/\gamma(\omega)\big) \tag{2.7}$$

a.e. in (a, b) for any increasing sequence $\mathbf{v} := \{\nu_k\}$ of positive integers.

If, in addition, for every fixed m,

$$\varrho_m(\omega, r) = o\big((g(\omega)/\gamma(\omega))^p\big), \quad \text{as} \quad \omega \to \infty, \tag{2.8}$$

then the O_x in (2.7) can be replaced by o_x.

We mention that the most important special case of Theorem 1 is when both (2.5) and (2.8) are satisfied with $g(\omega)\equiv 1$. In this case we get that

$$A_\omega(f, p, \mathbf{v};\ x) = o_x\big(\gamma(\omega)^{-1}\big) \tag{2.9}$$

holds a.e. in (a, b).

3. Lemmas. In order to prove Theorem 1 and for its applications we need a number of results; some were proved earlier, others will be proven here. In what follows K will denote absolute constants and $K(.)$ constants depending only on those parameters as indicated.

Lemma 1. [13]. *If $\{a_m\}$ is a sequence of non-negative numbers, then*

$$\sum_{m=1}^{n} a_m \leqq K a_n, \quad n = 1, 2, \dots$$

hold if and only if there exist positive integers N and s so that

$$a_{m+1} \leqq N a_m \quad \text{and} \quad a_{m+s} \geqq 2a_m, \quad m = 1, 2, \ldots .$$

Lemma 2. [7]. *Let $\{\lambda_m\}$ be an increasing sequence of positive integers, let $\{\gamma_m\}$ be a non-decreasing sequence of positive numbers so that*

$$\sum_{m=1}^{n} \gamma_{\lambda_m}^2 \leqq K \gamma_{\lambda_n}^2, \quad n = 1, 2, \ldots . \tag{3.1}$$

If

$$\sum_{n=1}^{\infty} c_n^2 \gamma_n^2 < \infty, \tag{3.2}$$

then

$$s_{\lambda_n}(x) - f(x) = o_x(\gamma_{\lambda_n}^{-1}) \tag{3.3}$$

a.e. in (a, b).

Lemma 3. [10]. *Let $\delta > 0$ and $\{\delta_n\}$ an arbitrary sequence of positive numbers. Suppose that for any orthonormal system the condition*

$$\sum_{n=1}^{\infty} \delta_n \Big(\sum_{k=n}^{\infty} c_k^2 \Big)^{\delta} < \infty$$

implies that the sequence $\{s_n(x)\}$ possesses a property P, then any subsequence $\{s_{\nu_n}(x)\}$ also possesses property P.

Lemma 4. *Let $\sigma_k(x) := (k+1)^{-1} \sum_{i=0}^{k} s_i(x)$, $k = 0, 1, \ldots$. If*

$$\sum_{n=0}^{\infty} c_n^2 < \infty,$$

then

$$\sum_{n=1}^{\infty} n \int_a^b (\sigma_n(x) - \sigma_{n-1}(x))^2 \, dx \leqq K \sum_{n=0}^{\infty} c_n^2; \tag{3.4}$$

and for every $p > 0$

$$\int_a^b \Big\{ \sup_{n \geqq 0} \Big((n+1)^{-1} \sum_{k=0}^{n} |s_k(x) - \sigma_k(x)|^p \Big)^{1/p} \Big\}^2 dx \leqq K(p) \sum_{n=0}^{\infty} c_n^2. \tag{3.5}$$

Inequality (3.4) can be found in [1] and (3.5) was proved in [16].

Lemma 5. *Let $p > 0$ and $M < N$ positive integers. Let*

$$\bar{\sigma}_n(x) = \begin{cases} 0, & \text{if } n \leqq M, \\ (n+1)^{-1} \sum_{k=M+1}^{n} (s_k(x) - s_M(x)), & \text{if } M < n \leqq N, \\ (n+1)^{-1} \sum_{k=M+1}^{N} (s_k(x) - s_M(x)), & \text{if } n > N. \end{cases} \tag{3.6}$$

Then

$$\sum_{n=M+1}^{N} \int_a^b n(\bar\sigma_n(x)-\bar\sigma_{n-1}(x))^2\,dx \leqq K \sum_{n=M+1}^{N} c_n^2, \tag{3.7}$$

and

$$\int_a^b \left\{\frac{1}{N+1}\sum_{n=M+1}^{N} |s_n(x)-s_M(x)-\bar\sigma_n(x)|^p\right\}^{2/p} \leqq K(p) \sum_{n=M+1}^{N} c_n^2. \tag{3.8}$$

Proof. Let

$$\bar c_k = \begin{cases} c_k, & \text{if } M<k\leqq N,\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to see that for the corresponding partial sums $\bar s_n(x)$ of (1.1) we have

$$\bar s_n(x) = \begin{cases} 0, & \text{if } n\leqq M,\\ s_n(x)-s_M(x), & \text{if } M<n\leqq N,\\ s_N(x)-s_M(x), & \text{if } N<n,\end{cases}$$

and therefore the $(C, 1)$-means $\bar\sigma_n(x)$ of $\{\bar s_n(x)\}$ are given by (3.6). The application of (3.4) to $\{\bar c_k\}$ clearly implies (3.7), the application of (3.5) to $\{\bar c_k\}$ implies (3.8).

Lemma 6. *Let $p>0$ and let $\sigma_n^*(x)$ be defined by*

$$\sigma_n^*(x) := \sigma_n^*(\mu;\, x) := \frac{1}{n+1}\sum_{k=\mu_m}^{n} (s_k(x)-s_{\mu_m}(x)) \tag{3.9}$$

for $\mu_m\leqq n<\mu_{m+1}$, $m=0, 1, \dots$. If (2.6) holds, then

$$\Delta_m(x) := \left\{\frac{1}{\mu_{m+1}}\sum_{k=\mu_m}^{\mu_{m+1}-1} |s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^p\right\}^{1/p} = o_x(\gamma(\mu_m)^{-1}) \tag{3.10}$$

a.e. in (a, b).

Proof. We set $M=\mu_m$ and $N=\mu_{m+1}-1$ with $m=0, 1, \dots$ successively into (3.8) and observe that for $\mu_m\leqq n<\mu_{m+1}$, $\sigma_n^*(x)$ equals $\bar\sigma_n(x)$ of (3.6). Multiplying by $\gamma(\mu_m)^2$ and summing over m, we get

$$\sum_{m=0}^{\infty}\int_a^b \gamma(\mu_m)^2\Delta_m^2(x)\,dx \leqq K(p)\sum_{m=0}^{\infty}\gamma(\mu_m)^2\sum_{k=\mu_m}^{\mu_{m+1}-1} c_k^2.$$

The sum on the right-hand side is finite on account of (2.2) and (2.6). This implies the required result.

Lemma 7. *Let $\sigma_n^*(x)$ be as defined by (3.9). If (2.6) holds, then*

$$\sigma_n^*(x) = o_x(\gamma(n)^{-1})$$

a.e. in (a, b).

Proof. Since $\sigma^*_{\mu_m}(x)=0$ for every m, we have

$$\max_{\mu_n\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2 \leqq \max_{\mu_m\leqq k<\mu_{m+1}} \Big(\sum_{j=\mu_m+1}^{k} |\sigma^*_j(x)-\sigma^*_{j-1}(x)|\Big)^2 \leqq \tag{3.11}$$

$$\leqq \Big(\sum_{j=\mu_m+1}^{\mu_{m+1}-1} |\sigma^*_j(x)-\sigma^*_{j-1}(x)|\Big)^2 \leqq K \sum_{j=\mu_m+1}^{\mu_{m+1}-1} j\,|\sigma^*_j(x)-\sigma^*_{j-1}(x)|^2,$$

where the last inequality is the consequence of the Schwarz inequality and (2.1). To the last expression we may apply (3.7) with $M=\mu_m$ and $N=\mu_{m+1}-1$, since in the required range $\sigma^*_j(x)=\bar\sigma_j(x)$. Thus we obtain from (3.11)

$$\int_a^b \max_{\mu_m\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2\,dx \leqq K \sum_{j=\mu_m+1}^{\mu_{m+1}-1} c_j^2. \tag{3.12}$$

It follows now from (3.12) that

$$\sum_{m=0}^{\infty} \gamma(\mu_m)^2 \int_a^b \max_{\mu_m\leqq k<\mu_{m+1}} |\sigma^*_k(x)|^2\,dx \leqq K \sum_{m=0}^{\infty} \gamma(\mu_m)^2 \sum_{j=\mu_m+1}^{\mu_{m+1}-1} c_j^2 \leqq K \sum_{k=0}^{\infty} \gamma(k)^2 c_k^2 < \infty,$$

on account of (2.2) and (2.6). The last inequality implies the required result using (2.2) once again.

Lemma 8. [4]. *Let $\{\alpha_k(n)\}$, the coefficients of a regular Hausdorff transformation, be given by*

$$\alpha_k(n) = \int_0^1 \binom{n}{k} t^k(1-t)^{n-k}\,\phi(t)\,dt, \tag{3.13}$$

where $\phi(t)\in L^r(0,1)$ for some $r>1$. Then

$$\sum_{k=0}^{n} |\alpha_k(n)|^r \leqq K(r)(n+1)^{1-r}. \tag{3.14}$$

Lemma 9. *Let $\{\alpha_k(\omega)\}$, the coefficients of a regular $[J,f]$-transformation, be given by*

$$\alpha_k(\omega) = \frac{\omega^k}{k!} \int_0^1 t^{\omega} (\log(1/t))^k\,\phi(t)\,dt, \tag{3.15}$$

where $\phi(t)\in L^r(0,1)$ for some $r>1$. Then for $l=0,1,\dots$

$$\sum_{k=l}^{\infty} |\alpha_k(\omega)|^r \leqq K(r)\big((1+\omega)^{-1} e^{-l/1+\omega}\big)^{r-1}. \tag{3.16}$$

Proof. Denote $\lambda_k(\omega, t)=(k!)^{-1}(\log(1/t))^k t^\omega$ and let $r^{-1}+s^{-1}=1$. By Hölder's inequality we get from (3.15)

$$|\alpha_k(\omega)|^r \leqq \Big(\int_0^1 \lambda_k(\omega, t)\,dt\Big)^{r-1} \cdot \int_0^1 \lambda_k(\omega, t)\,|\phi(t)|^r\,dt. \tag{3.17}$$

Now, we find by an easy calculation that for $k=0, 1, \ldots$

$$\int_0^1 \lambda_k(\omega, t)\,dt = \omega^k(1+\omega)^{-k-1} \leqq (1+\omega)^{-1}e^{-k/1+\omega} \tag{3.18}$$

and that

$$\sum_{k=0}^{\infty} \lambda_k(\omega, t) \equiv 1.$$

Inequality (3.16) is therefore a consequence of (3.17) and (3.18).

4. Proof of Theorem 1. First we carry out the proof when $\nu_k=k$. Using elementary considerations we see that

$$\Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)\,|s_k(x)-f(x)|^p\Big\}^{1/p} \leqq K(p)\big(\textstyle\sum_1+\sum_2+\sum_3\big), \tag{4.1}$$

where

$$\textstyle\sum_1 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^p\Big\}^{1/p},$$

$$\textstyle\sum_2 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|s_{\mu_m}(x)-f(x)|^p\Big\}^{1/p}, \tag{4.2}$$

$$\textstyle\sum_3 = \Big\{\displaystyle\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)\,|\sigma_k^*(x)|^p\Big\}^{1/p}.$$

Let $r^{-1}+s^{-1}=1$. By Hölder's inequality, using (2.2) and (3.10) with ps in place of p, we get

$$\begin{aligned}\textstyle\sum_1^p &\leqq \sum_{m=0}^{\infty}\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\omega)^r\Big\}^{1/r}\cdot\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1} |s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^{ps}\Big\}^{1/s} \leqq \\ &\leqq K\sum_{m=0}^{\infty} \mu_m\varrho_m(\omega, r)\cdot\Big\{\frac{1}{\mu_{m+1}}\sum_{k=\mu_m}^{\mu_{m+1}-1} |s_k(x)-s_{\mu_m}(x)-\sigma_k^*(x)|^{ps}\Big\}^{1/s} \leqq \\ &\leqq K\sum_{m=0}^{\infty} \mu_m\varrho_m(\omega, r)\cdot o_x\big(\gamma(\mu_m)^{-p}\big),\end{aligned} \tag{4.3}$$

a.e. in (a, b).

In order to estimate $\sum_2$ we use (3.3) with $\lambda_n=\mu_n$ and $\gamma_n:=\gamma(n)$, observing that (3.1) is satisfied due to our assumptions (2.2) and (2.3) and Lemma 1. By Hölder's inequality

$$\sum\nolimits_2^p \leqq \sum_{m=0}^{\infty}\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\omega)^r\Big\}^{1/r}\cdot\Big\{\sum_{k=\mu_m}^{\mu_{m+1}-1}|s_{\mu_m}(x)-f(x)|^{ps}\Big\}^{1/s} \leqq \tag{4.4}$$

$$\leqq K\sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)o_x\big(\gamma(\mu_m)^{-p}\big),$$

taking (2.1) and (2.2) into account.

For estimating $\sum_3$ we use Lemma 7. By Hölder's inequality

(4.5)

$$\sum\nolimits_3^p \leqq \sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)\cdot\Big\{\frac{1}{\mu_m}\sum_{k=\mu_m}^{\mu_{m+1}-1}|\sigma_k^*(x)|^{ps}\Big\}^{1/s} \leqq K\sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)o_x\big(\gamma(\mu_m)^{-p}\big).$$

Collecting these estimations and taking account of assumption (2.5) we immediately get the required result (2.7) when $\nu_k=k$.

If (2.8) is also satisfied, then the proof runs as follows. By (4.1)—(4.5) we have, when $\nu_k=k$, that

$$A_\omega(f, p, \nu;\ x)^p \leqq K\sum_{m=0}^{\infty}\mu_m\varrho_m(\omega, r)o_x\big(\gamma(\mu_m)^{-p}\big) \tag{4.6}$$

holds a.e. in (a, b).

Let now $\varepsilon>0$ be given. If x is a point where (4.6) holds, then let $M(x)$ be a positive integer such that for $m>M(x)$ the inequality $o_x\big(\gamma(\mu_m)^{-p}\big)<\varepsilon^p\gamma(\mu_m)^{-p}$ is valid. For such x we get from (4.6) that

$$\big(\gamma(\omega)/g(\omega)\big)^p A_\omega(f, p, \nu;\ x)^p \leqq K(x)\Big\{\sum_{m=0}^{M(x)}\mu_m\varrho_m(\omega, r)\gamma(\mu_m)^{-p}\Big\}\big(\gamma(\omega)/g(\omega)\big)^p +$$

$$+K\varepsilon^p\big(\gamma(\omega)/g(\omega)\big)^p\sum_{m=M(x)+1}^{\infty}\mu_m\varrho_m(\omega, r)\gamma(\mu_m)^{-p}.$$

When $\omega\to\infty$, the first sum on the right converges to zero by (2.8); and the second sum remains $O\big((g(\omega)/\gamma(\omega))^p\big)$, by (2.5).

Hence, for $\nu_k=k$,

$$A_\omega(f, p, \nu;\ x) = o_x\big(g(\omega)/\gamma(\omega)\big), \quad \text{as} \quad \omega\to\infty \tag{4.7}$$

clearly follows. Since (4.6) holds a.e. in (a, b), it follows that (4.7) also holds a.e. in (a, b). This completes the proof when $\nu_k=k$.

The statements of Theorem 1 in their generality — for arbitrary $\nu:=\{\nu_k\}$ — follow from the results just proved and (2.6) using Lemma 3 with $\delta=1$ and $\delta_n:=\gamma(n)^2-\gamma(n-1)^2$.

5. Applications. First we treat those results which can be derived from Theorem 1 in the special case when $g(\omega)\equiv 1$ and both (2.5) and (2.8) are satisfied.

(i) If

$$p_{nk}(t) = \binom{n}{k} t^k(1-t)^{n-k}, \quad k = 0, 1, \dots, n; \; n = 1, 2, \dots \tag{5.1}$$

and $\phi(t) \in L^1(0, 1)$ is a non-negative function with $\|\phi\|_1 = 1$, then the matrix $\{\alpha_k(n)\}$ defined by

$$\alpha_k(n) = \int_0^1 p_{nk}(t)\phi(t)\,dt, \quad k = 0, 1, \dots, n; \; n = 1, 2, \dots \tag{5.2}$$

yields the coefficients of a regular Hausdorff transformation. For these transformations we have the following result.

Theorem 2. *Let $\gamma > 0$. Suppose that $\{\alpha_k(n)\}$ is given by* (5.2), *where $\phi(t) \in L^r(0,1)$ with some $r > 1$. If* (1.2) *holds and*

$$0 < p\gamma < 1 - r^{-1}, \tag{5.3}$$

then

$$\Big\{\sum_{k=0}^{n} \alpha_k(n) |s_{\nu_k}(x) - f(x)|^p\Big\}^{1/p} = o_x(n^{-\gamma}) \tag{5.4}$$

a.e. in (a, b) for any increasing sequence $\{\nu_k\}$ of positive integers.

Corollary 2.1. *If $\{\alpha_k(n)\}$ is the matrix of a Cesàro $(C, \varkappa)$ or a Hölder $(H, \varkappa)$ transformation, then* (5.4) *holds whenever $0 < p\gamma < \min(1, \varkappa)$.*

Remark. Although Theorem 2 does not include Theorem A for arbitrary $\varkappa > 0$, if we take into acoount the special properties of the $(C, \varkappa)$ transformation matrix, we find easily that Theorem 1 is applicable. For, in this case, (2.5) with $g(t) \equiv 1$ and $\gamma(t) = t^\gamma$ will be satisfied if we choose $r(>1)$ so that $\varkappa > 1 - r^{-1}$.

Proof of Theorem 2. We wish to show that conditions (2.5) and (2.8) of Theorem 1 are satisfied if $[\omega] = n$, $\gamma(t) = t^\gamma$ and $g(\omega) \equiv 1$. From (3.14) we get

$$\varrho_m(\omega, r)\gamma(\omega)^p \leqq K(r)\mu_m^{-1/r}\omega^{(1/r)-1+p\gamma},$$

whence (2.8) follows by (5.3). Now we observe that in this case $\varrho_m(\omega, r) = 0$ if $\mu_m > \omega$. Hence, again from (3.14),

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K\omega^{(1/r)-1} \sum \mu_m^{1-(1/r)-p\gamma},$$

where the summation on the right is for $\mu_m \leqq \omega$. Because of the assumptions made on the sequence $\{\mu_m\}$, this last sum is $O(\omega^{1-(1/r)-p\gamma})$. This proves (2.5). The conclusion of Theorem 2 now follows from Theorem 1.

Proof of Corollary 2.1. Both the $(C, \varkappa)$ and $(H, \varkappa)$ transforms are Hausdorff transforms with $\phi_1(t)=\varkappa(1-t)^{\varkappa-1}$ and $\phi_2(t)=\Gamma(\varkappa)^{-1}(\log 1/t)^{\varkappa-1}$, respectively. If $\varkappa\geqq 1$, then $\phi_i(t)\in L^r(0, 1)$, for arbitrary large r, hence (5.3) will hold whenever $0<p\gamma<1$ and r is large enough. If $0<\varkappa<1$, then $\phi_i(t)\in L^r(0, 1)$ if $r^{-1}>1-\varkappa$, hence in this case (5.3) holds whenever $0<p\gamma<1-\frac{1}{r}<\varkappa$.

(ii) If

$$\lambda_k(\omega, t) = \frac{(\omega \log(1/t))^k}{k!} t^\omega, \quad k = 0, 1, \ldots \tag{5.5}$$

and $\phi(t)\in L^1(0, 1)$ is a non-negative function with $\|\phi\|_1=1$, then the function-sequence $\{\alpha_k(\omega)\}$ defined by

$$\alpha_k(\omega) = \int_0^1 \lambda_k(\omega, t)\phi(t)\, dt, \quad k = 0, 1, \ldots \tag{5.6}$$

yields the coefficients of a regular $[J, f]$-transformation. For this transformation we have the following result.

Theorem 3. *Let $\gamma>0$. Suppose that $\{\alpha_k(\omega)\}$ is given by* (5.6), *where $\phi(t)\in L^r(0, 1)$ with some $r>1$. If* (1.2) *holds and*

$$0 < p\gamma < 1-r^{-1}, \tag{5.7}$$

then

$$\Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)|s_{\nu_k}(x)-f(x)|^p\Big\}^{1/p} = o_x(\omega^{-\gamma}) \tag{5.8}$$

a.e. in (a, b) for any increasing sequence $\{\nu_k\}$ of positive integers.

Corollary 3.1. *If $\{\alpha_k(\omega)\}$ is the coefficient-sequence of the Abel transformation, then* (5.8) *holds whenever $0<p\gamma<1$.*

Proof of Theorem 3. We shall show that the conditions (2.5) and (2.8) of Theorem 1 are satisfied in this case with $\gamma(t)=t^\gamma$ and $g(\omega)\equiv 1$. From (3.16) we obtain

$$\varrho_m(\omega, r)\gamma(\omega)^p \leqq K(r)\mu_m^{-1/r}\omega^{(1/r)-1+p\gamma},$$

whence (2.8) follows by (5.7). Also from (3.16)

$$\sum_{\mu_m\leqq\omega} \mu_m\varrho_m(\omega, r)\mu_m^{-p\gamma} \leqq \frac{K(r)}{(1+\omega)^{1-1/r}}\cdot \sum_{\mu_m\leqq\omega} \mu_m^{1-(1/r)-p\gamma} \leqq K(r)\omega^{-p\gamma}, \tag{5.9}$$

due to the assumptions concerning $\{\mu_m\}$. Finally, again from (3.16),

$$\sum_{\mu_m>\omega}\mu_m\varrho_m(\omega, r)\mu_m^{-p\gamma} \leqq K(r)\cdot\sum_{\mu_m>\omega}\left(\frac{\mu_m}{1+\omega}\right)^{1-1/r} e^{-\frac{\mu_m}{1+\omega}\left(1-\frac{1}{r}\right)}\mu_m^{-p\gamma} \leqq \tag{5.10}$$

$$\leqq K(r)\sum_{\mu_m>\omega}\mu_m^{-p\gamma} \leqq K(r)\omega^{-p\gamma},$$

due to the fact that $xe^{-x}<1$ for $x>0$ and the properties of $\{\mu_m\}$.

Inequalities (5.9) and (5.10) prove (2.5), hence Theorem 3 is a consequence of Theorem 1.

Proof of Corollary 3.1. If $\phi(t)\equiv 1$, then $\alpha_k(\omega)=\omega^k/(1+\omega)^{k+1}$ for $k=0, 1, \ldots$, which yield the classical Abel transformation. In this case, clearly, $\phi(t)\in L^r(0, 1)$ for any $r>0$, hence the result follows from Theorem 3.

(iii) If the function $\phi(t)$ in (5.2) satisfies

$$0 \leqq \phi(t) \leqq K(\beta)t^{\beta-1},$$

with $\beta>0$, then it is easy to see that

$$\alpha_k(n) \leqq K(\beta)\frac{(k+1)^{\beta-1}}{(n+1)^\beta} \tag{5.11}$$

for $0\leqq k\leqq n$, $n=1, 2, \ldots$. Using (5.11) one can establish by easy estimations that in these cases (2.5) and (2.8) hold whenever $\gamma(t)=t^\gamma$, $g(t)\equiv 1$ and $0<p\gamma<\beta$. For example if $\phi(t)=\beta t^{\beta-1}$, then

$$\alpha_k(n) = \beta\frac{n!}{\Gamma(n+\beta+1)}\frac{\Gamma(k+\beta)}{k!}, \quad k = 0, 1, \ldots, n,$$

which yield, essentially, the Riesz transformation of order β. Hence Theorem B follows from Theorem 1.

(iv) If the function $\phi(t)$ in (5.6) satisfies

$$0 \leqq \phi(t) \leqq K(q)\left(\log\frac{1}{t}\right)^q$$

with $q\geqq 0$, then easy calculations yield that

$$\alpha_k(\omega) \leqq K(q)\frac{(k+1)^q}{(\omega+1)^{q+1}}\left(\frac{\omega}{\omega+1}\right)^k \tag{5.12}$$

for $k=0, 1, \ldots$. Using (5.12) it is not difficult to show that in these cases (2.5) and (2.8) hold whenever $0<p\gamma<q+1$. For example, if $\phi(t)=\frac{1}{\Gamma(q+1)}\left(\log\frac{1}{t}\right)^q$,

$q \geqq 0$, then

$$\alpha_k(\omega) = (\omega+1)^{-q-1}\binom{k+q}{k}\cdot\left(\frac{\omega}{\omega+1}\right)^k, \quad k = 0, 1, \dots$$

which yields the generalized Abel transforms of order $q+1$, $q \geqq 0$. Hence the first statement of Theorem C with the relaxed condition $0<p\gamma<q+1$ follows from Theorem 1 for all $q \geqq 0$.

It seems worthwhile mentioning that Theorem 1 with suitable choices of $\gamma(t)$ and $\{\mu_m\}$ can also be applied to strong approximation by certain Nörlund and Riesz means having the form

$$N_n^\lambda(f, p;\ x) := \left\{\frac{1}{\lambda(n)}\sum_{k=0}^{n-1}(\lambda(n-k)-\lambda(n-k-1))|s_k(x)-f(x)|^p\right\}^{1/p}$$

and

$$R_n^\lambda(f, p;\ x) := \left\{\frac{1}{\lambda(n)}\sum_{k=0}^{n-1}(\lambda(k+1)-\lambda(k))|s_k(x)-f(x)|^p\right\}^{1/p}$$

where $\lambda = \{\lambda(n)\}$ denotes an increasing unbounded sequence of positive numbers satisfying

$$\lambda(n) \geqq cn^\varepsilon \quad \text{or} \quad \lambda(n)-\lambda(n-1) \leqq \lambda(n)n^{-\varepsilon}$$

with $c>0$ and $\varepsilon>0$, respectively.

Furthermore, the function $\gamma(t)$ chosen as t^γ in Theorems 2 and 3 could be replaced by functions of the form $\gamma(t) = t^\gamma(\log t)^\beta$.

Next, without proof, we mention some further applications of Theorem 1 with $g(\omega) := (\log(1+\omega))^{1/p}$. The proofs would run as in the previous cases. These special cases of Theorem 1 include certain parts of the so-called limit-case theorems. For example Theorems D and E, moreover the second part of Theorem C quoted in this paper, belong to these cases.

(v) Let $\{\alpha_k(n)\}$ denote the coefficient matrix of a regular Hausdorff transformation with $\phi(t) \in L^r(0, 1)$ for some $r>1$.

Theorem 2*. *If* (1.2) *holds and* $p\gamma = 1-r^{-1}$, *then*

$$\left\{\sum_{k=0}^{n}\alpha_k(n)|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma})$$

a.e. in (a, b) *for any increasing sequence* $\{\nu_k\}$ *of positive integers.*

This result ($r=\infty$) includes the special case $\varkappa=1$ of Theorem D. Similarly it includes the special case $\beta \leqq 1$ of Theorem E.

(vi) Let $\{\alpha_k(\omega)\}$ denote the function-sequence of coefficients of a regular $[J, f]$-transformation with $\phi(t) \in L^r(0, 1)$ for some $r>1$.

Theorem 3*. *If* (1.2) *holds and* $p\gamma = 1 - r^{-1}$, *then*

$$\Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)\,|s_{\nu_k}(x) - f(x)|^p\Big\}^{1/p} = o_x((\log(1+\omega))^{1/p}\omega^{-\gamma})$$

a.e. in (a, b) *for any increasing sequence* $\{\nu_k\}$ *of positive integers.*

Theorems 2* and 3*, because of their generality, do not yield the limit-cases included in Theorems C and E. However, if we take into account the special properties of the coefficients of the Riesz and the generalized Abel summation methods, as appear under (5.11) and (5.12), then our main result, Theorem 1, yields the results for the above mentioned cases as well.

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