On the central limit theorem for series with respect to periodical multiplicative systems. I

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Introduction. It is well known that many important properties of independent random variables are transfered on broad classes of various orthonormal systems. The questions concerning the statistical properties of lacunary subsystems of orthonormal systems have been studied by many authors. For the trigonometric systems the first result in this direction is due to SALEM and ZYGMUND.

Theorem ([13]). *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$, *where* $\{n_k\}$ *is an infinite sequence of positive integers satisfying the condition* $\frac{n_{k+1}}{n_k} \geqq \lambda$ *for certain* $\lambda>1$ *(so-called Hadamard's lacunarity); furthermore let* $\{a_k\}$ *be a sequence of real numbers such that*

$$A_N = \Big\{\sum_{k=1}^{N} a_k^2\Big\}^{1/2} \to \infty, \quad a_N = o(A_N) \quad as \quad N \to \infty,$$

and $\{\alpha_k\}$ *be an arbitrary sequence of real numbers. Then for any set* $E\subset[0, 1]$ *of positive measure and for any* $x\in\mathbf{R}$ *we have*

$$\lim_{N\to\infty} \frac{1}{|E|} \big|\{t\colon\ t\in E,\ S_N(t) \leqq x\cdot A_N\}\big| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\Big(-\frac{z^2}{2}\Big)\,dz,$$

where $|E|$ *denotes the Lebesgue measure of* E.

This result is called central limit theorem (abbrev. CLT for lacunary trigonometric series and it has been generalized by many ways ([1], [15]—[16]).

For Walsh—Paley's system $\{W_n(x)\}$ the first analogous result was achieved in [12] and afterwards it was extended in [2]—[3], [7].

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Theorem ([3]). *Let us assume that a sequence* $\{n_k\}$ *satisfies the conditions*

$$(1)\qquad \frac{n_{k+1}}{n_k} \geqq 1+\frac{c}{k^\alpha}, \quad c>0, \quad 0 \leqq \alpha \leqq \frac{1}{2}, \quad k=1,2,\ldots;$$

and $\{a_k\}$ *has the properties*

$$(2)\qquad A_N = \Big\{\sum_{k=1}^{N} a_k^2\Big\}^{1/2} \to \infty, \quad a_N = o(A_N \cdot N^{-\alpha}).$$

Then for any $x \in \mathbf{R}$ *we have*

$$(3)\qquad \lim_{N\to\infty} \Big|\Big\{t:\ t\in[0,1],\ \sum_{k=1}^{N} a_k W_{n_k}(t) \leqq x\cdot A_N\Big\}\Big| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\Big(-\frac{z^2}{2}\Big)\, dz.$$

In [7] it was remarked that under hypothesis (1) the second condition of (2) is necessary for the validity of (3).

The purpose of the present work is to study the CLT for weakly lacunary series with respect to the generalized Walsh's functions, i.e. for so-called periodical multiplicative orthonormal systems (abbrev. PMONS).

We recall the definition of PMONS following the survey paper [6].

A sequence of functions $X=\{\chi_k(x)\}_{k=0}^{\infty}$ is called multiplicative system if the following conditions are fulfilled:

a) if $\chi_k(x),\ \chi_l(x)\in X$ then the product $\chi_k(x)\cdot\chi_l(x)=\chi(k,l,x)$ also belongs to X;

b) if $\chi_k(x)\in X$ then $\{\chi_k(x)\}^{-1}$ belongs to X, too.

The system X is called periodical if for every $n=0, 1, \ldots$ there exists an integer k_n such that $\{\chi_n(x)\}^{k_n}\equiv 1$.

We shall define a periodical, multiplicative and orthonormal system X which will be considered later on the interval $[0, 1]$. This system can be numerated in the following way (see [6]): there exist integers

$$0=m_{-1}<1=m_0<m_1<m_2\ldots$$

and functions $\chi_0(x)\equiv 1,\ \chi_{m_0}(x),\ \chi_{m_1}(x), \ldots$ such that the quotients $\frac{m_{n+1}}{m_n}=p_{n+1}$ are prime numbers *) and every functions $\chi_k(x)$ of the system X has the representation

$$\chi_k(x) = \prod_{j=0}^{n} \{\chi_{m_j}(x)\}^{\alpha_j},$$

*) We remark that p_n has not to be a prime number necessarily.

provided that k is expressed in the from

$$k = \sum_{j=0}^{n} \alpha_j \cdot m_j, \quad \text{where} \quad 0 \leqq \alpha_j < p_{j+1}, \quad k > 0.$$

The choice of $\{\chi_{m_n}(x)\}$ may be also ambiguous, but we suppose that it is made by certain determined manner.

We shall study the properties of the series having the form $\sum_{k=1}^{\infty} a_k \chi_{n_k}(x)$, where $\{n_k\}$ is a sequence of positive integers such that

$$\frac{n_{k+1}}{n_k} \geqq 1 + \omega(k) \quad \text{for} \quad k = 1, 2, \ldots, \tag{4}$$

and $\{\omega(k)\}$ is a non-negative, non-increasing sequence such that

$$k^{\alpha} \cdot \omega(k) \uparrow \infty \quad \text{for some} \quad \alpha, 0 < \alpha < 1. \tag{5}$$

Finally we assume that the sequence of the coefficients $\{a_k\}$ satisfies the condition

$$A_N = \Big\{\sum_{k=1}^{N} a_k^2\Big\}^{1/2} \to \infty. \tag{6}$$

We shall consider the following sum

$$T_N(x) := \frac{1}{A_N} \sum_{k=1}^{N} a_k \chi_{n_k}(x). \tag{7}$$

Further on the sequence of the complex-valued functions $T_N(x)$ will be understood as a sequence of two-dimensional random vectors. These vectors are defined on the probability space $(\Omega, \mathscr{F}, P)$, where Ω is the square $[0, 1] \times [0, 1]$, $\mathscr{F}$ is the σ-field of all Borel-measurable sets on Ω and P is the Lebesgue measure on $\mathscr{F}$. The components of the vector $T_N(x)$ are the real part and the imaginary part of the function $T_N(x)$. If it will be necessary, we shall represent the vector $T_N(x)$ in the form

$$T_N(x) = (\xi_N^1(x), \xi_N^2(x)),$$

where

$$\xi_N^1(x) = \operatorname{Re}\{T_N(x)\}, \quad \xi_N^2(x) = \operatorname{Im}\{T_N(x)\}.$$

In the case of the trigonometric system (or the Walsh's system) the CLT was proved by a direct proof showing the convergence of the sequence $\{T_N(x)\}$ to the normal distributions. But in our case the corresponding distributions have two-dimensional character and it requires a special approach.

We shall require some informations from the theory of probability. The terminology and the facts are taken from [14].

Definition 1. A random vector $\xi=(\xi_1, \xi_2, ..., \xi_n)$ is called *normally distributed (Gaussian)* if its characteristic function $\varphi_\xi(t)$ has the form

$$\varphi_\xi(t) = \exp\left\{i\cdot\langle t, m\rangle - \frac{1}{2}\langle \mathbf{R}t, t\rangle\right\},$$

where $m=(m_1, m_2, ..., m_k)$, $|m_k|<\infty$; $\mathbf{R}=\|r_{kl}\|$ is a symmetrical, positive semidefinite matrix, the dimension of which is equal to $n\times n$; $\langle .,.\rangle$ denotes a scalar product. For brevity we shall use the notation $\xi\sim\mathcal{N}(m, \mathbf{R})$.

In this connection m is a vector of mean value, i.e.

$$m_k = M\xi_k \quad \text{for} \quad k = 1, 2, ..., n;$$

and $\mathbf{R}$ is a covariance matrix, i.e.

$$r_{kl} = M\{(\xi_k - M\xi_k)\cdot(\xi_l - M\xi_l)\} = \operatorname{cov}(\xi_k, \xi_l); \quad k, l = 1, 2, ..., n.$$

Here the symbol $M\xi$ denotes the mathematical expectation of random variable ξ and r_{kl} are the elements of $\mathbf{R}$.

Definition 2. If there exists a two-dimensional Gaussian vector $T(x)= =(\xi^1(x), \xi^2(x))$ such that the sequence of random vectors $T_N(x)$ weakly converges to $T(x)$ as $N\to\infty$ (in distribution) then the subsystem $\{a_k\chi_{n_k}(x)\}$ is called *the subject to CLT*. We denote these facts as follows:

$$T_N(x)\xrightarrow{d} T(x) \quad \text{and} \quad \{a_k\chi_{n_k}(x)\}\subset CLT,$$

where the symbol $\xrightarrow{d}$ means the weak convergence.

In other words, there exist a vector $m=(m_1, m_2)$ and a covariance matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$T_N(x)\xrightarrow{d}\mathcal{N}(m, \mathbf{R}) \quad \text{as} \quad N\to\infty.$$

1. The main theorem. Let the PMONS $X=\{\chi_n(x)\}_{n=0}^{\infty}$ be defined by means of the sequence $\{p_n\}$. As earlier, we assume that $m_0=1$, $m_{n+1}=m_n\cdot p_{n+1}$; $n=1, 2, \dots$. The functions $\chi_{m_n}(x)$ are used as "basis" elements in the system X. The set of the functions $\chi_k(x)$ having the index-number from m_n to $m_{n+1}-1$ (inclusively) will be called the "n-th block of X" and denoted by $[m_n, m_{n+1})$. Also let us define the operations of addition and subtraction on the group of non-negative integers according to the following rules:

$$m = k \dotplus l, \quad \text{if} \quad \chi_m(x) = \chi_k(x)\cdot\chi_l(x);$$

$$m = k \dot{-} l, \quad \text{if} \quad \chi_m(x) = \chi_k(x)\cdot\overline{\chi_l(x)},$$

where $\overline{\chi_l(x)}=\{\chi_l(x)\}^{-1}$ denotes the complex conjugate function of $\chi_l(x)$.

To formulate the further results we shall introduce some additional concepts. Let $\chi_k(x)\in X$. The number s is called conjugate to the number k, if $s+k=0$ (i.e. $\chi_s(x)=\overline{\chi_k(x)}$). The coefficients at the conjugate functions $\chi_{n_k}(x)$ and $\overline{\chi_{n_k}(x)}$ (if such pair there will be in our subsystem $\{\chi_{n_k}(x)\}$) will be denote by a_k and $\hat{a}_k$, respectively.

Furthermore, let the numbers q, r be given such that $m_n\leqq q,\ r<m_{n+1}$ for some n. Suppose that $q+r\neq 0$ and let $l=\min\{i\colon 0<q+r<m_{i+1}\}$ (l can be equal to $0, 1, \ldots, n$). In this case we shall call the numbers q and r (l, n)-adjoint.

If a sequence $\{n_k\}$ is given, then, in general, there exist both conjugate and (l, n)-adjoint numbers in $\{n_k\}$. The quantity of the conjugate pairs (n_q, n_r), where $m_n\leqq q,\ r<m_{n+1}$ will be denoted by λ_n (in addition, we suppose that the pairs (n_q, n_r) and (n_r, n_q) are distinct if $q\neq r$). The value $\lambda_n^l(q)$ will be defined as quantity of the numbers n_r being (l, n)-adjoint with n_q for a fixed q.

Finally, for given sequences $\{n_k\}$ and $\{a_k\}$ we put

$$f(0) = 0, \quad f(k) = \max\{i\colon n_i < m_k\}, \quad k = 1, 2, \ldots$$

$$\Delta_k(x) = \sum_{i=f(k)+1}^{f(k+1)} a_i\chi_{n_i}(x); \; B_k = A_{f(k+1)}; \; k = 0, 1, \ldots \tag{1.1}$$

$$b_k = \max\{|a_j|\colon f(k)+1 \leqq j < f(k+1)\}; \; \delta_k = f(k+1)-f(k).$$

Remark 1.1. If for some k $f(k)=f(k+1)$, then we assume that $\Delta_k(x)\equiv 0$; $B_k=B_{k-1}$; $\delta_k=b_k=0$.

Now we can formulate the main statement of our work.

Theorem A. *Suppose that for a given system X the corresponding sequence $\{p_n\}$ is bounded. We also assume that the sequences $\{n_k\}$, $\{\omega(k)\}$ and $\{a_k\}$ satisfy conditions* (4)—(6), *respectively. Additionally if*

a)

$$a_k = o(A_k\omega(k)); \tag{1.2}$$

b) *there exists a real number η, $0\leqq\eta\leqq 1$ such that*

$$\lim_{N\to\infty}\frac{1}{B_N^2}\sum_{j}^{f(N+1)}(a_j\cdot\hat{a}_j) = \eta, \tag{1.3}$$

where the summation is taken for all conjugate numbers being not greater than $f(N+1)$;

c) *there exists a constant $C>1$ (independent of q, j) such that for any fixed q and for any j, $0\leqq j\leqq k-1$*

$$\lambda_k^j(q)\cdot\omega(f(k)) = O(C^{j-k}) \quad \text{as} \quad k\to\infty \tag{1.4}$$

holds, then the subsystem $\{a_k\chi_{n_k}(x)\}$ is the subject to CLT.

It is easy to see that Theorem A will be proved if we can show the existence of a vector $m=(m_1, m_2)$ and a matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$T_N(x) \xrightarrow{d} \mathcal{N}(m, \mathbf{R}) \quad \text{as} \quad N \to \infty. \tag{1.5}$$

2. Lemmas. First we shall recall some further facts of the probability.

Definition 3 ([14], pp. 467—474). Let $\{l_n\}$ be a certain sequence of indices and $\{X_{n,i};\ n=0, 1, \ldots;\ 0 \leqq i \leqq l_n\}$ be an array of random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_{n,i};\ n=0, 1, \ldots;\ 0 \leqq i \leqq l_n\}$ be any triangular array of sub σ-fields of $\mathcal{F}$ such that

$$\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i} \quad \text{for all} \quad n = 0, 1, \ldots;\ 1 \leqq i \leqq l_n.$$

Then we shall call the array $\{X_{n,i}\}$ a *martingale difference array* (briefly MDA) with respect to $\{\mathcal{F}_{n,i}\}$ if $X_{n,i}$ is $\mathcal{F}_{n,i}$-measurable and $M\{|X_{n,i}|\}<\infty$, $M\{X_{n,i}|\mathcal{F}_{n,i-1}\}=0$ almost everywhere (a.e.) for all n and $i \geqq 1$ (the definition of the conditional expectation with respect to σ-field can be found in [14], p. 227).

Definition 4 ([14], p. 204). The class of random variables is called *uniformly integrable* if

$$\sup_n M\{|\xi_n| \cdot I_{[|\xi_n|>c]}\} \to 0 \quad \text{as} \quad c \to \infty.$$

Now we shall prove some auxiliary assertions.

Lemma 2.1. *On the probability space* $(\Omega, \mathcal{F}, P)$ *let the sequences* $\{T_n\}$ *and* $\{\xi_n\}$ *of random variables be given such that*

a) $\{T_n\}$ *is uniformly integrable*;

b) $\xi_n \xrightarrow{P} 0$ *as* $n \to \infty$;

c) $\{T_n \cdot \xi_n\}$ *is uniformly integrable.*

Then $T_n \cdot \xi_n \xrightarrow{L_1} 0$ *(here* $\xrightarrow{P}$ *and* $\xrightarrow{L_1}$ *denote the convergence with respect to probability and* L_1*-metric, respectively).*

Proof. Let $\varepsilon>0$ be fixed. By virtue of condition c) there exists $\delta(\varepsilon)>0$ such that for any $n \in \mathrm{N}$ and $A \subset \mathcal{F}$ we have

$$\int_A |T_n \cdot \xi_n|\, dP \leqq \varepsilon \tag{2.1}$$

if $P(A)<\delta(\varepsilon)$.

Furthermore, by condition b) there exists N such that for all $n>N$

$$P\{|\xi_n| > \varepsilon\} \leqq \delta(\varepsilon). \tag{2.2}$$

From (2.1) and (2.2) we conclude that for $n>N$

$$\int_{\{|\xi_n|>\varepsilon\}} |T_n\xi_n|\, dP \leqq \varepsilon.$$

Therefore by $n>N$

(2.3)

$$\int_\Omega |T_n\xi_n|\, dP = \int_{\{|\xi_n|>\varepsilon\}} + \int_{\{|\xi_n|\leqq\varepsilon\}} \leqq \int_{\{|\xi_n|>\varepsilon\}} |T_n\xi_n|\, dP + \varepsilon\cdot\int_\Omega |T_n|\, dP < \varepsilon+\varepsilon\int_\Omega |T_n|\, dP$$

Since $\{T_n\}$ is uniformly integrable, therefore

$$\sup_n \int_\Omega |T_n|\, dP < \infty$$

(see [14], p. 206). Hence, taking into account (2.3), we obtain the assertion of Lemma 2.1.

Now let $X_{n,j}=(\mu_{n,j};\, \nu_{n,j})$; $n=0, 1, \ldots;\ 0\leqq j\leqq n$ be the set of random vectors on the probability space $(\Omega, \mathscr{F}, P)$; $\mathscr{F}_{n,j}$ be the set of sub σ-fields of $\mathscr{F}$ such that for all n, j $(n=0, 1, \ldots;\ 0\leqq j\leqq n)$ the variables $X_{n,j}$ are $\mathscr{F}_{n,j}$-measurable and $\mathscr{F}_{n,j-1}\subset \subset\mathscr{F}_{n,j}$.

Put

$$T_n := \prod_{j=0}^{n} \left(1+i\langle t, X_{n,j}\rangle\right), \tag{2.4}$$

where symbol i denotes the imaginary unit, $t=(t_1, t_2)$ is any vector, and $\langle\cdot, \cdot\rangle$ denotes scalar product.

Lemma 2.2. *Let the sequence $\{X_{n,j}\}$ satisfy the following conditions:*

a) $\max_{j\leqq n} |X_{n,j}| \xrightarrow{P} 0$ *as* $n\to\infty$;

b) *there exist constants* μ, ν, ξ, *such that*

$$\sum_{j=0}^{n} (\mu_{n,j})^2 \xrightarrow{P} \mu;\quad \sum_{j=0}^{n} (\nu_{n,j})^2 \xrightarrow{P} \nu;$$

$$\sum_{j=0}^{n} (\mu_{n,j}\cdot\nu_{n,j}) \xrightarrow{P} \xi;$$

c) *for any vector* $t=(t_1, t_2)$ *the sequence* $\{T_n\}$ *is uniformly integrable and* $M\{T_n\}\to 1$.

Then $S_n=\sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R})$, *where* $\mathbf{R}=\|r_{kl}\|=\begin{pmatrix}\mu & \xi\\ \xi & \nu\end{pmatrix}$.

Proof. We use the relation

$$\exp(ix) = (1+ix)\left(\exp\left\{-\frac{x^2}{2}+r(x)\right\}\right), \quad \text{where} \quad |r(x)| \leqq |x|^3$$

for all $|x|<1$.

Let

$$V_n := \exp\{i\cdot\langle t, S_n\rangle\}$$

and

$$U_n := \exp\left\{-\frac{1}{2}\sum_{j=0}^{n}(\langle t, X'_{n,j}\rangle)^2+\sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right\}.$$

We have

$$V_n = \exp\{i\cdot\langle t, \sum_{j=0}^{n} X_{n,j}\rangle\} = \exp\{i\cdot\sum_{j=0}^{n}\langle t, X_{n,j}\rangle\} =$$

$$= T_n\cdot\exp\left\{-\frac{1}{2}\sum_{j=0}^{n}(\langle t, X_{n,j}\rangle)^2+\sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right\} =$$

$$= T_n\cdot\exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}+T_n\left(U_n-\exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right).$$

By virtue of a theorem about the connection between the pointwise convergence and the convergence of corresponding distributions (see [14], p. 343) for the proof of Lemma 2.2 it will be sufficient to show that for any $t=(t_1, t_2)$

$$M\{|V_n|\} \to \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}. \tag{2.5}$$

Since $M\{T_n\}\to 1$ thus we have to verify only that

$$T_n\left(U_n-\exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right) \xrightarrow{L_1} 0. \tag{2.6}$$

First we show that

$$U_n-\exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\} \xrightarrow{P} 0. \tag{2.7}$$

According to b)

$$\sum_{j=0}^{n}(\langle t, X_{n,j}\rangle)^2 \xrightarrow{P} \langle t, \mathbf{R}t\rangle.$$

Furthermore

$$\left|\sum_{j=0}^{n} r(\langle t, X_{n,j}\rangle)\right| \leqq |t|^3\cdot\sum_{j=0}^{n}|X_{n,j}|^3 \leqq |t|^3\cdot\max_{j\leqq n}|X_{n,j}|\cdot\sum_{j=0}^{n}|X_{n,j}|^2 \xrightarrow{P} 0,$$

so long as

$$\sum_{j=0}^{n}|X_{n,j}|^2 = \sum_{j=0}^{n}(\mu_{n,j}^2+\nu_{n,j}^2) \xrightarrow{P} \mu+\nu$$

and $\max_{j \geqq n} |X_{n,j}| \xrightarrow{P} 0$ by condition a). This implies that

$$U_n \xrightarrow{P} \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\},$$

i.e. (2.7) is valid.

Since $\{T_n\}$ and $\{V_n\}$ are uniformly integrable (the uniform integrability of $\{V_n\}$ follows from $M\{|V_n|^2\}=1$), thus the sequence of values

$$\eta_n := V_n - T_n \cdot \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\} = T_n\left(U_n - \exp\left\{-\langle t, \mathbf{R}t\rangle\right\}\right)$$

is also uniformly integrable as a convex set of uniformly integrable sequences (e.g. see [illegible] p. 27).

By condition c), relation (2.7) and the uniform integrability of $\{\eta_n\}$ we can see that for the sequences $\{T_n\}$ and $\left\{U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}\right\}$ all of the conditions of Lemma 2.1 are fulfilled. $\left(\text{For it is sufficient to put } \xi_n = U_n - \exp\left\{-\frac{1}{2}\langle t, \mathbf{R}t\rangle\right\}.\right)$ Applying Lemma 2.1 we obtain (2.6) and moreover (2.5). Consequently the proof is complete.

The next lemma is basic for the proof of Theorem A.

Lemma 2.3. *Let* $\{X_{n,j}; \mathscr{F}_{n,j}\}$ *be an* MDA *satisfying the conditions:*

a) $\max_{j \leqq n} |X_{n,j}|$ *is uniformly bounded (in* L_2*-norm);*

b) $\max_{j \leqq n} |X_{n,j}| \xrightarrow{P} 0$;

c) *there exist constants* μ, ν, ξ *such that*

$$\sum_{j=0}^{n} (\mu_{n,j})^2 \xrightarrow{P} \mu; \quad \sum_{j=0}^{n} (\nu_{n,j})^2 \xrightarrow{P} \nu; \quad \sum_{j=0}^{n} (\mu_{n,j} \cdot \nu_{n,j}) \xrightarrow{P} \xi,$$

where $\mu_{n,j}; \nu_{n,j}$ *are the components of random vector* X_n. *Then*

$$S_n = \sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathcal{N}(0, \mathbf{R}) \quad \text{where} \quad \mathbf{R} = \begin{pmatrix} \mu & \xi \\ \xi & \eta \end{pmatrix}.$$

Proof. Let us define the sequence $\{Z_{n,j}\}$ in the following way:

$$Z_{n,j} := X_{n,j} \cdot I\left(\sum_{k=0}^{j-1} |X_{n,k}|^2 \leqq 2(\mu+\nu)\right),$$

where $I(A)$ denotes the characteristic function of A.

It is clear that $\{Z_{n,j}; \mathscr{F}_{n,j}\}$ also represents an MDA and

$$P\{Z_{n,j} \neq X_{n,j} \text{ for some } j \leqq n\} \leqq P\Big\{\sum_{j=0}^{n} |X_{n,j}|^2 > 2(\mu+\nu)\Big\} \to 0, \tag{2.8}$$

since $|X_{n,j}|^2 = (\mu_{n,j})^2 + (\nu_{n,j})^2$, and according to c)

$$\sum_{j=0}^{n} |X_{n,j}|^2 \xrightarrow{p} \mu+\nu.$$

Therefore $\{Z_{n,j}\}$ also satisfies the conditions a), b), c) of Lemma 2.3. Now for any $t=(t_1, t_2)$ we put

$$T_n := \prod_{j=0}^{n} (1+i\cdot\langle t, Z_{n,j}\rangle).$$

Then $M\{T_n\}=1$ for all n, because $\{Z_{n,j}; \mathscr{F}_{n,j}\}$ is an MDA.

Put

$$J_n := \begin{cases} \min\Big\{j \leqq n\colon \sum_{k=0}^{j} |X_{n,k}|^2 > 2(\mu+\nu)\Big\}, & \text{if } \sum_{k=0}^{n} |X_{n,k}|^2 > 2(\mu+\nu); \\ n, & \text{otherwise.} \end{cases}$$

We have

$$M\{|T_n|^2\} = M\Big\{\prod_{j=0}^{n} (1+(\langle t, Z_{n,j}\rangle)^2)\Big\} \leqq M\Big\{\exp\Big[|t|^2 \cdot \sum_{j=0}^{J_n-1} |X_{n,j}|^2\Big] \times$$
$$\times [1+(\langle t, X_{n,J_n}\rangle)^2]\Big\} \leqq \exp\big\{2|t|^2\cdot(\mu+\nu)\cdot[1+|t|^2\cdot M\{|X_{n,J_n}|^2\}]\big\}.$$

The right side of the last inequality is uniformly bounded (by condition a)). Therefore the set $\{T_n\}$ is uniformly integrable (see [14], p. 207). Taking into account b) and c), we can see that for $\{Z_{n,j}\}$ all of the conditions of Lemma 2.2 are fulfilled. Therefore

$$\sum_{j=0}^{n} Z_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R}),$$

whence, by means of (2.8), we obtain

$$S_n = \sum_{j=0}^{n} X_{n,j} \xrightarrow{d} \mathscr{N}(0, \mathbf{R}),$$

which completes our proof.

Remark 2.1 (see [10]). Lemma 2.3 is the two-dimensional extension of Mc. Leish's theorem.

3. Preparation to the proof of Theorem A. We shall suppose that the sequence $\{p_n\}$ is bounded. Using notations (1.1), we can select for any N a number k such that

$$f(k) < N \leqq f(k+1). \tag{3.1}$$

Then

$$T_n(x) = \frac{1}{A_N}\sum_{m=1}^{N} a_m \chi_{n_m}(x) = \frac{B_{k-1}}{A_N}\cdot\frac{1}{B_{k-1}}\sum_{i=0}^{k-1}\Delta_i(x) + \frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x).$$

Put

(3.2) $$X_{k,i} := \frac{\Delta_i(x)}{B_k}, \quad S_k := \sum_{i=0}^{k} X_{k,i}; \quad k = 0, 1, \ldots; \; i = 0, 1, \ldots, k.$$

We rewrite $T_N(x)$ in the following form:

$$T_N(x) = \frac{B_{k-1}}{A_N}\sum_{i=0}^{k-1} X_{k-1,i} + \frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x) =$$

$$= \frac{B_{k-1}}{A_N}\cdot S_{k-1} + \frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x).$$

Hence, in order to prove Theorem A, it is enough to show that

(3.3) there exist a vector $m=(m_1, m_2)$ and a symmetrical positive semi-definite matrix $\mathbf{R}=\|r_{kl}\|$; $k, l=1, 2$; such that

$$S_k \xrightarrow{d} \mathcal{N}(m, \mathbf{R});$$

(3.4) $$\frac{B_{k-1}}{A_N} \to 1;$$

(3.5) $$\frac{1}{A_N}\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x) \to 0.$$

as $N \to \infty$.

Assertions (3.4) and (3.5) follow from conditions (4)—(6) and (1.2). Now we show our assertions.

Lemma 3.1 ([8]). *Let sequences* $\{n_k\}$ *and* $\{\omega(k)\}$ *satisfy the conditions* (4) *and* (5), *respectively. Then*

(3.6) $$\delta_k = O\left\{\frac{1}{\omega(f(k))}\right\},$$

(3.7) $$f(k+1) \sim f(k),$$

(3.8) $$\omega(f(k)) = O\{\omega(k+1)\}.$$

as $k \to \infty$.

Further, using (3.1), we have

$$1 \geqq \frac{B_{k-1}}{A_N} \geqq \frac{B_{k-1}}{B_k} \geqq 0.$$

In order to verify (3.4) it is sufficient to prove that

(3.9) $$\frac{B_{k-1}}{B_k} \to 1.$$

By (1.2) and (3.3) we get

$$1-\left(\frac{B_{k-1}}{B_k}\right)^2 = \frac{1}{B_k^2}(B_k^2-B_{k-1}^2) = \frac{1}{B_k^2}\sum_{m=f(k)+1}^{f(k+1)} a_m^2 \leqq \frac{1}{B_k^2}\left\{\max_{f(k)<m\leqq f(k+1)} |a_m|^2\cdot\delta_k\right\} =$$

$$= \frac{1}{B_k^2}\cdot o\left\{\frac{B_k^2\cdot(\omega(f(k)))^2}{\omega(f(k))}\right\} = o\{\omega(f(k))\} = o(1),$$

consequently (3.9) and (3.4) are proved.

Furthermore (1.2) and (3.9) imply that

$$\frac{1}{A_N}\cdot\left|\sum_{m=f(k)+1}^{N} a_m\chi_{n_m}(x)\right| \leqq \frac{1}{B_{k-1}}\cdot\sum_{m=f(k)+1}^{f(k+1)} |a_m| \leqq \frac{1}{B_{k-1}}\cdot\{b_k\cdot[f(k+1)-f(k)]\} =$$

$$= \frac{1}{B_{k-1}}\cdot o\{B_k\cdot\omega(f(k))\}\cdot O\left\{\frac{1}{\omega(f(k))}\right\} = \frac{1}{B_{k-1}}\cdot o(B_k) = o(1), \quad \text{as} \quad k\to\infty,$$

and by the previous reason (3.5) is proved.

Since the functions $\chi_n(x)$ are two-dimensional random variables, defined on $(\Omega, \mathscr{F}, P)$, we shall denote by $\mathscr{F}_{k,i}$ $(k=0, 1, \dots;\ 0\leqq i\leqq k)$ the sub σ-field of $\mathscr{F}$ generated by random variables $\{\chi_{m_s}(x)\colon 0\leqq s\leqq i\}$. In this case the values $X_{k,i}(x)$, defined by (3.2), are $\mathscr{F}_{k,i}$-measurable, $\mathscr{F}_{k,i-1}\subset\mathscr{F}_{k,i}$ and $M\{X_{k,i}|\mathscr{F}_{k,i-1}\}=0$ a.e. for all k, i $(1\leqq i\leqq k)$. These evidently follow from the properties of our system.

In addition, we remark that

$$M\{|X_{k,i}|\} = \int_0^1 |X_{k,i}|\,dx = \frac{1}{B_k}\int_0^1 |\Delta_i(x)|\,dx \leqq \frac{1}{B_k}\int_0^1 \sum_{j=f(i)+1}^{f(i+1)} |a_j|\,dx \leqq$$

$$\leqq \frac{1}{B_k}\cdot b_i\cdot\delta_i = \frac{1}{B_k}\cdot o\{B_i\cdot\omega(f(i))\}\cdot o\left\{\frac{1}{\omega(f(i))}\right\} = O\left(\frac{B_i}{B_k}\right) = o(1) \quad \text{as} \quad k\to\infty.$$

Therefore $M\{|X_{k,i}|\}<\infty$ for all k, i. Thus, the sequence $\{X_{k,i};\mathscr{F}_{k,i}\}$ represents an MDA and in order to prove (3.4) it is sufficient to verify the validity of the conditions of Lemma 2.3 for the sequence $\{X_{k,i}\}$.

Using the multiplicative property and the orthogonality of the system X, we get

$$M\left\{\max_{0\leqq i\leqq k}|X_{k,i}|^2\right\} = \frac{1}{B_k^2}\int_0^1 \max_{0\leqq i\leqq k}|\Delta_i(x)|^2\,dx \leqq$$

$$\leqq \frac{1}{B_k^2}\int_0^1\sum_{i=0}^{k}|\Delta_i(x)|^2\,dx = \frac{1}{B_k^2}\sum_{i=0}^{k}\int_0^1|\Delta_i(x)|^2\,dx = \frac{B_k^2}{B_k^2} = 1,$$

which means that condition a) is satisfied.

Furthermore,

$$\max_{0\le i\le k}|X_{k,i}| = \frac{1}{B_k}\cdot\max_{0\le i\le k}|\Delta_i(x)| \le \frac{1}{B_k}\cdot\max_{0\le i\le k}\sup_x|\Delta_i(x)| \le$$

$$\le \frac{1}{B_k}\cdot\max_{0\le i\le k}\sum_{m=f(i)+1}^{f(i+1)}|a_m| \le \frac{1}{B_k}\cdot\max_{0\le i\le k}\left\{\max_{f(i)<m\le f(i+1)}|a_m|\cdot\delta_i\right\} =$$

$$= \frac{1}{B_k}\cdot\max_{0\le i\le k}\left\{o\big(B_i\cdot\omega(f(i))\big)\cdot O\left(\frac{1}{\omega(f(i))}\right)\right\} = \frac{1}{B_k}\cdot\max_{0\le i\le k}\{o(B_i)\} = o(1), \quad \text{as} \quad k\to\infty,$$

and this proves b).

For the direct proof of condition c) we require some lemmas.

Lemma 3.2. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{a_k\}$ *satisfy conditions* (4)—(6), (1.2), *respectively. Then*

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^N|\Delta_k(x)|^2-1\right|^2 dx = o(1) \quad as \quad N\to\infty.$$

The proof of this lemma can be found in [8] (replacing only the symbol O by o),

Lemma 3.3. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{a_k\}$ *fulfil conditions* (4)—(6) (1.2)—(1.4), *respectively. Then*

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^N\big((\Delta_k(x))^2-\eta\big)\right|^2 dx = o(1) \quad as \quad N\to\infty.$$

Proof. The next equalities are evident

(3.10)

$$\int_0^1\left|\frac{1}{B_N^2}\sum_{k=0}^N(\Delta_k(x))^2-\eta\right|^2 dx = \int_0^1\frac{1}{B_N^4}\cdot\Big\{\sum_{k=0}^N(\Delta_k)^2-\eta B_N^2\Big\}\cdot\Big\{\sum_{k=0}^N(\bar\Delta_k)^2-\eta B_N^2\Big\}\,dx =$$

$$= \int_0^1\Big\{\frac{1}{B_N^4}\Big[\sum_{k=0}^N(\Delta_k)^2\cdot\sum_{k=0}^N(\bar\Delta_k)^2-\eta B_N^2\cdot\Big(\sum_{k=0}^N(\Delta_k)^2+\sum_{k=0}^N(\bar\Delta_k)^2\Big)+\eta^2B_N^4\Big]\Big\}\,dx =$$

$$= \frac{1}{B_N^4}\Big\{\int_0^1\Big|\sum_{k=0}^N(\Delta_k)^2\Big|^2dx-\eta B_N^2\int_0^1\sum_{k=0}^N[(\Delta_k)^2+(\bar\Delta_k)^2]\,dx+\eta^2B_N^4\Big\},$$

since

$$\overline{\sum_{k=0}^N(\Delta_k)^2} = \sum_{k=0}^N(\bar\Delta_k)^2 \quad \text{and} \quad \bar\eta = \eta.$$

Let us evaluate the values in the brace. We have

$$(3.11)\qquad \int_0^1 \Big|\sum_{k=0}^{N}(\Delta_k)^2\Big|^2 dx = \sum_{k=0}^{N}\sum_{j=0}^{N}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx.$$

The terms of the type $\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx$, in turn, consist of the summands containing the expressions of the species

$$(3.12)\qquad \int_0^1 (\chi_{n_q}\cdot\chi_{n_r}\cdot\bar{\chi}_{n_h}\cdot\bar{\chi}_{n_i})\,dx$$

(with corresponding coefficients), where

$$f(k) < q, r \leqq f(k+1);\; f(j) < h, i \leqq f(j+1);\; 0 \leqq k, j \leqq N.$$

Each of the integrals is equal to zero or one (by virtue of the multiplicative property and the orthogonality of the system X). We have to estimate the quantity of the non-zero summands. Let $k>j$ (the case $k<j$ can be treated similarly). Arguing the same way as in the proof of Lemma 2.4 in [8], we conclude that the functions $\chi_p(x)=\chi_{n_q}(x)\cdot\chi_{n_r}(x)$ belong to a block, number of which is not larger than j (otherwise the integral (3.12) will become zero). Therefore the numbers n_q and n_r have to be conjugate or (l, k)-adjoint (moreover $0\leqq l\leqq j$).

Now we rewrite the previous equality in the form

$$\int_0^1 \Big|\sum_{k=0}^{N}(\Delta_k)^2\Big|^2 dx = \sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx + \sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx,$$

where the symbol $\sum'$ denotes the set of that summands, for which the numbers n_q and n_r are conjugate in the fourfold product $\chi_{n_q}\cdot\chi_{n_r}\cdot\bar{\chi}_{n_h}\cdot\bar{\chi}_{n_i}$ and the symbol $\sum''$ denotes the set of all other summands.

In the sum $\sum'$ we have to consider only the summands, for which $n_q \dotplus n_r=0$ and $n_h \dotplus n_i=0$ simultaneously; other summands, are equal to zero, because for them the equality

$$\chi_{n_q}\cdot\chi_{n_r}\cdot\bar{\chi}_{n_h}\cdot\bar{\chi}_{n_i} \equiv 1$$

does not fulfil.

Therefore,

$$\sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = \sum_{k}^{f(N+1)}\sum_{j}^{f(N+1)} (a_k\cdot\hat{a}_k)\cdot(a_j\cdot\hat{a}_j) = \Big\{\sum_{k}^{f(N+1)} (a_k\cdot\hat{a}_k)\Big\}^2.$$

Using (1.3) we get

$$(3.13)\qquad \sum_{k,j=0}^{N}{}' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = \eta^2\cdot B_N^4 + o(B_N^4) \quad \text{as} \quad N\to\infty.$$

In the sum $\sum''$ we select the summands for the cases $k=j$, $k>j$ and $k<j$:

$$\sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx = \sum_{k=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_k^2)\,dx + \sum_{k=0}^{N}{}'' \sum_{j=0}^{k-1} \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx + \tag{3.14}$$

$$+ \sum_{j=1}^{N}{}'' \sum_{k=0}^{j-1} \int_0^1 (\bar{\Delta}_j^2 \cdot \Delta_k^2)\,dx = L_N^{(1)} + L_N^{(2)} + L_N^{(3)}.$$

Now we have

$$L_N^{(1)} = \sum_{k=0}^{N}{}'' \int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_k^2)\,dx \leqq \sum_{k=0}^{N} \int_0^1 |\Delta_k^4|\,dx = \sum_{k=0}^{N} \int_0^1 (|\Delta_k|^2)^2\,dx. \tag{3.15}$$

Since

$$|\Delta_k|^2 = \sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q} \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r \bar{\chi}_{n_r} = \sum_{q=f(k)+1}^{f(k+1)} a_q \Big\{ \sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_q} \cdot \bar{\chi}_{n_r} \Big\},$$

therefore, applying Minkowski's inequality, we obtain

$$\Big\{\int_0^1 (|\Delta_k|^2)^2\,dx\Big\}^{1/2} \leqq \sum_{q=f(k)+1}^{f(k+1)} |a_q| \cdot \Big\{\int_0^1 \Big|\sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_q} \cdot \bar{\chi}_{n_r}\Big|^2 dx\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=f(k)+1}^{f(k+1)} |a_q| \cdot \Big\{\int_0^1 \Big|\sum_{r=f(k)+1}^{f(k+1)} a_r \bar{\chi}_{n_r}\Big|^2 dx\Big\}^{1/2} = \sum_{q=f(k+1)}^{f(k+1)} |a_q| \cdot \Big\{ \sum_{r=f(k)+1}^{f(k+1)} a_r^2 \Big\}^{1/2}.$$

Hence,

(3.16)

$$\sum_{k=0}^{N} \int_0^1 (|\Delta_k|^2)^2\,dx = \sum_{k=0}^{N} \Big\{ \sum_{q=f(k)+1}^{f(k+1)} |a_q| \Big\}^2 \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r^2 \leqq \sum_{k=0}^{N} (b_k \delta_k)^2 \times \sum_{r=f(k)+1}^{f(k+1)} a_r^2 =$$

$$= \sum_{k=0}^{N} \Big\{ o\big(B_k \cdot \omega(f(k))\big) \cdot \frac{1}{\omega\big(f(k)\big)} \Big\}^2 \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r^2 = \sum_{k=0}^{N} o(B_k^2) \cdot \sum_{r=f(k+1)}^{f(k+1)} a_r^2 =$$

$$= o(B_N^2) \cdot \sum_{k=0}^{N} \sum_{r=f(k)+1}^{f(k+1)} a_r^2 = o(B_N^2) \cdot B_N^2 = o(B_N^4) \quad \text{as} \quad N \to \infty$$

(we used relations (1.2) and (3.3)).

Passing on to the estimation of $L_N^{(2)}$ in (3.14), we remark that

$$\int_0^1 (\Delta_k^2 \cdot \bar{\Delta}_j^2)\,dx = \int_0^1 \Big(\sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q} \Big)^2 \cdot \Big(\sum_{h=f(j)+1}^{f(j+1)} a_h \bar{\chi}_{n_h} \Big)^2 dx =$$

$$= \int_0^1 \Big(\sum_{q=f(k)+1}^{f(k+1)} a_q \chi_{n_q} \cdot \sum_{r=f(k)+1}^{f(k+1)} a_r \chi_{n_r} \Big) \cdot \Big(\sum_{h=f(j)+1}^{f(j+1)} a_h \bar{\chi}_{n_h} \cdot \sum_{i=f(j)+1}^{f(j+1)} a_i \bar{\chi}_{n_i} \Big)\,dx = \tag{3.17}$$

$$= \sum_{q=f(k)+1}^{f(k+1)} a_q \sum_{h=f(j)+1}^{f(j+1)} a_h \sum_{r=f(k)+1}^{f(k+1)} a_r \sum_{i=f(j)+1}^{f(j+1)} a_i \int_0^1 (\chi_{n_q} \chi_{n_r} \bar{\chi}_{n_h} \bar{\chi}_{n_i})\,dx.$$

As it was mentioned above, for any non-zero term of $\sum''$ there should exist an (l, k)-adjoint of the numbers n_q and n_r. Therefore the total quantity of the appropriate pairs (n_q, n_r) is not more than $\lambda_k^j(q)$.

Under fixed indices q, h and for any selected number n_r there exists not more than one number n_i such that

$$\chi_{n_q}\cdot\chi_{n_r}\cdot\bar\chi_{n_h}\cdot\bar\chi_{n_i}\equiv 1.$$

Thus, by (3.17), we get the estimation

$$\int_0^1 (\Delta_k^2\cdot\bar\Delta_j^2)\,dx \le b_k\cdot\sum_{q=f(k)+1}^{f(k+1)}|a_q|\cdot\lambda_k^j(q)\cdot b_j\sum_{h=f(j)+1}^{f(j+1)}|a_h|. \tag{3.18}$$

By Cauchy—Bunjakowski's inequality and by (1.2) and (3.3):

$$b_k\cdot\sum_{q=f(k)+1}^{f(k+1)}|a_q| = o(B_k\cdot\omega(f(k)))\cdot\Big\{\sum_{q=f(k)+1}^{f(k+1)}a_q^2\Big\}^{1/2}\cdot(\delta_k)^{1/2} =$$

$$= o(B_k\cdot\omega(f(k)))\cdot O\left\{\frac{1}{\sqrt{\omega(f(k))}}\right\}\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2} =$$

$$= o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2}\quad\text{as}\quad k\to\infty.$$

This implies that under the realization of (1.4) the next relation holds

$$\int_0^1(\Delta_k^2\cdot\bar\Delta_j^2)\,dx = o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot\Big\{\int_0^1|\Delta_k(x)|^2\,dx\Big\}^{1/2}\times$$

$$\times O\left(\frac{C^{j-k}}{\omega(f(k))}\right)\cdot o\big(B_j\cdot\sqrt{\omega(f(j))}\big)\cdot\Big\{\int_0^1|\Delta_j|^2\,dx\Big\}^{1/2}\quad\text{as}\quad k\to\infty.$$

Hence,

$$L_N^{(2)} = \sum_{k=0}^{N}\sum_{j=0}^{k-1} o\big(B_k\cdot\sqrt{\omega(f(k))}\big)\cdot O\left(\frac{C^{j-k}}{\omega(f(k))}\right)\cdot o\big(B_j\cdot\sqrt{\omega(f(j))}\big)\times$$

$$\times\Big\{\int_0^1|\Delta_k^2|\,dx\cdot\int_0^1|\Delta_j|^2\,dx\Big\}^{1/2} = o(B_N^2)\cdot\sum_{k=1}^{N}\sum_{j=0}^{k-1}C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\times$$

$$\times\Big\{\int_0^1|\Delta_k|^2\,dx\cdot\int_0^1|\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2}\quad\text{as}\quad N\to\infty.$$

Let us show that $L_N^{(2)}=o(B_N^4)$ as $N\to\infty$. It is sufficient to show that

$$\sum_{k=1}^{N}\sum_{j=0}^{k-1}C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\cdot\Big\{\int_0^1|\Delta_k|^2\,dx\cdot\int_0^1|\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2} = O(B_N^2).$$

Consider the sum $\sum_{j=0}^{k-1} C^{j-k}\cdot\omega(f(j))$. Since $k\cdot\omega(k)\uparrow\infty$, thus for any natural $\tau\geqq 2$ we have

$$k\cdot\omega(k)\leqq\left[\frac{\tau}{2}k\right]\cdot\omega\left(\left[\frac{\tau}{2}k\right]\right)\leqq\frac{\tau}{2}k\cdot\omega\left(\left[\frac{\tau}{2}k\right]\right),$$

from which

(3.19) $$\omega\left(\left[\frac{\tau k}{2}\right]\right)\leqq\frac{2\omega(k)}{\tau},\quad k=1,2,\ldots,$$

(the symbol $[x]$ denotes the integral part of x).

By (3.7) there exists a number M such that if $k>M$ then

(3.20) $$f(k+1)<\left[\frac{[C+1]}{2}f(k)\right].$$

Relations (3.19) and (3.20) imply that if $\tau=[C+1]\geqq 2$ and $k>M$, then

$$\omega(f(k+1))>\omega\left(\left[\frac{[C+1]}{2}\cdot f(k)\right]\right)\geqq\frac{2\omega(f(k))}{[C+1]}\geqq\frac{2\omega(f(k))}{C+1},$$

i.e. $\omega(f(k))<\frac{C+1}{2}\cdot\omega((f+1))$ as $k>M$.

Therefore

$$\omega(f(j))<\left(\frac{C+1}{2}\right)^{k-j}\cdot\omega(f(k))\quad\text{if}\quad k>j>M,$$

and thus

(3.21)

$$\sum_{j=M+1}^{k-1} C^{j-k}\cdot\omega(f(j))<\sum_{j=M+1}^{k-1}\left(\frac{C+1}{2C}\right)^{k-j}\cdot\omega(f(k))=O\{\omega(f(k))\}\quad\text{as}\quad k\to\infty.$$

On the other hand

$$\sum_{j=0}^{M} C^{j-k}\cdot\omega(f(j))=O(C^{-k})\quad\text{as}\quad k\to\infty.$$

At the same time, by (3.20), we have

$$f(k)<C\cdot f(k-1)<\ldots<C^{k-M-1}\cdot f(M+1)\quad\text{if}\quad k>M+1,$$

hence

$$C^{-k}\leqq\frac{f(M+1)}{C^{M+1}}\cdot\frac{1}{f(k)}=O\left\{\frac{1}{f(k)}\right\}=O\{\omega(f(k))\}$$

since $f(k)\cdot\omega(f(k))\uparrow\infty$ as $k\to\infty$.

So

$$\sum_{j=0}^{M} C^{j-k}\cdot\omega(f(j)) = O\{\omega(f(k))\} \quad \text{as} \quad k\to\infty. \tag{3.22}$$

By (3.21) and (3.22) we obtain that

$$\sum_{j=0}^{k-1} C^{j-k}\cdot\omega(f(j)) = O\{\omega(f(k))\} \quad \text{as} \quad k\to\infty. \tag{3.23}$$

Applying Cauchy—Bunjakowski's inequality, by (3.23),

$$\sum_{k=1}^{N}\sum_{j=0}^{k-1} C^{j-k}\cdot\frac{1}{\sqrt{\omega(f(k))}}\cdot\Big\{\int_0^1 |\Delta_k|^2\,dx\cdot\int_0^1 |\Delta_j|^2\,dx\cdot\omega(f(j))\Big\}^{1/2} \leqq$$

$$\leqq \sum_{k=1}^{N}\{\omega(f(k))\}^{-1/2}\cdot\Big\{\int_0^1 |\Delta_k|^2\,dx\Big\}^{1/2}\cdot\Big(\sum_{j=1}^{k=1} C^{j-k}\cdot\omega(f(j))\Big)^{1/2}\times$$

$$\times\Big\{\sum_{j=0}^{k-1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big\}^{1/2} = O\Big\{\sum_{k=1}^{N}\Big(\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\cdot\Big(\sum_{j=0}^{k+1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big)^{1/2}\Big\} =$$

$$= O\Big(\Big(\sum_{k=0}^{N}\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\cdot\Big(\sum_{k=1}^{N}\sum_{j=0}^{k-1} C^{j-k}\cdot\int_0^1 |\Delta_j|^2\,dx\Big)^{1/2}\Big) =$$

$$= O\Big((B_N^2)^{1/2}\cdot\Big(\sum_{k=0}^{N-1}\int_0^1 |\Delta_k|^2\,dx\Big)^{1/2}\Big) = O(B_N^2).$$

Thus, the relation $L_N^{(2)}=O(B_N^4)$ is proved. The proof of $L_N^{(3)}=O(B_N^4)$ runs similarly.

Using these relations, by (3.15) and (3.16), we get

$$\sum_{k,j=0}^{N}{}'' \int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = o(B_N^4) \quad \text{as} \quad N\to\infty. \tag{3.24}$$

By (3.13) and (3.24) we have

$$\int_0^1 \Big\{\Big|\sum_{k=0}^{N}(\Delta_k)^2\Big|^2\Big\}\,dx = \eta B_N^4 + o(B_N^4) \quad \text{as} \quad N\to\infty. \tag{3.25}$$

Now let us consider the value

$$\int_0^1 \Big\{\sum_{k=0}^{N}(\Delta_k^2+\bar{\Delta}_k^2)\Big\}\,dx = \sum_{k=0}^{N}\Big\{\int_0^1 (\Delta_k)^2\,dx + \int_0^1 (\bar{\Delta}_k^2)\,dx\Big\}.$$

The terms of the type $\int_0^1 (\Delta_k)^2 dx$ consist of the summands of the species

$$\int_0^1 (a_q a_r \chi_{n_q}\chi_{n_r})\,dx,$$

where the functions χ_{n_r} and χ_{n_q} belong to the k-th block. The quantity of non-zero summands of this species depends on the number of conjugate pairs (n_q, n_r) in the k-th block. Therefore, using (1.3), we have

$$\sum_{k=0}^{N} \int_0^1 (\Delta_k)^2 dx = \sum_{k}^{f(N+1)} (a_k \cdot \hat{a}_k) = \eta B_N^2 + o(B_N^2) \quad \text{as} \quad N \to \infty. \tag{3.26}$$

Analogously

$$\sum_{k=0}^{N} \int_0^1 (\bar{\Delta}_k^2)\, dx = \eta B_N^2 + o(B_N^2) \quad \text{as} \quad N \to \infty. \tag{3.27}$$

Finally, substituting estimations (3.25)—(3.27) into (3.10), we receive that

$$\int_0^1 \left| \frac{1}{B_N^2} \sum_{k=0}^{N} (\Delta_k)^2 - \eta \right|^2 dx = \frac{1}{B_N^4} \{\eta^2 \cdot B_N^4 + o(B_N^4) - \eta B_N^2 \cdot (2\eta B_N^2 + o(B_N^2)) + B_N^4 \cdot \eta^2\} =$$

$$= \frac{1}{B_N^4} \{\eta^2 B_N^4 - 2\eta^2 B_N^4 + \eta^2 B_N^4 + o(B_N^4)\} = o(1) \quad \text{as} \quad N \to \infty.$$

Thus Lemma 3.3 is proved.

4. The proof of Theorem A. Lemmas 3.2 and 3.3 imply that if the conditions of Theorem A are fulfilled then

$$\frac{1}{B_k^2} \sum_{j=0}^{k} |\Delta_j|^2 - 1 \xrightarrow{P} 0 \quad \text{and} \quad \frac{1}{B_k^2} \sum_{j=0}^{k} (\Delta_j)^2 - \eta \xrightarrow{P} 0.$$

Regarding the definitions (3.2), we obtain

$$\sum_{j=0}^{k} |X_{k,j}|^2 \xrightarrow{P} 1 \quad \text{and} \quad \sum_{j=0}^{k} (X_{k,j})^2 \xrightarrow{P} \eta. \tag{4.1}$$

Now we show that (4.1) implies the realization of condition c) of Lemma 2.3 (for the sequence $\{x_{k,j}; \mathscr{F}_{k,j}\}$).

Let $X_{k,j} = (\mu_{k,j}; \nu_{k,j})$; $k = 0, 1, \dots$; $0 \leq j \leq k$, where

$$\mu_{k,j} = \operatorname{Re}\{X_{k,j}\}; \; \nu_{k,j} = \operatorname{Im}\{X_{k,j}\}.$$

Then

$$|X_{k,j}|^2 = (\mu_{k,j})^2 + (\nu_{k,j})^2,$$

$$(X_{k,j})^2 = \{(\mu_{k,j})^2 - (\nu_{k,j})^2\} + 2i \cdot (\mu_{k,j} \cdot \nu_{k,j})$$

(here i denotes the imaginary unit).

Substituting it into (4.1), we get

$$\sum_{j=0}^{k} \{(\mu_{k,j})^2 + (\nu_{k,j})^2\} \xrightarrow{P} 1 \tag{4.2}$$

$$\sum_{j=0}^{k} \{(\mu_{k,j})^2 - (\nu_{k,j})^2\} + 2i(\mu_{k,j} \cdot \nu_{k,j}) \xrightarrow{P} \eta. \tag{4.3}$$

Adding and subtracting equalities (4.2) and (4.3) we conclude that

$$\text{(4.4)}\qquad \begin{cases} \sum_{j=0}^{k} (\mu_{k,j})^2 \xrightarrow{p} \dfrac{1+\eta}{2}, \\ \sum_{j=0}^{k} (\nu_{k,j})^2 \xrightarrow{p} \dfrac{1-\eta}{2}, \\ \sum_{j=0}^{k} (\mu_{k,j}\cdot \nu_{k,j}) \xrightarrow{p} 0. \end{cases}$$

Relations (4.4) show that Lemma 2.3 is applicable for the sequence $\{X_{k,j}; \mathscr{F}_{k,j}\}$. It implies the validity of (3.6). Thus we have

$$S_k = \sum_{j=0}^{k} X_{k,j} \xrightarrow{d} \mathscr{N}(m, \mathbf{R}),$$

where $m=(0, 0)$, $\mathbf{R}=\dfrac{1}{2}\begin{pmatrix} 1+\eta & 0 \\ 0 & 1-\eta \end{pmatrix}$.

Finally, taking into account the definition of the value $T_N(x)$ and relations (3.4)—(3.6), we obtain that

$$T_N(x) \xrightarrow{d} \mathscr{N}(m, \mathbf{R}),$$

where $m=(0, 0)$, $\mathbf{R}=\|r_{kl}\|$, $r_{11}=\dfrac{1}{2}(1+\eta)$, $r_{12}=r_{21}=0$, $r_{22}=\dfrac{1}{2}(1-\eta)$.

Herewith Theorem A is proved completely.

Remark 4.1. The foregoing proof implies that if our system $X=\{\chi_n(x)\}_{n=0}^{\infty}$ is real-valued then $\overline{\Delta_k(x)}=\Delta_k(x)$ and the assertions of Lemmas 3.2 and 3.3 coincide, therefore we have $\eta=1$ (because $\overline{\chi_n(x)}=\chi_n(x)$ for all n). Then, in the case of Walsh—Paley's system, the realization of condition (1.2) already is sufficient. Condition (1.3) is fulfilled automatically ($\eta=1$), and condition (1.4) is furnished by conditions (4)—(6) and (1.2) (see, e.g., the proof of Lemma 2.4 in [8]). The covariant matrix in this case is the following:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(it conforms to the normal distribution of a vector such that one of its components — the imaginary part in our case — equals zero identically).

Remark 4.2. Since the divergence of the series $\sum_{k=1}^{\infty} a_k^2$ implies the divergence of the series $\sum_{k=1}^{\infty} \frac{a_k^2}{A_k^2}$, thus the sequence $\{\omega(k)\}$ in Theorem A must satisfy the condition: $\sum_{k=1}^{\infty} (\omega(k))^2 = \infty$.

As a sample example realizing the conditions of Theorem A for complex-valued PMONS we bring the next one.

Let us consider the Chrestenson—Levy's system generated by $p_k \equiv 3$. Then $m_0 = 1, m_1 = 3, \ldots, m_k = 3^k; k = 1, 2, \ldots$; the functions $\chi_{m_k}(x)$ are "basis" in the blocks $[m_k, m_{k+1})$. Put $n_1 = m_1$, $n_2 = 2m_1, \ldots, n_{2i-1} = m_i$, $n_{2i} = 2m_i$; $i = 1, 2, \ldots$. Let $a_k = 1$ for all k. So, for our sequence $\{n_k\}$ $\frac{n_{k+1}}{n_k} \geqq \frac{3}{2}$, $k = 1, 2, \ldots$ hold, and we can put $\omega(k) \equiv \frac{1}{2}$.

Then the conditions of Theorem 1 are fulfilled trivially. Indeed, $A_k = \sqrt{k}$ for all k. It is also clear that $\overline{\chi_{m_k}(x)} = (\chi_{m_k}(x))^2 = \chi_{2m_k}(x)$. Consequently the quantity of the conjugate pairs is equal to 2 in each block (we remind that the pairs (n_q, n_r) and (n_r, n_q) are considered as distinct if $q \neq r$).

At the same time $B_k^2 = f(k+1) = 2k$. Thus,

$$\eta = \lim_{N \to \infty} \frac{1}{B_N^2} \sum_{j}^{f(N+1)} (a_j \cdot \hat{a}_j) = \lim_{N \to \infty} \frac{2N}{2N} = 1.$$

The validity of (1.4) follows from the fact that there are no (l, k)-adjoint numbers in our sequence $\{n_k\}$, i.e. $\lambda_k^l(q) = 0$ for all k, l, q $(0 \leqq l \leqq k)$. Therefore, the constructed subsystem $\chi_{n_k}(x)$ is a subject to CLT with the covariant matrix $\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We also note that taking the subsystem $\{\chi_{n_k}(x)\}$ such that $n_k = m_k$ (i.e. $\{\chi_{n_k}(x)\}$ consists of the "basis" functions), then all of the conditions of Theorem A are fulfilled and we evidently have $\eta = 0$ in this case. So the covariant matrix is $\mathbf{R} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (as before we put $a_k = 1$ for all k).

5. The sharpness of conditions of Theorem A. The following theorem shows that conditions (1.2)—(1.4) in Theorem A cannot be weakened, generally.

Theorem B. *There exist sequences* $\{a_k\}$, $\{n_k\}$ *and* $\{\omega(k)\}$, *satisfying conditions* (4)—(6) *and there exists a* PMONS $\{\chi_{n_k}(x)\}$ *such that if even any of conditions* (1.2)—(1.4) *is broken then the subsystem* $\{a_k \chi_{n_k}(x)\}$ *is not the subject to* CLT.

In the proof of Theorem B we shall use the following fact (see e.g. [9], pp. 195—198):

Let the sequence $\{\xi_n\}$ of random vectors be given, where $\xi_n=(\xi_n^1, \xi_n^2, \dots, \xi_n^k)$ weakly converges to some random vector $\xi=(\xi^1, \xi^2, \dots, \xi^k)$. Also let $\mu_n^{(r)}$ and $m_n^{(\nu)}$ denote the r-th absolute moment and the ν-th $(\nu=(\nu_1, \nu_2, \dots, \nu_k))$ mixed moment of random variable ξ_n, respectively; i.e.

$$M_n^{(r)} = M|\xi_n|^r \quad \text{and} \quad m_n^{(\nu)} = M\{(\xi_n^1)^{\nu_1}\cdot(\xi_n^2)^{\nu_2} \dots (\xi_n^k)^{\nu_k}\},$$

where $\nu_i \geqq 0$, $i=1, 2, \dots, k$.

In this case, if the sequence $\{\mu_n^{(r_0+\delta)}\}$ is bounded for some $\delta>0$, then the sequences $\{\mu_n^{(\nu)}\}$ and $\{m_n^{(\nu)}\}$ of the moments converge to the corresponding moments of the distribution of vector ξ for all r, $|\nu|\leqq r_0$ (here $|\nu|=\sum_{i=1}^{k}\nu_i$), i.e.

$$\mu_n^{(r)} \to \mu^{(r)}, \quad m_n^{(\nu)} = m_n^{(\nu_1, \nu_2, \dots, \nu_k)} \to m^{(\nu)} \quad \text{as} \quad n\to\infty. \tag{5.1}$$

Moreover, the mentioned limits are finite.

The direct proof of Theorem B requires constructions of counterexamples, that are showing the necessity of conditions (1.2)—(1.4), by turns. First we notice that the necessity of (1.2) was shown in [7].

6. Counterexamples. Passing to the proof of the necessity of (1.3), let us choose the Chrestenson—Levy's system generated by $p_k\equiv 3$. Put $a_k=1$ for all k. The sequence $\{n_k\}$ is constructed in the following way:

$$n_1 = m_1, \quad n_2 = 2m_1, \quad n_{2k-1} = m_k \quad (m_k = 3^k, k = 1, 2, \dots)$$

$$n_{2k} = \begin{cases} 2m_k - 1, & \text{if} \quad 10^{2l} < k \leqq 10^{2l+1}; \\ 2m_k & \text{if} \quad 10^{2l+1} < k \leqq 10^{2l+2}; \end{cases} \quad l = 0, 1, \dots \tag{6.1}$$

Thus there exists one pair of the terms of $\{\chi_{n_k}(x)\}$ in every block $[m_k, m_{k+1})$: The terms of n_k can be conjugate if $n_{2k}=2n_{2k-1}$. Let us check the fulfilment of the conditions of Theorem A for $\{n_k\}$.

Since $n_{k+1}\geqq\frac{3}{2}n_k$ for all k, it is clear that we can put $\omega(k)\equiv\frac{1}{2}$. Further, $A_k=\sqrt{k}\to\infty$ and $a_k=1=o\big(A_k\omega(k)\big)$ as $k\to\infty$. For these reasons conditions (4)—(6) and (1.2) are fulfilled.

The verification of condition (1.4) is trivial, because $\lambda_k^j(q)=0$ for all q, k and $0\leqq j\leqq k-1$ and $\lambda_k^k(q)=1$ for $q=2k$, $10^{2l}+1\leqq k\leqq 10^{2l+1}$.

In the same time the value

$$C_N = \frac{1}{B_N^2}\sum_{j}^{f(N+1)} (a_j\cdot \hat{a}_j)$$

has no limit, since

$$C_{10^{2l+1}} < \frac{2\cdot 10^{2l}}{2\cdot 10^{2l+1}} = \frac{1}{10} \quad \text{for} \quad l = 0, 1, \dots,$$

but

$$C_{10^{2l+2}} \geqq \frac{2\cdot 9\cdot 10^{2l+1}}{2\cdot 10^{2l+2}} = \frac{9}{10} \quad \text{for} \quad l = 0, 1, \ldots.$$

So, (1.3) is failed. Now we shall show that CLT for $\{\chi_{n_k}(x)\}$ does not hold. Let us estimate the absolute moment of the 4-th power of the random variable

$$T_N(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \chi_{n_k}(x).$$

We have

$$\mu_N^{(4)} = \int_0^1 |T_N(x)|^2\,dx = \frac{1}{N^2} \sum_{1\leqq i,\,j,\,p,\,q\leqq N} \int_0^1 (\chi_{n_q}\chi_{n_p}\bar{\chi}_{n_j}\bar{\chi}_{n_i})\,dx. \tag{6.2}$$

The summands in the right side of (6.2) are distinct from zero if and only if

$$\chi_{n_p}\cdot\chi_{n_q}\cdot\bar{\chi}_{n_i}\cdot\bar{\chi}_{n_j} = 1. \tag{6.3}$$

Arguing the same way as in the proof of Lemma 2.4 in [8], we conclude that (6.3) holds only if the fourfold product $\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j}$ has a decomposition of two pairs such that each of the pairs belongs to certain block $[m_k, m_{k+1})$, perhaps, to the same. Since the number of the blocks is not more than $\left[\frac{N}{2}\right]+1$, thus the number of the non-zero summands in the right side of (6.2) does not exceed the value $4!\left(\left[\frac{N}{2}\right]+1\right)^2$. Consequently,

$$\mu_N^{(4)} \leqq \frac{1}{N^2}\,4!\left(\left[\frac{N}{2}\right]+1\right)^2 = O(1).$$

Thus, the sequence $\{\mu_N^{(4)}\}$ is bounded.

Now let us use relations (5.1). If the sequence $\{T_N(x)\}$ weakly converged to some (Gaussian) random variable $T(x)=(\xi^1(x), \xi^2(x))$, then the following limits would exist:

$$\lim_{N\to\infty} m_N^{(2,0)} = \lim_{N\to\infty} M\{(\xi_N^1(x))^2\} = \lim_{N\to\infty} M\{(\mathrm{Re}\,(T_N(x)))^2\},$$

$$\lim_{N\to\infty} m_N^{(0,2)} = \lim_{N\to\infty} M\{(\xi_N^2(x))^2\} = \lim_{N\to\infty} M\{(\mathrm{Im}(T_N(x)))^2\},$$

$$\lim_{N\to\infty} m_N^{(1,1)} = \lim_{N\to\infty} m\{(\xi_N^1(x))\cdot(\xi_N^2(x))\} = \lim_{N\to\infty} M\{(\mathrm{Re}(T_N(x)))(\mathrm{Im}(T_N(x)))\}.$$

These relations imply that under the assumption $T_N(x)\xrightarrow{d} T(x)$ there exists a finite limit of the value

$$M\{\mathrm{Re}^2(T_N(x)) - \mathrm{Im}^2(T_N(x)) + 2i\cdot\mathrm{Re}\,(T_N(x))\cdot\mathrm{Im}\,(T_N(x))\}$$

(where i denotes the imaginary unit).

In other words, the limit ought to exist

$$\lim_{N\to\infty} M\{(T_N(x))^2\} = \lim_{N\to\infty} \int_0^1 (T_N(x))^2\,dx = \lim_{N\to\infty} \frac{1}{N} \sum_{1\le i,j\le N} \int_0^1 (\chi_{n_i}\cdot\chi_{n_j})\,dx. \tag{6.4}$$

The quantity of the non-zero summands of $\int_0^1 (\chi_{n_i}\chi_{n_j})dx$ is given by the number of the conjugate pairs (n_i, n_j) in our sequence $\{n_k\}$. By (6.1) it is easy to see that

$$\int_0^1 (T_{20}(x))^2\,dx = \frac{2}{20} = \frac{1}{10},$$

$$\int_0^1 (T_{200}(x))^2\,dx = \frac{182}{200} > \frac{9}{10};$$

and so on, generally,

$$\int_0^1 (T_{2\cdot 10^{2l-1}}(x))^2\,dx \leqq \frac{1}{10}, \quad \int_0^1 (T_{2\cdot 10^{2l}}(x))^2\,dx > \frac{9}{10} \quad \text{for all} \quad l = 1, 2, \dots. \tag{6.5}$$

Inequalities (6.5) show that limit (6.4) for the subsystem $\{\chi_{n_k}(x)\}$ does not exist. Therefore, the sequence $\{T_N(x)\}$ cannot converge (with respect to distribution) to any random variable $T(x)$. This contradiction proves that CLT does not hold for the subsystem $\{\chi_{n_k}(x)\}$, and this completes the proof.

Furthermore, for the proof of the necessity of (1.4), let us take the Chrestenson—Levy's system generated by $p_k\equiv 5$. As before we put $a_k=1$ for all k. The sequence $\{n_k\}$ will be defined in the following way:

$$n_1 = 2m_1, \quad n_2 = 4m_1, \quad n_{2k-1} = 2m_k \quad (m_k = 5^k, k = 1, 2, \dots)$$

$$n_{2k} = \begin{cases} 3m_k+1, & 10^{2l} < k \leqq 10^{2l+1}; \\ 4m_k, & 10^{2l+1} < k \leqq 10^{2l+2} \end{cases} \quad l = 0, 1, \dots. \tag{6.6}$$

Let us verify the fulfilment of Theorem A. We have $n_{k+1}\geqq\frac{5}{4}n_k$ for all k, consequently we can put $\omega(k)\equiv 1/4$. Conditions (4)—(6) and (1.2) in this case are also fulfilled evidently. Condition (1.3) is fulfilled because there are no conjugate numbers in our sequence $\{n_k\}$ and by the same reason the limit in (1.3) is equal to zero.

But condition (1.4) does not fulfil. Indeed, the numbers $2m_k$ and $3m_k+1$ are $(0, k)$-adjoint, therefore

$$\lambda_k^0(q) = 1 \quad \text{for} \quad q = 2k \quad \text{and} \quad 10^{2l}+1 \leqq k \leqq 10^{2l+1},$$

hence

$$\forall C > 1 \quad \lambda_k^0(2k)\cdot C^k \neq O(1) = O\left\{\frac{1}{\omega(f(k))}\right\} \quad \text{as} \quad k\to\infty.$$

Let us show that CLT for $\{\chi_{n_k}(x)\}$ also does not hold. Now we consider the 6-th absolute moment of the random variable $T_N(x)$. We have

$$\mu_N^{(6)} = \int_0^1 |T_N(x)|^6\,dx = \frac{1}{N^3}\sum_{1\le i,j,h,p,q,r\le N}\int_0^1 (\chi_{n_p}\chi_{n_q}\chi_{n_r}\bar{\chi}_{n_h}\bar{\chi}_{n_i}\bar{\chi}_{n_j})\,dx.$$

Arguing as in the proof of the previous counterexample of the boundedness of $\mu_N^{(4)}$, we can see that the sequence $\{\mu_N^{(6)}\}$ is bounded. Now if we assume that $\{T_N(x)\}$ weakly converges to some Gaussian random variable $T(x)$ then (5.1) implies that the limit of the sequence $\{\mu_N^{(4)}\}$ exists. So we get

$$\mu_N^{(4)} = \int_0^1 |T_N(x)|^4\,dx = \frac{1}{N^2}\sum_{1\le i,j,p,q\le N}\int_0^1 (\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j})\,dx. \tag{6.7}$$

The definition of $\{n_k\}$ (see (6.6)) shows that the summands of the type $\int_0^1 (\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j})dx$ differ from zero if the fourfold product $\chi_{n_p}\chi_{n_q}\bar{\chi}_{n_i}\bar{\chi}_{n_j}$ either consists of the factors belonging to the same block or this product decomposes into two pairs of the factors such that each of the pairs belong to different block. Therefore, if $N=2M$ then we can rewrite (6.7) in the following form:

$$\mu_N^{(4)} = \frac{1}{4M^2}\int_0^1 \{|\sum_{k=0}^{M}\Delta_k|^2\}^2\,dx = \frac{1}{4M^2}\int_0^1 \{\sum_{k=1}^{M}\Delta_k\cdot\sum_{j=1}^{M}\bar{\Delta}_j\}^2\,dx = \tag{6.8}$$

$$= \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\Big\{\int_0^1 (|\Delta_k|^2\cdot|\bar{\Delta}_j|^2+(\Delta_k)^2\cdot(\bar{\Delta}_j)^2)\,dx\Big\} =$$

$$= \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\int_0^1 (|\Delta_k|^2\cdot|\bar{\Delta}_j|^2)\,dx + \frac{1}{4M^2}\sum_{k=1}^{M}\sum_{j=1}^{M}\int_0^1 (\Delta_k^2\cdot\bar{\Delta}_j^2)\,dx = L_M^{(1)}+L_M^{(2)}.$$

By a direct calculation it is possible to see that

$$\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_j|^2\,dx = \begin{cases}4, & \text{if } k\ne j,\\ 6, & \text{if } k=j.\end{cases}$$

Hence

$$L_M^{(1)} = \frac{1}{4M^2}\Big\{\sum_{k=1}^{M}\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_k|^2\,dx + \sum_{k=2}^{M}\sum_{j=1}^{k-1}\int_0^1 |\Delta_k|^2\cdot|\bar{\Delta}_j|^2\,dx\Big\} = \tag{6.9}$$

$$= \frac{1}{4M^2}\Big(6M+4\cdot\frac{M(M-1)}{2}\Big) = \frac{1}{2}+\frac{1}{M}.$$

So, if $M\to\infty$ (i.e. as $N\to\infty$)

$$\lim_{M\to\infty} L_M^{(1)} = \frac{1}{2}. \tag{6.9}$$

At the same time we have

(6.10)

$$\int_0^1 (\Delta_k)^2\cdot(\Delta_j)^2\,dx=\begin{cases}6, & \text{if } k=j;\\ 4, & \text{if } \begin{cases}10^{2l}<k\le 10^{2l+1};\\ 10^{2m}<j\le 10^{2m+1};\end{cases} \quad l,m=0,1,\dots;\ l\ne m:\\ 0, & \text{othervise}\end{cases}$$

Therefore

$$L_{10}^{(2)}\ge\frac{1}{400}\sum_{k=2}^{10}\sum_{j=2}^{10}\int_0^1(\Delta_k^2\cdot\bar\Delta_j^2)\,dx=\frac{4\cdot 9\cdot 9}{400}=\frac{81}{100},$$

$$L_{100}^{(2)}=\frac{1}{40\,000}(6\cdot 100+4\cdot 9\cdot 9)=\frac{231}{10\,000}<\frac{3}{100},$$

and so on, generally,

(6.11)

$$L_{10^{2l+1}}^{(2)}\ge(4\cdot 10^{4l+2})^{-1}\cdot\sum_{k=10^{2l}+1}^{10^{2l+1}}\sum_{j=10^{2l}+1}^{10^{2l+1}}\int_0^1(\Delta_k^2\cdot\bar\Delta_j^2)\,dx=\frac{4\cdot(9\cdot 10^{2l})^2}{4\cdot 10^{4l+2}}=\frac{81}{100};$$

(6.12) $$L_{10^{2l+2}}^{(2)}\le\frac{1}{4\cdot 10^{4l+4}}\cdot\Big(6\cdot\sum_{k=10^{2l}+1}^{10^{2l+1}}\sum_{j=10^{2l}+1}^{10^{2l+1}}\int_0^1(\Delta_k^2\cdot\bar\Delta_j^2)\,dx+6\cdot 10^{2l+2}\Big)=$$

$$=\frac{6\cdot(10^{2l+2}+10^{4l+2})}{4\cdot 10^{4l+4}}\le\frac{12\cdot 10^{4l+2}}{4\cdot 10^{4l+4}}=\frac{3}{100}$$

for all $l=0,1,2,\dots$.

Inequalities (6.11)—(6.12) show that the value $L_M^{(2)}$ has no limit if $M\to\infty$, and so if $N\to\infty$. Comparing it with (6.9) and (6.10) we can conclude that the sequence $\{\mu_N^{(4)}\}$ also diverges as $N\to\infty$. The obtained contradiction (with assumption about the weak convergence of $\{T_N(x)\}$) implies that the subsystem $\{\chi_{n_k}(x)\}$ is not subjected to CLT as desired.

Theorem B is proved completely.

Remark 6.1. Theorems A and B demonstrate that the known results on CLT with respect to real-valued orthonormal systems (for example, trigonometric system or Walsh's system) have no direct analogues in the case of general PMONS. Namely, in order to prove the validity of CLT in our case, it is not sufficient to know the ratio of the lacunarity of $\{n_k\}$ and the magnitude of the coefficients $\{a_k\}$ but we have to know certain facts about the existence and the regularity of the conjugate and the (l,k)-adjoint numbers in the sequence $\{n_k\}$. We also mention that in our case it can occur, despite a very good lacunarity of $\{n_k\}$, that conditions (1.3) and (1.4) are not fulfilled independently of each other.

It should be noted that some problems, closely connected with them here, were studied in [4]—[5], but they were formulated in a different way; in addition, for the sequences $\{n_k\}$ there were assumed certain "arithmetical" conditions.

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