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On commutativity of left s-unital rings

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1. Introduction. In this paper we study the commutativity of a left s-unital ring R satisfying the polynomial identity

(1) $$x^t[x^n, y] = \pm y^r[x, y^m]y^s \quad \text{for all} \quad x, y \in R,$$

where m, n, r, s and t are fixed non-negative integers. To establish commutativity, we need some extra conditions. The results of this paper generalize some of the well-known commutativity theorems.

2. Preliminary results. Throughout the present paper, R will represent an associative ring (not necessarily with unity 1), $Z(R)$ the center of R, $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements in R, $N'(R)$ the set of all zero-divisors in R, and R^+ the additive group of R. As usual, for each $x, y \in R$, we write $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $(GF(p))_2$, for a prime p.

Definition 1. A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, R is called *s-unital* if it is both left as well as right s-unital, that is, $x \in xR \cap Rx$ for each $x \in R$.

Definition 2. If R is an s-unital (resp. a left or right s-unital) ring, then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e is called the *pseudo* (resp. *pseudo left* or *pseudo right*) *identity* of F in R.

Definition 3. For any positive integer n, the ring R is said to have *property $Q(n)$* if for all $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

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The property $Q(n)$ is an H-property in the sense of [9]. It is obvious that every n-torsion free ring R has the property $Q(n)$, and every ring has the property $Q(1)$. Also, it is clear that if a ring R has the property $Q(n)$, then R has the property $Q(m)$ for every divisor m of n.

In the proof of our results, we shall require the following well-known results.

Lemma 1 ([3, Lemma 2]). *Let R be a ring with unity 1, and let x and y be elements in R. If $kx^m[x, y]=0$ and $k(x+1)^m[x, y]=0$ for some integers $m\geqq 1$ and $k\geqq 1$, then necessarily $k[x, y]=0$.*

Lemma 2 ([14, Lemma 3]). *Let x and y be elements in a ring R. If $[x, [x, y]]=0$, then $[x^k, y]=kx^{k-1}[x, y]$ for all integers $k\geqq 1$.*

Lemma 3 ([18, Lemma 3]). *Let R be a ring with unity 1, and let x and y be elements in R. If $(1-y^k)x=0$, then $(1-y^{km})x=0$ for some integers $k>0$ and $m>0$.*

Lemma 4. *Let x and y be elements in a ring R. Suppose that there exists relatively prime positive integers m and n such that $m[x, y]=0$ and $n[x, y]=0$. Then $[x, y]=0$.*

Lemma 5 ([4, Theorem 4 (C)]). *Let R be a ring with unity 1. Suppose that for each $x\in R$ there exists a pair n and m of relatively prime positive integers for which $x^n\in Z(R)$ and $x^m\in Z(R)$. Then R is commutative.*

Following results play an important role in proving the main results of this paper. The first is due to KEZLAN [10, Theorem] and BELL [3, Theorem 1] (also see [9, Proposition 2]), the second and third are due to Herstein.

Theorem KB. *Let f be a polynomial in n non-commuting indeterminates $x_1, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(1) *For any ring satisfying the polynomial identity $f=0$, $C(R)$ is a nil ideal.*

(2) *For every prime p, $(GF(p))_2$ fails to satisfy $f=0$.*

(3) *Every semi-prime ring satisfying $f=0$ is commutative.*

Theorem H ([7, Theorem 18]). *Let R be a ring and let $n>1$ be an integer. Suppose that $x^n-x\in Z(R)$ for all $x\in R$. Then R is commutative.*

Theorem H′ ([8, Theorem]). *If for every x and y in a ring R we can find a polynomial $p_{x,y}(t)$ with integral coefficients which depends on x and y such that $[x^2p_{x,y}(x)-x, y]=0$, then R is commutative.*

3. Main Results. Now, we present our results.

Theorem 1. *Let* $n>1$, m, r, s *and* t *be fixed non-negative integers, and let* R *be a left s-unital ring satisfying the polynomial identity* (1). *Further, if* R *possesses* $Q(n)$, *then* R *is commutative.*

Following lemma shows that the ring considered in Theorem 1 is in fact an *s*-unital ring. According to Proposition 1 of [9] this lemma enables us to reduce the proof of Theorem 1 to a ring with unity 1.

Lemma 6. *Let* $n>0$, m, r, s *and* t *be fixed non-negative integers such that* $(r, n, s, m, t) \neq (0, 1, 0, 1, 0)$, *and let* R *be a left s-unital ring satisfying the polynomial identity* (1). *Then* R *is s-unital.*

Proof. Let x and y be arbitrary elements in R. Suppose that R is a left *s*-unital ring. Then there exists an element $e\in R$ such that $ex=x$ and $ey=y$. Replace x by e in (1). Then $e^{t+n}y-e^tye^n=\pm(y^rey^{m+s}-y^{r+m}ey^s)$. Thus $y=ye^n\in yR$ for all $y\in R$. Therefore, R is *s*-unital.

Lemma 7. *Let* $n>0$, m, r, s *and* t *be fixed non-negative integers, and let* R *be a ring satisfying the polynomial identity* (1). *Then* $C(R)$ *is nil.*

Proof. Let $x=e_{11}$ and $y=e_{12}$. Then x and y fail to satisfy the polynomial identity (1) whenever $n>0$ except for $r=s=0$, $m=1$. In this later case one can choose $x=e_{12}$ and $y=e_{21}$. By Theorem KB,

$$C(R) \subseteqq N(R). \tag{2}$$

Combining Lemma 7 with Theorem KB gives the following commutativity theorem for semi-prime rings.

Theorem 2. *Let* $n>0$, m, r, s *and* t *be fixed non-negative integers. If* R *is a semi-prime ring satisfies the polynomial identity* (1), *then* R *is commutative.*

Lemma 8. *Let* $n>1$, m, r, s *and* t *be fixed non-negative integers, and let* R *be a ring with unity* 1. *Suppose that* R *satisfies the polynomial identity* (1). *Further, if* R *has* $Q(n)$, *then* $N(R)\subseteqq Z(R)$.

Proof. If $a\in N(R)$, then there exists a positive integer p such that

$$a^k\in Z(R) \quad \text{for all } k \geqq p, \quad \text{and } p \text{ minimal.} \tag{3}$$

Let $p=1$. Then $a\in Z(R)$. Suppose that $p>1$ and $b=a^{p-1}$. Replace x by b in (1) to get $b^t[b^n, y]=\pm y^r[b, y^m]y^s$. In view of (3) and the fact that $(p-1)n\geqq p$ for $n>1$,

$$y^r[b, y^m]y^s = 0 \quad \text{for all} \quad y\in R. \tag{4}$$

Now, replace x by $1+b$ in (1) to get $(1+b)^t[(1+b)^n, y]=\pm y^r[1+b, y]y^s$. As $1+b$

is invertible, using (4), the last identity gives

$$[(1+b)^n, y] = 0 \quad \text{for all} \quad y \in R. \tag{5}$$

Combining (3) and (5) yield $0=[(1+b)^n, y]=[1+nb, y]=n[b, y]$. Now, $Q(n)$ implies that $[b, y]=0$ for all $y \in R$, that is $a^{p-1} \in Z(R)$. This contradicts the minimality of p. So, $p=1$ and $a \in Z(R)$. Therefore,

$$N(R) \subseteqq Z(R). \tag{2'}$$

Remark 1. Combining (2) and (2'), one gets

$$C(R) \subseteqq N(R) \subseteqq Z(R), \tag{6}$$

for any ring R with unity 1 which satisfies the polynomial identity (1) for all fixed non-negative integers $n>1$, m, r, s and t and whenever R has $Q(n)$. Hence, in view of (6), $[x, [x, y]]=0$ for all $x, y \in R$ and thus the conclusion of Lemma 2 holds. In the proof of Theorem 1, we shall therefore routinely use Lemma 2 without explicit mention.

Proof of Theorem 1. According to Lemma 6, R is s-unital. Therefore, in view of Proposition 1 of [9], it suffices to prove the theorem for R with unity 1.

If $m=0$, then (1) gives $x^t[x^n, y]=0$. Thus, $nx^{t+n-1}[x, y]=0$. Replace x by $x+1$ and apply Lemma 1 to obtain $n[x, y]=0$ which by $Q(n)$, we get $[x, y]=0$ for all $x, y \in R$. Therefore, R is commutative.

Suppose that $m \geqq 1$. Let $q=(p^{t+n}-p)$ (for a prime p). Then from (1) we have $qx^t[x^n, y]=(p^{t+n}-p)x^t[x^n, y]=p^{t+n}x^t[x^n, y]-px^t[x^n, y]=(px)^t[(px)^n, y] \mp \mp py^r[x, y^m]y^s=(px)^t[(px)^n, y] \mp y^r[(px), y^m]y^s=0$. Therefore, $qnx^{t+n-1}[x, y]=0$. If we set $k=qn$, then $k[x, y]=0$ and thus $[x^k, y]=kx^{k-1}[x, y]=0$. So

$$x^k \in Z(R) \quad \text{for all} \quad x \in R. \tag{7}$$

We consider two cases:

Case (a): If $m>1$, then $x^t[x^n, y]=\pm m[x, y]y^{r+s+m-1}$ and $x^t[x^n, y^m]=\pm \pm m[x, y^m]y^{m(r+s+m-1)}$. So $mx^t[x^n, y]y^{m-1}=-m[x, y^m]y^{m(r+t+m-1)}$. By using (1), we obtain $my^r[x, y^m]y^{s+m-1}=m[x, y]y^{m(r+s+m-1)}$. Using Lemma 3, we get

$$m[x, y^m]y^{r+s+m-1}(1-y^{k(m-1)(r+s+m-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{8}$$

Now, by (6), the polynomial identity (1) becomes

$$nx^{t+n-1}[x, y] = \pm my^{r+s+m-1}[x, y] = \pm m[x, y]y^{r+s+m-1} \quad \text{for all} \quad x, y \in R. \tag{9}$$

It is well-known that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$, the index set). Each R_i satisfies (1), (6), (7), (8) and (9). We consider the ring R_i ($i \in I$). Let S be the intersection of all non-zero ideals of R_i. Then $S \neq (0)$, and $Sd=0$ for any central zero-divisor d.

Let $a \in N'(R_i)$. By (8), $m[x, a^m]a^{r+s+m-1}(1-a^{k(m-1)(r+s+m-1)})=0$. If $m[x, a^m]a^{r+s+m-1} \neq 0$, then $a^{k(m-1)(r+s+m-1)}$ and $1-a^{k(m-1)(r+s+m-1)}$ are central zero-divisors. So $(0)=S(1-a^{k(m-1)(r+s+m-1)})=S \neq 0$, which is a contradiction. Thus

$$(10) \qquad m[x, a^m]a^{r+s+m-1} = 0 \quad \text{for all} \quad x \in R_i.$$

From (9) and (10), $nx^{t+n-1}[x, a^m] = \pm m[x, a^m]a^{m(r+s+m-1)}=0$ and $n[x, a^m]=0$. Therefore, $nm[x, a]a^{m-1}=0$. Now,

$$n^2x^{t+n-1}[x, a] = n(nx^{t+n-1}[x, a]) = \pm nm[x, a]a^{r+s+m-1} = 0$$

and $n^2[x, a]=0$. But $[x^{n^2}, a]=n^2x^{n^2-1}[x, a]$. Therefore,

$$(11) \qquad [x^{n^2}, a] = 0 \quad \text{for all} \quad x \in R_i.$$

If $c \in Z(R_i)$, then by (1), $(c^{t+n}-c)x^t[x^n, y]=(cx)^t[(cx)^n, y]-cx^t[x^n, y]=(cx)^t[(cx)^n, y] \mp y^r[(cx), y^m]y^s=0$ and thus $n(c^{t+n}-c)x^{t+n-1}[x, y]=0$. By Lemma 1 $n(c^{t+n}-c)[x, y]=0$. So

$$(12) \qquad (c^{t+n}-c)[x^n, y] = 0 \quad \text{for all} \quad x, y \in R_i.$$

Using (7), we get

$$(13) \qquad (y^{k(t+n)}-y^k)[x^n, y] = 0 \quad \text{for all} \quad x, y \in R_i.$$

Suppose that $y \in R_i$. If $[x^n, y]=0$, then $[x^{n^2}, y^j-y]=0$ for all positive integers j and $x \in R_i$. If $[x^{n^2}, y] \neq 0$, then $[x^n, y] \neq 0$, for $[x^n, y]=0$ implies that $[x^{n^2}, y]=0$, which is a contradiction. If $[x^n, y] \neq 0$, then (13) implies that $y^{k(t+n)}-y^k$ is a zero-divisor. Therefore, $y^{k(t+n-1)+1}-y$ is also a zero-divisor. By (11), we have

$$(14) \qquad [x^{n^2}, y^{k(t+n-1)+1}-y] = 0 \quad \text{for all} \quad x, y \in R_i.$$

As each R satisfies (14), the original ring R also satisfies (14). But R has $Q(n)$. Combining (14) with Lemma 2, we obtain $[x, y^{k(t+n-1)+1}-y]=0$. Therefore, R is commutative by Theorem H.

Case (b): Let $m=1$. Then $x^t[x^n, y]=\pm y^r[x, y]y^s$ and $nx^{t+n-1}[x, y]=\pm[x, y]y^{r+s}$. Replace x by x^n to get

$$nx^{n(t+n-1)}[x^n, y] = \pm[x^n, y]y^{r+s} = \pm nx^{n-1}[x, y]y^{r+s} = \pm nx^{t+n-1}[x^n, y].$$

Thus $n(1-x^{(n-1)(t+n-1)})x^{t+n-1}[x^n, y]=0$, which in view of Lemma 3, we get

$$(15) \qquad n(1-x^{k(n-1)(t+n-1)})x^{t+n-1}[x^n, y] = 0 \quad \text{for all} \quad x, y \in R.$$

As in case (a) if $a \in N'(R\ )$, then by (15), $n(1-a^{k(n-1)(t+n-1)})a^{t+n-1}[a^n, y]=0$. Also, we can prove that

$$(16) \qquad na^{t+n-1}[a^n, y] = 0 \quad \text{for all} \quad y \in R_i.$$

Now, we have $\pm[a^n, y]y^{r+s} = na^{n(t+n-1)}[a^n, y] = 0$, and thus $[a^n, y] = 0$. Therefore, $[a, y]y^{r+s} = a^t[a^n, y] = 0$. So

$$(17) \qquad [a, y] = 0 \quad \text{for all} \quad y \in R_i.$$

If $c \in Z(R_i)$, then as in case (a), we get $(c^{t+n} - c)[x, y] = 0$. In particular, by (7), $(x^{k(t+n)} - x^k)[x, y] = 0$ for all $x, y \in R_i$. If $[x, y] = 0$ for all $x, y \in R_i$, then R satisfies $[x, y] = 0$ for all $x, y \in R$. Therefore, R is commutative. Now, if for each $x, y \in R_i$, $[x, y] \neq 0$, then $x^{k(t+n-1)+1} - x \in N'(R_i)$, and hence $x^{k(t+n-1)+1} - x \in N'(R)$. But the identity (17) is satisfied by R. So $[x^{k(t+n-1)+1} - x, y] = 0$ for each $x, y \in R$. Therefore, R is commutative by Theorem H.

In Theorem 1, $Q(n)$ is essential. To see this, we consider the following example:

Example 1. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over $\mathbf{Z}_2$, the ring of integers mod 2. If R is the ring generated by the matrices A_1, B_1 and C_1, then using Dorroh construction with $\mathbf{Z}_2$ (see [4, Remark]), we obtain a ring R with unity 1. Then R is non-commutative and satisfies $[x^2, y] = [x, y^2]$ for all $x, y \in R$.

The presence of the identity in Theorem 1 is not superfluous. To see this we consider the following example.

Example 2. Let

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over $\mathbf{Z}_2$. If R is the ring generated by the matrices A_2, B_2 and C_2, then for each integer $n \geqq 1$, the ring R satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in R$, but R is not commutative.

Corollary 1 ([4, Theorem 5]). *Let R be a ring with unity* 1, *and $n > 1$ be a fixed integer. If R^+ is n-torsion free and R satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in R$, then R is commutative.*

Corollary 2 ([15, Theorem 2]). *Let $n \geqq m \geqq 1$ be fixed integers such that $mn > 1$, and let R be an s-unital ring. Suppose that every commutator in R is $m!$-torsion free.*

Further, if R satisfies the polynomial identity

$$[x^n, y] = [x, y^m] \quad \text{for all} \quad x, y \in R, \tag{18}$$

then R is commutative.

Corollary 3 ([16, Theorem 1]). *Let $n>1$ and m be positive integers, and let s and t be any non-negative integers. Let R be an associative ring with unity* 1. *Suppose*

$$x^t[x^n, y] = [x, y^m]y^s \quad \textit{for all} \quad x, y \in R. \tag{19}$$

Further, if R is n-torsion free, then R is commutative.

In the following result we show that the conclusion of Theorem 1 is still valid if $Q(n)$ is replaced by requiring m and n to be relatively prime positive integers.

Theorem 3. *Let $m>1$, and $n>1$ be relatively prime integers, and let r, s, and t be non-negative integers. If R is a left s-unital ring satisfies the polynomial identity* (1), *then R is commutative.*

Proof. According to Lemma 6, R is s-unital. Therefore, in view of Proposition 1 of [9], it is sufficient to prove the theorem for R with unity 1.

Without loss of generality, we assume that R is subdirectly irreducible. Let $a \in N(R)$. Consider p and b as in Lemma 7. Following the proof of Lemma 7, we obtain $n[b, y]=0$ and $m[b, y]=0$. By Lemma 4, $[b, y]=0$. So $a^{p-1} \in Z(R)$, which contradicts the minimality of p. Therefore $p=1$ and $a \in Z(R)$. Thus $N(R) \subseteqq Z(R)$. By Lemma 6,

$$C(R) \subseteqq N(R) \subseteqq Z(R). \tag{20}$$

The proof of (7) also works in the present situation. So there exists an integer k (as in the proof of Theorem 1) such that

$$x^k \in Z(R) \quad \text{for all} \quad x \in R. \tag{21}$$

Let $u \in N'(R)$. Using argument similar to one as in the proof of Theorem 1 (see (11)), we get $[x^{n^2}, u]=0$ and $[x^{m^2}, u]=0$. By Lemma 4,

$$[x, u] = 0 \quad \text{for all} \quad x \in R. \tag{22}$$

If $c \in Z(R)$, then, as observed in the proof followed by (11), we can prove that $n(c^{t+n}-c)[x, y]=0$ and $m(c^{t+n}-c)[x, y]=0$. By Lemma 4,

$$(c^{t+n}-c)[x, y] = 0 \quad \text{for all} \quad x, y \in R. \tag{23}$$

By (21), $(y^{k(t+n)}-y^k)[x, y]=0$. Arguing as in the proof of Theorem 1, we finally get $y^{k(t+n-1)+1}-y \in N'(R)$. Hence (22) yields $y^{k(t+n-1)+1}-y \in Z(R)$ for all $y \in R$. By Theorem H, R is commutative.

Corollary 4 ([16, Theorem 2]). *Let m and n be relatively prime positive integers, and let s and t be any non-negative integers. Suppose that R is an associative ring with unity* 1 *satisfies the polynomial identity* (19). *Then R is commutative.*

Next result deals with the commutativity of R satisfying (1) for the case $n=1$.

Theorem 4. *Let R be a left s-unital ring, and let m, r, s and t be fixed non-negative integers such that* $(t, m, r, s)\neq(0, 1, 0, 0)$. *If R satisfies the polynomial identity*

$$x^t[x, y] = \pm y^r[x, y^m]y^s \quad \textit{for all} \quad x, y\in R, \tag{24}$$

then R is commutative.

Proof. According to Lemma 6, R is an s-unital ring. In view of proposition 1 of [9], we prove the result for R with unity 1.

Case (I): If $m=0$, then the identity (24) becomes $x^t[x, y]=0$. By Lemma 1, $[x, y]=0$ for each $x, y\in R$. Therefore, R is commutative.

Case (II): Let $m>1$, $x=e_{11}$, and $y=e_{12}$. Then x and y fail to satisfy the identity (24). By Theorem KB, $C(R)\subseteq N(R)$. If $a\in N(R)$, then there exists a positive integer p such that

$$a^k\in Z(R) \quad \text{for all } k \geqq p, \text{ and } p \text{ minimal.} \tag{25}$$

If $p=1$, then $a\in Z(R)$. Now, let $p>1$, and let $b=a^{p-1}$. Replace y by b in (24) to get $x^t[x, b]=\pm b^r[x, b^m]b^s$. In view of (25), $x^t[x, b]=0$. By Lemma 1, $[x, b]=0$ for all $x\in R$. Therefore, $a^{p-1}\in Z(R)$ which is a contradiction. Thus $p=1$, and hence $N(R)\subseteq Z(R)$. So $C(R)\subseteq N(R)\subseteq Z(R)$. The method of proof of Theorem 1 enables us to establish the commutativity of R.

Case (III): Let $m=1$. Then (24) becomes

$$x^t[x, y] = \pm y^r[x, y]y^s \quad \text{for all} \quad x, y\in R. \tag{26}$$

We consider the following cases.

(i): Let $r=0$. Then (26) becomes

$$x^t[x, y] = \pm [x, y]y^s \quad \text{for all} \quad x, y\in R. \tag{27}$$

If $s=0$, then $t>0$. Thus, $x^t[x, y]=\pm[x, y]$ for all $x, y\in R$. Therefore, R is commutative by [11, Theorem]. Similarly, if $t=0$ in (27), then R is commutative by [11, Theorem]. Let $t>0$ and $s>0$. Then $x=e_{11}$, and $y=e_{12}$ fail to satisfy the identity (27). By Theorem KB, $C(R)\subseteq N(R)$. Now, for any positive integer q, we can easily see that

$$x^{qt}[x, y] = \pm [x, y]y^{qt} \quad \text{for all} \quad x, y\in R. \tag{28}$$

If $a \in N(R)$, then for sufficiently large q, we get $x^{qt}[x, a]=0$ for all $x, y \in R$. By Lemma 1, $a \in Z(R)$. Therefore $C(R) \subseteqq N(R) \subseteqq Z(R)$.

Let $l=(p^{s+1}-p)>0$ for $s>0$ (p is a prime). Then we can prove that

$$(29) \qquad x^l \in Z(R) \quad \text{for all} \quad x \in R.$$

By (28) and (29), $[x^{lt+1}, y]=\pm[x, y^{ls+1}]$ for all $x, y \in R$. In view of Proposition 3 (ii) of [9], there exists positive integer j such that $[x, y^{(ls+1)^j}]=0$ for each $x, y \in R$. But $(ls+1)^j=lk+1$. Then (28) yields $[x, y]y^{lk}=0$, and so by Lemma 1, we obtain $[x, y]=0$ for all $x, y \in R$. Therefore, R is commutative.

(ii): If $s=0$, then (26) becomes

$$(30) \qquad x^t[x, y] = \pm y^r[x, y] \quad \text{for all} \quad x, y \in R,$$

and so either $t>0$ or $r>0$. Without loss of generality, we can suppose that $r>0$. Clearly, $x=e_{11}$ and $y=e_{12}$ fail to satisfy (30). By Theorem KB, $C(R) \subseteqq N(R)$. Following the same argument as in (i) we can prove the commutativity of R.

(iii): If $t=0$, then (26) gives

$$(31) \qquad [x, y] = \pm y^r[x, y]y^s \quad \text{for all} \quad x, y \in R.$$

Then either $r>0$ or $s>0$. Clearly $x=e_{11}$ and $y=e_{12}$ fail to satisfy (31). Therefore, $C(R) \subseteqq N(R)$. Let p and b as defined in case (II). Then (31) holds and $[x, b]= =\pm b^r[x, b]b^s=0$ for all $x \in R$, which is a contradiction. Therefore $a \in Z(R)$ and $N(R) \subseteqq Z(R)$. Thus

$$(32) \qquad C(R) \subseteqq N(R) \subseteqq Z(R).$$

By (32) and Lemma 2, $[x, y]=\pm y^{r+s}[x, y]$ for all $x, y \in R$. Therefore, R is commutative by [11, Theorem].

(iv): Let $r>0, s>0$ and $t>0$. Then $x=e_{11}$ and $y=e_{12}$ fail to satisfy (26). Therefore, $C(R) \subseteqq N(R)$. If p and b are as defined in case (II), then $x^t[x, b]=\pm \pm b^r[x, b]b^s=0$. So by Lemma 1, $[x, b]=0$, which contradicts the minimality of p. Therefore, $N(R) \subseteqq Z(R)$, and thus

$$(33) \qquad C(R) \subseteqq N(R) \subseteqq Z(R).$$

By (33), the identity (26) becomes

$$(34) \qquad x^t[x, y] = \pm [x, y]y^{r+s} \quad \text{for all} \quad x, y \in R.$$

Following the proof of (i), we can establish the commutativity of R.

Corollary 5 ([12, Theorem]). *Let t and m be two fixed non-negative integers. Suppose that R satisfies the polynomial identity*

$$(35) \qquad x^t[x, y] = [x, y^m] \quad \textit{for all} \quad x, y \in R.$$

(i) *If R is left s-unital, then R is commutative except when* $(m, t)=(1, 0)$.

(ii) *If R is right s-unital, then R is commutative except when* $(m, t)=(1, 0)$; *and* $m=0$ *and* $t>0$.

Remark 2. In Corollary 5, for $m>1$, R is commutative by Theorem 1. However, for $m=0$ (resp. $m=1$ and $t>0$), it is easy to prove the commutativity of R.

Corollary 6. *Let $n>0$ and m (resp. $m>0$, and n) be fixed non-negative integers. Suppose that a left (resp. right) s-unital ring R satisfies the polynomial identity*

(36) $$[xy, x^n \pm y^m] = 0 \quad \text{for all} \quad x, y \in R.$$

If R has $Q(n)$, then R is commutative.

Proof. Actually, R satisfies the identity $x[x^n, y]=\pm[x, y^m]y$ for all $x, y\in R$. Therefore, R is commutative.

Corollary 7. *Let $m>1$ and $n>1$ be relatively prime integers, and let R be a left s-unital ring satisfying the polynomial identity* (36). *Then R is commutative.*

In [6, Theorem B], Harmanci proved that "If $n>1$ is a fixed integer and R is a ring with unity 1 which satisfies the identities $[x^n, y]=[x, y^n]$ and $[x^{n+1}, y]=[x, y^{n+1}]$ for each $x, y\in R$, then R must be commutative." In [5, Theorem 6] Bell generalized this result. The following theorem further extends the result of Bell.

Theorem 5. *Let $m>1$ and $n>1$ be fixed relatively prime integers, and let r, s and t be fixed non-negative integers. If R is a left s-unital ring satisfies both the identities*

(37) $$x^t[x^n, y] = \pm y^r[x, y^n]y^s \quad \text{and} \quad x^t[x^m, y] = \pm y^r[x, y^m]y^s \quad \text{for all} \quad x, y\in R,$$

then R is commutative.

Proof. According to Proposition 1 of [9], we prove the theorem for R with 1. Let b as in the proof of Lemma 8. Following the proof of Theorem 1 and Theorem 2 of [16], we can prove that $n[b, y]=0$ and $m[b, y]=0$. By Lemma 4, $[b, y]=0$ for all $y\in R$. The argument in the proof of Lemma 8, gives $N(R)\subseteq Z(R)$. Also, $x=e_{22}$ and $y=e_{21}+e_{22}$ fail to satisfy the polynomial identities in (37). Hence, by Theorem KB, $C(R)\subseteq N(R)$, and thus $C(R)\subseteq N(R)\subseteq Z(R)$. The argument of subdirectly irreducible rings can then be carried out for n and m, yielding integers $j>1$ and $k>1$ such that $[x^j-x, y^{n^2}]=0$ and $[x^k-x, y^{m^2}]=0$ for all $x, y\in R$. Let $f(x)=(x^j-x)^k-(x^j-x)$. Then $0=[f(x), y^{n^2}]=n^2[f(x), y]y^{n^2-1}$, and $0=[f(x), y^{m^2}]=m^2[f(x), y]y^{m^2-1}$. By Lemma 4 and Lemma 5, $[f(x), y]y^r=0$ for all $x, y\in R$, and $r=\max\{m^2-1, n^2-1\}$. Therefore, $f(x)\in Z(R)$. Since $f(x)=x^2g(x)-x$ with $g(x)$ having integral coefficients, Theorem H$'$ shows that R is commutative.

Corollary 8 ([4, Theorem 6]). *Let* $m>1$ *and* $n>1$ *be relatively prime positive integers. If* R *is any ring with unity* 1 *satisfies both the identities* $[x^m, y]=[x, y^m]$ *and* $[x^n, y]=[x, y^n]$ *for all* $x, y \in R$, *then* R *is commutative.*

Remark 3. In case $m=0$ and $n \geqq 1$, Theorem 1 need not be true for right s-unital ring. Also, when $m=0$ and $t=1$, Corollary 4 is not valid for s-unital ring. In fact we have the following example.

Example 3. Let K be a field. Then, the non-commutative ring $R=\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$, has a right identity element and satisfies the polynomial identity $x[x, y]=0$ for all $x, y \in R$. Hence, in the case $m=0$ and $n>0$, Theorem 1 need not be true for right s-unital rings. As a matter of fact, Example 3 disproves Theorems 1, 3, 4, and 5 for right s-unital case whenever both r and t are positive.

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