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Additive functions satisfying congruences

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Dedicated to János Galambos on his fiftieth birthday

1. Let A (A^*) denote the class of additive (completely additive) functions having real values, A_G (A_G^*) be the set of additive (completely additive) functions defined on the set of nonzero Gaussian integers and taking on complex values.

It seems to us very probable that a condition

$$\sum_{j=0}^{R} F_j(n+j) \equiv 0 \pmod 1 \quad (\forall n \in \mathbf{N}) \tag{1.1}$$

for $F_j \in A^*$ $(j=0, \ldots, k)$ implies that the F_j may take on only integer values, and similarly if $G_0, G_1, \ldots, G_k \in A_G^*$ and if

$$\sum_{j=0}^{k} G_j(\alpha+j) \in G \tag{1.2}$$

holds for all $a \in \mathbf{G}$ with the exception of $\alpha=0, -1, \ldots, -k$, then $G_j(\alpha) \in \mathbf{G}$ for every $\alpha \in G \setminus \{0\}$ and $j=0, \ldots, k$. In [1] the rational case was considered for $k=3$, while in [2] the Gaussian case for $k=3$, and the results support the above conjectures.

We should like to mention that our conjecture is not true in general for the wider class of additive functions. We say that an additive function F is of a finite support mod 1, if $F(p^\alpha) \equiv 0 \pmod 1$ holds for all but finitely many primes p and every $\alpha \geqq 1$. Similarly, we say that a function $G \in A_G$ is of a finite support mod G if $F(\Pi^a) \in \mathbf{G}$ holds for all prime powers Π^a with the exception of at most finitely many primes $\Pi \in G$. We guess that the conditions (1.1), (1.2) for additive functions imply that the F_j are of finite support mod 1, and G_j are of finite support mod $\mathbf{G}$. It is quite easy to determine the additive functions F or G having finite support under

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the conditions (1.1), (1.2), respectively. A recent result due to Robert Styer supports this conjecture (case $k=2$, (1.1) is assumed, $F_0, F_1, F_2 \in A$).

If $k=1$ then much more is known. Several years ago it was proved by E. Wirsing that $F \in A$, $\|F(n+1)-F(n)\| \to 0$ $(n \to \infty)$ implies that $F(n)=\tau \log n + H(n)$, where τ is a suitable real number and $H(n)$ is an integer valued additive function. Here $\|x\|$ denotes the distance of x to the nearest integer. This was a conjecture of the first named author. By his method one can get that $\|F_0(n)+F_1(n+1)\| \to 0$ $(n \to \infty)$, $F_0, F_1 \in A$ implies that $F_0(n)=\tau \log n + H_0(n)$, $F_1(n)=-\tau \log n + H_1(n)$ and $H_0(n) \equiv 0 \pmod 1$ identically.

It is quite plausible to believe that $F_0, F_1, \ldots, F_k \in A$,

$$\Big\|\sum_{j=0}^{k} F_j(n+j)\Big\| \to 0 \quad \text{as} \quad n \to \infty$$

implies that

$$F_j(n) = \tau_j \log n + H_j(n) \quad (j = 0, \ldots, k),$$

$$\sum_{j=0}^{k} \tau_j = 0, \quad \text{and} \quad \sum_{j=0}^{k} H_j(n+j) \equiv 0 \pmod 1.$$

Our purpose in this paper is to determine all those functions $G_0, \ldots, G_5 \in A_G^*$ for which (1.2) $(k=5)$ holds true. This is formulated in Theorem 3 which is an easy consequence of Theorem 1 and 2.

Theorem 1. *Let* $H_j \in A^*$ $(j=0, 1, 2)$,

(1.3) $$V(n) := H_0(n)+H_1(n+1)+H_2(n+2)-H_2(n+4)-H_1(n+5)-H_0(n+6).$$

Assume that $V(n) \equiv 0 \pmod 1$ *for every* $n \in \mathbf{N}$. *Then* $H_j(n) \equiv 0 \pmod 1$ *holds for every* $j=0, 1, 2$ *and* $n \in \mathbf{N}$.

Theorem 2. *Let* $H_j \in A_G^*$ $(j=0, 1, 2)$, *and assume that*

(1.4) $$V(\alpha) := H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6)$$

is a Gaussian integer for all $\alpha \in G \setminus \{0, -1, -2, -4, -5, -6\}$. *Then* $H_j(\alpha) \in G$ *for all* $\alpha \in G \setminus \{0\}$ *and* $j=0, 1, 2$.

2. Proof of Theorem 1.

Lemma 1. *If* $V(n) \equiv 0 \pmod 1$ *holds for every* $n \in \mathbf{N}$, *then* $H_j(n) \equiv 0 \pmod 1$ *holds for* $n \leq 17$ *and* $j=0, 1, 2$.

Proof. The following ten expressions are integers and they are linear combinations of $H_j(2)$, $H_j(3)$ and $H_j(5)$ for $j=0, 1, 2$.

1, $V(4)$

2, $V(10)-V(3)+V(6)+V(2)$

3, $V(20)+3V(1)+5V(2)-2V(3)+2V(6)-V(7)-2V(8)-2V(9)-$
$-2V(12)-2V(50)$

4, $V(28)+V(1)-V(2)-2V(3)-V(5)-V(6)-V(11)+V(24)$

5, $V(34)+3V(1)+2V(2)-2V(3)+V(5)+V(6)-V(7)-2V(8)-V(9)+V(11)-$
$-2V(12)-V(13)-V(15)-V(19)-V(21)-2V(50)$

6, $V(86)+8V(1)+4V(2)-3V(3)+3V(6)-V(7)-4V(8)-3V(9)-4V(12)-$
$-V(13)-V(14)-V(15)-V(17)-V(18)-V(19)-V(21)+V(24)-V(43)-$
$-V(45)-4V(50)$

7, $V(90)-7V(1)-3V(2)-V(3)-V(5)-3V(6)+4V(8)+V(9)-V(11)+2V(12)+$
$+V(18)-V(21)+V(45)+2V(50)$

8, $V(110)+2V(1)+5V(2)+V(3)+3V(6)-V(7)-V(8)-2V(9)-V(11)-V(12)-$
$-V(13)-V(14)-2V(15)-V(17)-V(18)-V(23)+V(32)-V(50)$

9, $V(184)+4V(1)+2V(2)+3V(3)-V(5)-2V(6)+V(7)+V(9)-V(11)+3V(12)+$
$+V(13)+V(14)+V(18)+3V(19)+3V(21)-V(23)-V(29)+V(32)+$
$+V(45)+2V(50)$

10, $V(203)+6V(1)-V(2)-2V(3)-V(5)+2V(6)+2V(7)-3V(8)+3V(9)+V(11)+$
$+V(12)+2V(13)+2V(14)+2V(15)+2V(17)+V(18)+3V(19)+4V(21)+$
$+2V(23)+V(25)+V(27)+V(29)+V(31)+V(37)+V(43)+V(45)+V(50)$

These conditions can be written in matrix form as $RH^T \equiv 0 \pmod 1$, where H^T is the transpose of the vector

$$H = [H_0(2), H_0(3), H_0(5), H_1(2), H_1(3), H_1(5), H_2(2), H_2(3), H_2(5)]$$

and **R** is the matrix with integer entries given by

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & 1 & -2 & 1 & 0 \\ -6 & 1 & 1 & 1 & 0 & -1 & 4 & 0 & -2 \\ -13 & 2 & -1 & 1 & -2 & -2 & 13 & 3 & -9 \\ 6 & 2 & -2 & 4 & -5 & 3 & -8 & 6 & 0 \\ -9 & 5 & -2 & -2 & -2 & 1 & 11 & 4 & -8 \\ -11 & 4 & -5 & 3 & -13 & 4 & 19 & 15 & -20 \\ 9 & 5 & 4 & 3 & 12 & -3 & -15 & -11 & 17 \\ -8 & -3 & 0 & -5 & 0 & -1 & 12 & 2 & -8 \\ -4 & -9 & -4 & -4 & -11 & 2 & 1 & 6 & -8 \\ -11 & 0 & -6 & 1 & -16 & 4 & 11 & 9 & -14 \end{bmatrix}.$$

Using Gaussian elimination over the integers, it follows that the third row is linearly dependent upon the others (but needed to perform the Gaussian elimination) and that $H_j(2)$, $H_j(3)$ and $H_j(5)$ are integer valued for $j=0, 1, 2$.

To show that the same is true for the other primes less than or equal to 17 we consider the following expressions, which are linear combinations of $H_j(2)$, $H_j(3)$ and $H_j(5)$ alone and are therefore integer valued. Since $V(n)\equiv 0 \pmod 1$ it follows that $H_j(p)\equiv 0 \pmod 1$ for $j=0, 1, 2$, $p=7, 11, 13, 17$.

i, $V(1)+H_0(7)$

ii, $V(2)+H_1(7)$

iii, $V(3)+H_2(7)$

iv, $V(5)-H_2(7)+H_0(11)$

v, $V(6)-H_1(7)+H_1(11)$

vi, $V(8)+H_0(7)+H_1(13)$

vii, $V(12)-H_2(7)-H_1(13)+H_1(17)$

viii, $V(50)+H_0(7)+H_1(11)-H_1(17)-H_2(13)$

ix, $V(11)-H_0(11)-H_2(13)+H_0(17)$

x, $V(18)+V(22)+V(14)-H_0(11)+H_2(13)+H_2(11)$

xi, $V(20)-H_1(7)-H_2(11)+H_0(13)$

xii, $V(26)+V(30)-H_0(13)+H_1(7)-H_2(7)+H_2(17)$.

This proves Lemma 1.

The proof of Theorem 1 is completed by verifying the inductive step which is done in the following

Lemma 2. *If* $V(n)\equiv 0$ (1) *for all* $n\in N$ *and* $H_j(n)\equiv 0$ (1) *for all* $n\leqq 17$ *and* $j=0, 1, 2$, *then* $H_j(n)\equiv 0$ (1) *for all* $n\in N$ *and* $j=0, 1, 2$.

Proof. We prove the lemma indirectly. Assume that there is some n which is smallest possible for which $H_j(n)\not\equiv 0$ (1) for $j=0$ or $j=1$ or $j=2$. Then $n=p$, p is a prime. Since $V(p-6)\equiv 0$ (1), and $V(p-5)\equiv 0$ (1), it follows that $H_2(p)\not\equiv 0$ (1). From $V(p-4)\equiv 0$ (1) it follows that p and $p+2$ must be prime, and therefore $p\equiv 2 \pmod 3$. From $V(p)\equiv 0$ (1) it follows that $p+6$ is a prime as well. Since $(p+10)$ is divisible by 3, $V(p+4)\equiv 0$ implies that $(p+8)$ must be a prime and therefore $p\equiv 1 \pmod 5$. We now consider $V(4p+6)$, which is equal to

$$H_0(4p+6)+H_1(4p+7)+H_2(4p+9)-H_2(4p+10)-H_1(4p+11)-H_0(4p+12).$$

But $(4p+6)\equiv 0 \pmod 5$ and $(4p+11)=0 \pmod 5$, while $(4p+10)=2(2p+5)$ and

$(2p+5)\equiv 0 \pmod 3$. Also $(4p+12)=4(p+3)$ with $(p+3)$ composite and $(4p+8)=$ $=4(p+2)$. Therefore $V(4p+6)\equiv 0$ (1) means that $H_1(4p+7)+H_2(p+2)\equiv 0$ (1). If $H_1(4p+7)\equiv 0$ (1) the lemma is proved, since $H_2(p)-H_2(p+2)\equiv 0$ (1), which follows from $V(p-2)\equiv 0$ (1). Since $p\equiv 2 \pmod 3$, it follows that $(4p+7)=3n$. If n is composite, $H_1(4p+7)\equiv 0$.

If n is prime, then $(n+9)<2p$ since $p>17$. Therefore $(n+k)$ is composite and less than $2p$ for $k=-1, 1, 3, 5, 7, 9$, from which it follows that $H_j(n+k)\equiv 0$ (1) for these values of k and $j=0, 1, 2$. Since $V(n-1)\equiv 0$ (1) and $V(n+3)\equiv 0$ (1) one concludes that $H_1(n)\not\equiv 0$ (1) means that n, $n+4$ and $n+8$ must all be prime which is impossible. Hence $H_1(n)\equiv 0$ (1) and therefore $H_2(p)\equiv 0$ (1), which concludes the proof of Lemma 2 and therefore the proof of Theorem 1.

3. Proof of Theorem 2. To prove Theorem 2, clearly we may assume that H_j are real valued functions. Let us observe furthermore that $H(\varepsilon\alpha)=H(\alpha)$ for each $H\in A_G^*$ and $\varepsilon=-1, i, -i$. We introduce the notations

$$V^{+1}(\alpha):=H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6),$$

$$V^{+1}(\alpha):=H_0(\alpha)+H_1(\alpha+i)+H_2(\alpha+2i)-H_2(\alpha+4i)-H_1(\alpha+5i)-H_0(\alpha+6i)$$

The norm of α is defined by $N(\alpha)=\alpha\bar{\alpha}$. The proof of Theorem 2 is also done by induction, this time using the norm of α. Because of Theorem 1, we need to prove Theorem 2 only for elements of **G** which are not real nor purely imaginary. The following lemma lists some properties of such elements.

Two Gaussian integers $\beta=u+iv$ and $\gamma=c+di$ are congruent mod 5 in the arithmetic of **G** if $u\equiv c \pmod 5$ and $v\equiv d \pmod 5$ hold simultaneously. This is denoted by $(u, v)\equiv(c, d) \pmod 5$.

Lemma 3. *Let α be a Gaussian integer such that:*

(i) *α is a prime number,*

(ii) *$\alpha=a+bi$ with $a>b>0$;*

(iii) $N(\alpha)>13$;

(iv) $(a, b)\not\equiv(4, 1) \pmod 5$.

Then

(A) *$N(\alpha-n)<N(\alpha)$ for $n=1, 2, 3, 4, 5, 6$*

and

(B) *both of $\alpha+1$ and $\alpha+i$ are composite numbers, and their norms are strictly less than $2N(\alpha)$.*

In addition at least one of the assertions C, D, E *holds true:*

(C) *$\alpha+2$ is composite and $N(\alpha+2)<2N(\alpha)$;*

(D) *$\alpha+2i$ is composite and $N(\alpha+2i)<2N(\alpha)$, furthermore $N(\alpha-4i)\leqq N(\alpha)$ and $N(\alpha-ni)<N(\alpha)$ is true for $n=2$ and $n=3$;*

(E) $N(\alpha-n+2i)<N(\alpha)$ *for* $n=2, 3, 4, 5$ *and* 6, *while* $\alpha-1+2i$ *is composite and* $N(\alpha-1+2i)<2N(\alpha)$. *Moreover* $N(\alpha-4i)\leqq N(\alpha)$ *while* $N(\alpha-ki)<N(\alpha)$ *for* $k=2$ *and* 3.

Proof. Since $N(\alpha)>13$ and (ii) holds, therefore $a\geqq 4$. Hence (A) follows easily. Also (B) is obviously true. Since α is a prime, α is not an associate of $1+i$, therefore $1+i$ is a divisor of $\alpha+1$ and of $\alpha+i$, and so they are composite numbers. $N(\alpha+1)=a^2+b^2+2a+1$, $N(\alpha+i)=a^2+b^2+2b+1$, therefore the second assertion in (B) holds as well.

We shall classify α according to its residue (mod 5). Let

$$M(C) = \{(0, 1), (0, 4), (1, 1), (1, 4), (2, 2). (2, 3), (3, 0)\};$$

$$M(D) = \{(0, 3), (1, 0), (3, 2), (4, 0)\};$$

$$M(E) = \{(0, 2), (2, 0), (3, 3), (4, 4).$$

Since α is a prime, therefore (a, b) (mod 5) belongs to exactly one of the sets $M(C)$, $M(D)$, $M(E)$, (4, 1) (mod 5) is excluded by the condition (iv). We shall prove that the assertions (C), (D) and (E) are true if (a, b) (mod 5) belongs to $M(C)$, $M(D)$, $M(E)$, respectively.

Case $M(C)$. If $(a, b)\in M(C)$ (mod 5), then $5|N(\alpha+2)$, which can be seen easily. This implies that $2+i|\alpha+2$, and so $\alpha+2$ is composite. Since $a>b>0$, therefore $a\geqq 4$. But $a=4$ cannot occur, therefore $a>4$ and $N(\alpha+2)=a^2+b^2+4(a+1)<<2N(\alpha)$ obviously holds.

Case $M(D)$. If $(a, b)\in M(D)$ (mod 5), then $5|N(\alpha+2i)$ which implies that $\alpha+2$ is composite. Since $b\neq 1$, therefore $N(\alpha-ni)=N(\alpha)+n(n-2b)$, and so $n-2b$ is negative for $n=2$ and 3, nonpositive if $n=4$. This completes the proof of Case $M(D)$.

Case $M(E)$. If $(a, b)\in M(E)$ (mod 5), then $a\geqq 5$, and $a\not\equiv b$ (mod 2), since α is a prime. The case $(a-b)=1$ cannot occur, furthermore $b\neq 1$, whence we have that $b\geqq 2$ and $a-b>2$. By using these inequalities, we can prove (E) easily.

Since the functions H_j under the condition (1.4) satisfy the conditions of Theorem 1, therefore we have that $H_j(\alpha\bar{\alpha})=H_j(\alpha)+H_j(\bar{\alpha})\equiv 0$ (mod 1). This implies that it is enough to prove Theorem 2 either for α or for $\bar{\alpha}$.

Lemma 4. *If* $V(\alpha)\equiv 0$ (mod 1) *for all* $\alpha\in\mathbf{G}\setminus\{0, -1, -2, -4, -5, -6\}$ *then* $H_j(\alpha)\equiv H_j(\bar{\alpha})$ (mod 1) *for all* $\alpha\in\mathbf{G}\setminus\{0\}$ *and* $j=0, 1, 2$.

Proof. Let $h_j(\alpha):=H_j(\alpha)-H_j(\bar{\alpha})$. Then $h_j(\bar{\alpha})=-h_j(\alpha)$. To prove the lemma, we prove that $h_j(\alpha)\equiv 0$ (mod 1) for $j=0, 1, 2$. We observe that, for $j=0, 1, 2$, $h_j(1\pm\ )=0$ and $h_j(n)=0$ for all rational integers.

The complete additivity of the function H_j and the fact that $H_j(\varepsilon\alpha)=H_j(\alpha)$ for $\varepsilon=-1,\ ,\ -$ allows us to obtain the following 9 congruences modulo 1, which prove the assertion for those Gaussian primes with norm less than 20. The 9 congruences are:

i, $h_2(2+i)\equiv V^{+i}(2-2i)+V^{+i}(2-4i)+V^{+i}(3-3i)-V^{+1}(-3+i)\equiv 0 \pmod 1$

ii, $h_0(2-3i)\equiv V^{+i}(2-3i)\equiv 0 \pmod 1$

iii, $h_1(3-2i)\equiv V^{+i}(3-3i)\equiv 0 \pmod 1$

vi, $h_1(4-i)\equiv V^{+1}(-1+i)-V^{+1}(-1-i)\equiv 0 \pmod 1$

v, $h_1(2+i)\equiv V^{+i}(4-4i)+V^{+i}(4-2i)-V^{+i}(1-3i)\equiv 0 \pmod 1$

vi, $h_0(2-i)\equiv V^{+i}(4-4i)+V^{+i}(4-2i)\equiv 0 \pmod 1$

vii, $h_2(4-i)\equiv V^{+i}(4-3i)\equiv 0 \pmod 1$

viii, $h_2(3-2i)\equiv V^{+i}(3-4i)+V^{+i}(3-2i)\equiv 0 \pmod 1$

ix, $h_0(4-i)\equiv V^{+i}(3-5i)+V^{+i}(3-i)\equiv 0 \pmod 1$.

We finish the proof by using induction. Let us assume that our Lemma 4 is not true. Let α be such an integer for which $h_j(\alpha)\not\equiv 0 \pmod 1$ for at least one of the $j\in\{0,1,2\}$. Let us choose that α for which $N(\alpha)$ is the smallest one. Then $N(\alpha)\geqq 20$, and α is a Gaussian prime. We may assume furthermore that condition (ii) of Lemma 3 true also.

It is clear that

$$0\equiv V^{+1}(\alpha-6)-V^{+1}(\bar\alpha-6)\equiv h_0(\alpha-6)+h_1(\alpha-5)+$$
$$+h_2(\alpha-4)-h_2(\alpha-2)-h_1(\alpha-1)-h_0(\alpha) \pmod 1,$$
$$0\equiv V^{+1}(\alpha-5)-V^{+1}(\bar\alpha-5)\equiv h_0(\alpha-5)+h_1(\alpha-4)+$$
$$+h_2(\alpha-3)-h_2(\alpha-1)-h_1(\alpha)+h_0(\alpha+1) \pmod 1.$$

Since $\alpha+1$ is a composite number, and $N(\alpha-k)<N(\alpha)$ for $1\leqq k\leqq 6$, we conclude that $h_0(\alpha)\equiv 0 \pmod 1$, and $h_1(\alpha)\equiv 0 \pmod 1$.

To prove that $h_2(\alpha)\equiv 0 \pmod 1$, we assume first that (iv) in Lemma 3 holds, i.e. that $(a,b)\not\equiv(4,1) \pmod 5$. We observe that

$$0\equiv V^{+1}(\alpha-4)-V^{+1}(\bar\alpha-4)\equiv -h_2(\alpha) \pmod 1$$

in Case $M(C)$,

(3.2) $$0\equiv V^{+i}(\alpha-4i)+V^{+i}(\bar\alpha-2i)\equiv$$
$$\equiv h_0(\alpha-4i)-h_0(\alpha+2i)-h_2(\alpha)+h_2(\alpha-2i)+h_1(\alpha-3i)-h_1(\alpha+i) \pmod 1.$$

which implies that $h_2(\alpha)\equiv 0 \pmod 1$ in Case D. In Case E we start from the relation

$$-h_0(\alpha+2i) \equiv V^{+1}(\alpha-6+2i)-V^{+1}(\overline{\alpha-6+2i}) \equiv 0 \pmod 1$$

whence, by (3.2) we deduce that $h_2(\alpha)\equiv 0 \pmod 1$.

Finally, we consider the case $(a, b)\equiv(4, 1) \pmod 5$. Since $N(\alpha)\geqq 20$, $\alpha\neq 4+i$. Since 5 is a divisor of $N(\alpha+2i)$, in the case $b\neq 1$, $\alpha+2i$ is a composite number and $N(\alpha+2i)<2N(\alpha)$, $N(\alpha-ki)<N(\alpha)$ $(k=1, 2, 3, 4)$ are satisfied. Hence, by (3.2) we obtain that $h_2(\alpha)\equiv 0 \pmod 1$. If $b=1$, then $a\geqq 14$. In this case $N(\alpha-k+4i)<N(\alpha)$ holds for every integer k in $1\leqq k\leqq 6$, and from $0\equiv V^{+1}(\alpha-6+4i)-V^{+1}(\overline{a-6+4i})\equiv 0 \pmod 1$ we deduce that $h_0(\alpha+4i)\equiv 0 \pmod 1$. Since $\alpha+3i$ and $\alpha+2i$ are composite numbers, and $N(\alpha+3i)<2N(\alpha)$, $N(\alpha+2i)<2N(\alpha)$, substituting first α by $\bar{\alpha}$, in (3.2), we get that $h_2(\alpha)\equiv 0 \pmod 1$.

By this the proof of Lemma 4 is completed.

Lemma 5. *If* $V(\alpha)\equiv 0 \pmod 1$ *holds for every* $\alpha\in G\setminus\{0, -1, -2, -4, -5, -6\}$ *then* $H_j(\alpha)\equiv 0 \pmod 1$ *for every* $\alpha\in G\setminus\{0\}$, *with* $N(\alpha)\leqq 13$, $j=0, 1, 2$. *Furthermore,* $H_2(4+i)\equiv 0 \pmod 1$.

Proof. The Gaussian primes π with $N(\pi)<17$ are $(1\pm i)$, $(2\pm i)$ and $(3\pm 2\ )$. By Lemma 4 it suffices to consider either π or $\bar{\pi}$. Also by Lemma 4,

$$H_j(\alpha)-H_j(\bar{\alpha}) \equiv H_j(\alpha)+H_j(\bar{\alpha}) \equiv 2H_j(\alpha) \equiv 0 \pmod 1.$$

This allows us to replace $H_j(\alpha)$ by $\pm H_j(\bar{\alpha})$, it means also that $2H_j(\alpha)\equiv 0 \pmod 1$ holds for every α.

The additivity of the functions H_j, together with the factorization of Gaussian integers, allows us to obtain the following ten congruences, in the given order, which, as can be seen easily, prove the lemma:

i, $H_1(2+i) \equiv V^{+i}(3-i)+V^{+1}(-2+i) \equiv 0 \pmod 1$.

ii, $H_2(2+i) \equiv V^{+1}(4+6i)+V^{+i}(5+i)+V^{+1}(2i)+V^{+1}(i)+$
$+V^{+1}(-2+i) \equiv 0 \pmod 1$

iii, $H_2(3-2i) \equiv V^{+1}(6+2i)+V^{+1}(i)+V^{+1}(-2+i)+V^{+1}(2+2i) \equiv 0 \pmod 1$

iv, $H_0(3+2i) \equiv V^{+i}(9+i)+V^{+1}(3+2i)+V^{+i}(5-2i)+$
$+V^{+1}(-2+i)+V^{+1}(-1+2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

v, $H_1(2+3i) \equiv V^{+1}(4+2i)+V^{+1}(2i)+V^{+1}(1+3i)+$
$+V^{+1}(-1+2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

vi, $H_0(2-i) \equiv V^{+i}(6)+V^{+1}(1+i)+V^{+i}(5+i)+V^{+1}(2i)+V^{+1}(i) \equiv 0 \pmod 1$

vii, $H_0(1+i) \equiv V^{+i}(4-2i)+V^{+1}(-1+i) \equiv 0 \pmod 1$

viii, $H_1(1+i) \equiv V^{+1}(3+i)+V^{+1}(-1+i)+V^{+1}(-1+2i)+V^{+i}(4-i)+$
$$+V^{+1}(1+3i)+V^{+1}(-2+i) \equiv 0 \pmod 1$$

ix, $H_2(1+i) = V^{+i}(5-i)+V^{+1}(1+3i)+V^{+i}(4)+V^{+1}(-1+i)+$
$$+V^{+1}(-1+2i) \equiv 0 \pmod 1$$

x, $H_2(4+i) \equiv V^{+i}(4)+V^{+1}(-1+i)+V^{+i}(5-i) \equiv 0 \pmod 1$.

The final step of the proof of Theorem 2 is contained in the next

Lemma 6. *If* $V(\alpha)\equiv 0 \pmod 1$ *holds for all* $\alpha\in \mathbf{G}\setminus\{0, -1, -2, -4, -5, -6\}$, *and* $H_j(\alpha)\equiv 0 \pmod 1$ *for all nonzero* α *with* $N(\alpha)<17$ *and* $j=0, 1, 2$, *then* $H_j(\alpha)\equiv 0 \pmod 1$ $(j=0, 1, 2)$ *holds for all nonzero Gaussian integer.*

Proof. Assume that the assertion is not true. Let α be such a Gaussian integer with smallest norm for which $H_j(\alpha)\not\equiv 0 \pmod 1$ for at least one j. By Lemma 4, we may assume that $\alpha=a+bi$, $a>b>0$. It is clear that α is a Gaussian prime.

Since $N(\alpha)>13$, taking into account the relations, $V^{+1}(\alpha-6)\equiv 0$, $V^{+1}(\alpha-5)\equiv \equiv 0 \pmod 1$, by Lemma 3 we deduce that $H_2(\alpha)\not\equiv 0 \pmod 1$.

Let us consider first the case $(a, b)\equiv(4, 1) \pmod 5$ which was excluded in Lemma 3. If $(a, b)\equiv(4, 1) \pmod 5$, then $\alpha+2i$ is composite and $N(\alpha+2i)< <2N(\alpha)$. If $b\neq 1$, then $b\geqq 6$ and $V^{+i}(\alpha-4i)\equiv 0 \pmod 1$ implies that $H_2(\alpha)\equiv \equiv 0 \pmod 1$. The case $\alpha=4+i$ was treated in Lemma 5, so we may assume that $\alpha\neq 4+i$. Thus we may assume that $b=1$ and $a\geqq 14$. Then $N(\alpha-k+4i)<N(\alpha)$ for $k=1, 2, 3, 4, 5, 6$, and $V^{+1}(\alpha-6+4i)\equiv 0 \pmod 1$ implies that $H_0(\alpha+4i)\equiv 0 \pmod 1$. Since $\alpha+2i$ and $\alpha+3i$ are composite numbers with norm less than $2N(\alpha)$, $V^{+i}(\alpha-2i)\equiv 0 \pmod 1$ implies that $H_2(\alpha)\equiv 0 \pmod 1$.

In all remaining cases Lemma 3 enables us to apply the induction hypothesis. In Case C we consider $V^{+1}(\alpha-4)\equiv 0 \pmod 1$, while in Case D we take $V^{+i}(\alpha-4i)\equiv 0 \pmod 1$, and hence deduce immediately that $H_2(\alpha)\equiv 0 \pmod 1$. If Case E is satisfied, then we start from $V^{+1}(\alpha-6+2i)\equiv 0 \pmod 1$, which implies that $H_0(\alpha+2i)\equiv 0 \pmod 1$, and consider $V^{+i}(\alpha-4i)\equiv 0$, whence we have that $H_2(\alpha)\equiv 0 \pmod 1$.

By this the proof of Lemma 6 and therefore of Theorem 2 is completed.

4. The next theorem is an easy consequence of our Theorem 2.

Theorem 3. *Let* $F_0, \ldots, F_5\in A_{\mathbf{G}}^*$ *which satisfy the relations*

$$F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)+F_4(\alpha+4)+F_5(\alpha+5) \equiv 0 \pmod{\mathbf{G}} \tag{4.1}$$

for all $\alpha\in G\setminus\{0, -1, -2, -3, -4, -5\}$. *Then* $F_j(\alpha)\equiv 0 \pmod{\mathbf{G}}$ *holds for all* $\alpha\in\mathbf{G}\setminus\{0\}$ *and* $j=0, \ldots, 5$.

Proof. It is enough to prove our theorem for functions F_j which take on real values.

Let us write (4.1) in the form

$$U(\alpha):=F_0(\alpha)+F_1(\alpha+1)+F_2(\alpha+2)+F_3(\alpha+3)+F_4(\alpha+4)+F_5(\alpha+5)+F_6(\alpha+6),$$

where $F_6\in A^*_G$ is identically zero. Then

$$0\equiv U(-6-\alpha)\equiv F_6(\alpha)+F_5(\alpha+1)+F_4(\alpha+2)+F_3(\alpha+3)+$$
$$+F_2(\alpha+4)+F_1(\alpha+5)+F_0(\alpha+6) \pmod 1.$$

Let

$$H_0(\alpha):=F_0(\alpha)-F_6(\alpha),\quad H_1(\alpha):=F_1(\alpha)-F_5(\alpha),\quad H_2(\alpha)=F_2(\alpha)-F_4(\alpha).$$

Since $U(\alpha)-U(-\alpha-6)\equiv 0 \pmod 1$, therefore

(4.2)

$$H_0(\alpha)+H_1(\alpha+1)+H_2(\alpha+2)-H_2(\alpha+4)-H_1(\alpha+5)-H_0(\alpha+6)\equiv 0 \pmod 1$$

is satisfied for all Gaussian integers α for which the sequence α, $\alpha+1$, $\alpha+2$, $\alpha+4$, $\alpha+5$, $\alpha+6$ does not contain the zero. Thus the conditions of Theorem 2 are satisfied, consequently $F_0(\alpha)-F_6(\alpha)\equiv 0$, $F_1(\alpha)\equiv F_5(\alpha)$, $F_2(\alpha)\equiv F_4(\alpha) \pmod 1$ holds for all nonzero Gaussian integers α. Especially $F_0(\alpha)\equiv 0 \pmod 1$. If we write now

$$V(\alpha):\equiv F_{-1}(\alpha)+F_0(\alpha)+\ldots+F_5(\alpha+5)$$

with $F_{-1}\in A^*_G$, $F_{-1}(\alpha)\equiv 0 \pmod 1$ identically, then we get similarly, that $F_{-1}(\alpha)\equiv$ $\equiv F_5(\alpha)$, $F_0(\alpha)\equiv F_4(\alpha)$, $F_1(\alpha)\equiv F_3(\alpha) \pmod 1$ which implies that $F_5(\alpha)\equiv F_4(\alpha)\equiv F_1(\alpha)\equiv$ $\equiv F_2(\alpha)\equiv 0 \pmod 1$, and the recursion (4.1) finally implies that $F_3(\alpha)\equiv 0 \pmod 1$ true as well.

By this our theorem is proved.

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