On equivalence of two variational problems in k-Lagrange spaces

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1. Introduction. In [3] we have considered generalization of the equivalence of two variational problems for single integrals treated by A. MoóR ([7]) in Lagrange spaces $\mathbf{L}^{*n}=(M, \mathscr{L}^*)$ and $\mathbf{L}^n=(M, \mathscr{L})$ ([6]). This problem has the following form

(1.1)
$$\mathbf{E}_i(\mathscr{L}^*(x,y)) = \lambda(x,y)\,\mathbf{E}_i(\mathscr{L}(x,y)), \quad \left(\mathbf{E}_i := \frac{d}{dt}\dot{\partial}_i - \partial_i,\ \dot{\partial}_i := \partial/\partial y^i,\ \partial_i := \partial/\partial x^i\right),$$
$$\lambda(x,y) \neq 0,$$

where y stands for $\dot{x}$, $\mathscr{L}$ and $\mathscr{L}^*$ are the two Lagrangians, and λ depends not only on x but on y too. We have given the transformation of the Lagrangians as a necessary and sufficient condition for this equivalence. Moreover, we have shown geometrical consequences of the equivalence relation (1.1).

In 1975 A. MoóR ([8]) gave a definition of equivalence of two variational problems for multiple integrals with the following relation

$$\mathbf{E}_i(\mathscr{L}^*(x^s, y^s_\alpha)) = \lambda^j_i(x^s)\,\mathbf{E}_j(\mathscr{L}(x^s, y^s_\alpha)), \quad \operatorname{rank}\|\lambda^j_i(x)\| = n \tag{1.2}$$

$$\left(y^s_\alpha := \partial x^s/\partial t^\alpha,\ \mathbf{E}_i := \frac{\partial}{\partial t^\alpha}\dot{\partial}^\alpha_i - \partial_i \text{ (summation over } \alpha);\ i,j,s = \overline{1,n};\ \alpha = \overline{1,k},\ k<n\right).$$

He investigated the properties of this relation but not in geometrical manner.

In [4] and [5] we have constructed a *geometrical model* for multiple integrals in the calculus of variations. Now we study a generalization of the Moór equivalence in geometrical manner using the theory of k-Lagrange geometry.

2. The Moór equivalence of multiple integral variational problems in k-Lagrange spaces. Consider the total space $E=\overset{k}{\underset{1}{\oplus}}TM=\overset{1}{\widetilde{TM}}\oplus\overset{2}{\widetilde{TM}}\oplus\ldots\oplus\overset{k}{\widetilde{TM}}$ of the vector

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bundle $\eta=(\overset{k}{\underset{1}{\oplus}} TM, \pi, M)$ with canonical coordinates (x^i, y^i_α) where i runs from 1 to n and α runs from 1 to k. By the theory of k-Lagrange spaces $\mathbf{L}^n_k$ ([4], [5]) we have a *regular* Lagrangian $\mathcal{L}$: $\overset{k}{\underset{1}{\oplus}} TM \to \mathbf{R}$ with the metric tensor field

$$(2.1)\qquad g^{\alpha\beta}_{ij}(x,y) = \partial^\alpha_i \partial^\beta_j \mathcal{L}(x,y);\ \operatorname{rank} \|g^{\alpha\beta}_{ij}\| = nk \quad (\partial^\alpha_i := \partial/\partial y^i_\alpha).$$

Now let $\mathcal{L}$ be defined on class C^2 of the *admissible submanifold* C_k, $\bar{C}_k$, ... on M, where

$$(2.2)\qquad C_k : x^i = x^i(t^\alpha), \quad \bar{C}_k : \bar{x}^j = \bar{x}^j(t^\alpha), \ldots$$

and they coincide with each other on the boundary ∂G_t of the parameter domain G_t ([9], [10]).

Then we can construct the k-fold integral

(2.3)

$$I(C_k) = \int_{G_t} \mathcal{L}(x^i(t^\beta), y^i_\alpha(t^\beta))\, d(t);\ d(t) := dt^1 \ldots dt^k;\ y^i_\alpha(t^\beta) := \partial x^i/\partial t^\alpha, \quad (\beta = \overline{1,k}).$$

This integral depends on the submanifold C_k by means of which it is defined. It is known from the classical calculus of variations of multiple integrals ([9]) that if a submanifold C_k is to afford an extreme value to I relative to other admissible submanifold it is necessary that the first variation δI of (2.3) should vanish. This implies that C_k must satisfy the system of n second order partial differential equations:

$$(2.4)\qquad \mathbf{E}_i(\mathcal{L}) := \frac{\partial}{\partial t^\alpha}(\partial^\alpha_i \mathcal{L}) - \partial_i \mathcal{L} = 0 \quad (\partial_i := \partial/\partial x^i),$$

where $\mathbf{E}_i$ are the components of the *Euler—Lagrange covariant vector* ([10]).

Let us consider a pair $\mathbf{L}^n_k=(M, \mathcal{L})$ and $\mathbf{L}^{*n}_k=(M, \mathcal{L}^*)$ of k-Lagrange spaces with the same base manifold M.

Definition 2.1. Two variational problems in $\mathbf{L}^n_k=(M, \mathcal{L})$ and $\mathbf{L}^{*n}_k=(M, \mathcal{L}^*)$ are called *equivalent in the sense of Moór* if

$$(2.5)\qquad \mathbf{E}_i(\mathcal{L}^*(x^j, y^j_\alpha)) = \lambda^s_i(x^j, y^j_\alpha)\, \mathbf{E}_s(\mathcal{L}(x^j, y^j_\alpha));\ \det \|\lambda^s_i(x,y)\| \neq 0$$

hold identically.

Remark. In (2.5) the tensor field λ depends on y too.

3. Some geometrical characters of the equivalence. Relation (2.5) has the following explicit form:

$$(3.1)\quad (\partial^\alpha_i \partial^\beta_s \mathcal{L}^* - \lambda^j_i \partial^\alpha_j \partial^\beta_s \mathcal{L}) y^s_{\alpha\beta} + (\partial^\alpha_i \partial_s \mathcal{L}^* - \lambda^j_i \partial^\alpha_j \partial_s \mathcal{L}) y^s_\alpha - (\partial_i \mathcal{L}^* - \lambda^j_i \partial_j \mathcal{L}) = 0,$$

$$y^s_{\alpha\beta} := \frac{\partial^2 x^s}{\partial t^\alpha \partial t^\beta}.$$

Using condition (2.1) for $\mathscr{L}$ and $\mathscr{L}^*$ we get from (3.1)

$$(3.2)\qquad (\overset{*}{g}{}^{\alpha\beta}_{is}-\lambda^j_i g^{\alpha\beta}_{js})y^s_{\alpha\beta}+(\partial^\alpha_i\partial_s\mathscr{L}^*-\lambda^j_i\partial^\alpha_j\partial_s\mathscr{L})y^s_\alpha-(\partial_i\mathscr{L}^*-\lambda^j_i\partial_j\mathscr{L})=0.$$

Since (3.2) is an identity in (x, y) it is necessary that the coefficients of $y^s_{\alpha\beta}$ should vanish. Hence we obtain from (3.2):

Theorem 3.1. *A necessary geometrical condition for equivalence of two variational problems of multiple integrals is that the k-Lagrange spaces $(M, \mathscr{L}^*)$ and $(M, \mathscr{L})$ be in „k-conformal" correspondence:*

$$(3.3)\qquad \overset{*}{g}{}^{\alpha\beta}_{is}(x,y)=\lambda^j_i(x,y)g^{\alpha\beta}_{js}(x,y).$$

From this Theorem it directly follows that the *metrical d-connections* (cf. [4])

$$LD^*=(\overset{*}{L}{}^i_{jm},\overset{*}{L}{}^{\alpha j}_{i\beta m},\overset{*}{C}{}^{i\alpha}_{jm},\overset{*}{C}{}^{\gamma i\beta}_{m\alpha j})\quad\text{and}\quad LD=(L^i_{jm},L^{\alpha j}_{i\beta m},C^{i\alpha}_{jm},C^{\gamma i\beta}_{m\alpha j}),$$

respectively, are related by the geometrical condition (3.3).

Proposition 3.1. *The d-tensor fields $\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{ijm}$ and $\tilde{C}^{\alpha\beta\gamma}_{ijm}$ are in the following relation*

$$(3.4)\qquad 2\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{ijm}=(\partial^\gamma_m\lambda^l_i)g^{\alpha\beta}_{lj}+2\lambda^l_i\tilde{C}^{\alpha\beta\gamma}_{ljm}.$$

Proof. We have

$$(3.5)\qquad \begin{aligned}&(a)\quad C^{\gamma i\beta}_{m\alpha j}=\frac{1}{2}g^{is}_{\alpha\varepsilon}\partial^\beta_j\partial^\gamma_m\partial^\varepsilon_s\mathscr{L}=\frac{1}{2}g^{is}_{\alpha\varepsilon}\partial^\varepsilon_s g^{\beta\gamma}_{jm},\\ &(b)\quad \tilde{C}^{\gamma\varepsilon\beta}_{msj}=g^{\varepsilon\alpha}_{si}C^{\gamma i\beta}_{m\alpha j}=\frac{1}{2}\partial^\gamma_m g^{\alpha\beta}_{ij},\end{aligned}$$

(cf. [4]). Hence a direct calculus leads to (3.4).

Using the result of the above Proposition we shall prove that our equivalence-problem can be reduced to the Moór one ([8]), i.e. his equivalence is a special case of relation (2.5).

Theorem 3.2. *If two variational problems of multiple integrals are equivalent in the sense of Moór then the k-conformal factor $\lambda^j_i(x, y)$ is necessarily independent of y^i_α.*

Proof. Differentiating (3.3) with respect to y^l_γ we obtain

$$(3.7)\qquad \partial^\gamma_l\overset{*}{g}{}^{\alpha\beta}_{is}=(\partial^\gamma_l\lambda^j_i)g^{\alpha\beta}_{js}+\lambda^j_i(\partial^\gamma_l g^{\alpha\beta}_{js})$$

and by virtue of (3.4) we have

$$(3.8)\qquad 2\overset{*}{\tilde{C}}{}^{\alpha\beta\gamma}_{isl}=(\partial^\gamma_l\lambda^j_i)g^{\alpha\beta}_{js}+2\lambda^j_i\tilde{C}^{\alpha\beta\gamma}_{jsl}.$$

Since the *d-tensor fields* $\overset{*}{\tilde{C}}$ and $\tilde{C}$ are totally symmetric ([4]), after the cyclic permutation of the indices we get

$$(3.9)\qquad (\partial_l^\gamma \lambda_i^j) g_{js}^{\alpha\beta} = (\partial_j^\alpha \lambda_i^j) g_{sl}^{\beta\gamma} = (\partial_s^\beta \lambda_i^j) g_{lj}^{\gamma\alpha}.$$

By using the symmetric property of the metric tensor $g_{is}^{\alpha\beta}$ from (3.9) it follows that

$$(3.10)\qquad (\partial_l^\gamma \lambda_i^j) g_{sj}^{\beta\alpha} - (\partial_s^\beta \lambda_i^j) g_{lj}^{\gamma\alpha} = 0.$$

Contracting by $g_{\alpha\beta}^{js}$ the last relation we get

$$(3.11)\qquad \begin{aligned} &\text{(a)} && (\partial_l^\gamma \lambda_i^j) nk - (\partial_s^\beta \lambda_i^j) \delta_{l\beta}^{\gamma s} = 0, \\ &\text{(b)} && (\partial_l^\gamma \lambda_i^j) nk - \partial_l^\gamma \lambda_i^j = 0, \end{aligned}$$

respectively. This means that

$$(3.12)\qquad (\partial_l^\gamma \lambda_i^j)(nk-1) = 0.$$

Because of $(nk-1) \neq 0$ the relation (3.12) holds iff

$$(3.13)\qquad \partial_l^\gamma \lambda_i^j (x, y) = 0.$$

Thus λ is independent of y_γ^l.

Corollary. *A geometrical character of the equivalence in* (2.5) *with the k-conformal factor* $\lambda_i^j(x)$ *is that the torsion tensor field* $C_{j\varepsilon m}^{\beta s\gamma}$ *of the metrical d-connection LD is invariant.*

Proof. Suppose that (2.5) holds with $\lambda_i^j(x)$. Using the relation $C_{j\varepsilon m}^{\beta s\gamma} = g_{\varepsilon\alpha}^{si} C_{jim}^{\beta\alpha\gamma}$, from Proposition 3.1 we directly get

$$(3.14)\qquad \overset{*}{C}{}_{j\varepsilon m}^{\beta s\gamma} = \overset{*}{g}{}_{\varepsilon\alpha}^{si} \tilde{C}_{jim}^{\beta\alpha\gamma} = \tilde{\lambda}_t^i g_{\varepsilon\alpha}^{st} \lambda_i^l \tilde{C}_{jlm}^{\beta\alpha\gamma} = \delta_t^l g_{\varepsilon\alpha}^{st} \tilde{C}_{jlm}^{\beta\alpha\gamma} = g_{\varepsilon\alpha}^{sl} \tilde{C}_{jlm}^{\beta\alpha\gamma} = C_{j\varepsilon m}^{\beta s\gamma},$$

where $\tilde{\lambda}_t^i \lambda_i^l = \delta_t^l$.

4. Transformation of the Lagrangians. We can easily check if the Lagrangians differ by a total derivative, i.e. $\mathscr{L}^*(x, y) = \mathscr{L}(x, y) + \partial_s^\beta A(x) y_\beta^s$, then $\mathrm{E}_i(\mathscr{L}^*) \equiv \equiv \mathrm{E}_i(\mathscr{L})$. This means that two variational problems of multiple integrals are equivalent in the sense of Moór with tensor field δ_i^j.

Now we examine the transformation of the Lagrangians under the equivalence relation in (2.5). First we prove

Proposition 4.1. *If the relation* (2.5) *holds and the k-conformal factor* λ *is independent of* y *then it is necessary that* $\lambda_i^j(x) = \delta_i^j \lambda(x)$.

Proof. Let us consider relation (3.3). Since the metric tensor fields g^* and g are symmetric in the indices $\binom{\alpha}{i}$ and $\binom{\beta}{s}$ we get

$$(4.1)\qquad \lambda_i^j g_{js}^{\alpha\beta} - \lambda_s^j g_{ji}^{\beta\alpha} = 0,$$

which can be written in the following form

(4.2) $$g_{jh}^{\varepsilon\gamma}(\lambda_i^j\delta_\gamma^\beta\delta_s^h\delta_\varepsilon^\alpha-\lambda_s^j\delta_i^h\delta_\varepsilon^\beta\delta_\gamma^\alpha)=0 \quad (\beta,\gamma,\varepsilon=\overline{1,k};\ i,j,s,h=\overline{1,n}).$$

We infer from the symmetry of the d-tensor field g that the coefficients of $g_{jh}^{\varepsilon\gamma}$ in (4.2) must be skewsymmetric in $\binom{\varepsilon}{j}$ and $\binom{\gamma}{h}$. This gives for the symmetric part:

(4.3) $$\lambda_i^j\delta_\gamma^\beta\delta_s^h\delta_\varepsilon^\alpha-\lambda_s^j\delta_i^h\delta_\varepsilon^\beta\delta_\gamma^\alpha+\lambda_i^h\delta_\varepsilon^\beta\delta_s^j\delta_\gamma^\alpha-\lambda_s^h\delta_i^j\delta_\gamma^\beta\delta_\varepsilon^\alpha=0.$$

Let $h=s$, $\beta=\gamma$, $\alpha=\varepsilon$, then we obtain

(4.4) $$\lambda_i^j k^2 n-\lambda_i^j k+\lambda_i^j k-\lambda_h^h\delta_i^j k^2=0.$$

Now putting

(4.5) $$\lambda(x)=\frac{1}{n}\lambda_h^h(x),$$

we get from (4.4)

(4.6) $$k^2 n\lambda_i^j(x)-k^2 n\delta_i^j\lambda(x)=0.$$

Thus

(4.7) $$\lambda_i^j(x)=\delta_i^j\lambda(x).$$

Proposition 4.2. *If relation* (2.5) *holds with* $\lambda_i^j(x)=\delta_i^j\lambda(x)$ *then the transformation beetwen the Lagrangians* $\mathscr{L}^*(x,y)$ *and* $\mathscr{L}(x,y)$ *is as follows*:

(4.8) $$\mathscr{L}^*(x,y)=\lambda(x)\mathscr{L}(x,y)+A_s^\beta(x)y_\beta^s+U(x).$$

Proof. By Theorem 3.1 we obtain $\overset{*}{g}{}_{is}^{\alpha\beta}=\delta_i^j\lambda(x)_{js}^{\alpha\beta}$. In view of property of Lagrangians we get

(4.9) $$\partial_i^\alpha\partial_s^\beta(\mathscr{L}^*-\lambda(x)\mathscr{L})=0.$$

Hence the function $\mathscr{L}^*-\lambda(x)\mathscr{L}$ is necessarily linear in y_β^s.

5. Some remarks about the normal form of the Euler—Lagrange equations in $\mathbf{L}_k^n$. It is known that in the equations of geodesics of Lagrange space the second derivatives $\ddot{x}^i$ appear explicitly and the functions $G^i(x,y)$ can be derived directly from the Lagrangians (cf. [6]). This suggests us to write the $\mathbf{E}_i(\mathscr{L}(x,y))$ in such form which is a generalization of that of geodesics. Hence we get

(5.1) $$\mathbf{E}_i(\mathscr{L}(x^j,y_\gamma^j))=g_{is}^{\alpha\beta}y_{\alpha\beta}^s+G_i(x^j,y_\gamma^j)\quad\left(y_{\alpha\beta}^s:=\frac{\partial^2x^s}{\partial t^\alpha\partial t^\beta}\right)$$

where the generalized $G_i(x^j,y_\gamma^j)$ are defined by

$$G_i:=(\partial_i^\gamma\partial_s\mathscr{L})y_\gamma^s-\partial_i\mathscr{L}.$$

By means of $g_{ih}^{\alpha\beta}g_{\beta\gamma}^{hl}=\delta_{i\gamma}^{\alpha l}$ equation (5.1) can be written in the following form

(5.2) $$\mathbf{E}_i(\mathscr{L}(x,y))=g_{is}^{\alpha\beta}(y_{\alpha\beta}^s+G_{\alpha\beta}^s(x,y)),$$

where the generalized $G^s_{\alpha\beta}$ are defined by

$$G^s_{\alpha\beta} := g^{is}_{\alpha\beta} G_i \quad (G_i := G^s_{\alpha\beta} g^{\alpha\beta}_{is}).$$

Finally we directly obtain

Proposition 5.1. *If two variational problems in* L^{*n}_k *and* L^n_k *are equivalent in the sense of Moór then*

$$\overset{*}{G}{}^s_{\alpha\beta} = G^s_{\alpha\beta}. \tag{5.3}$$

Indeed, from the equivalence relation (2.5) using Theorem 3.1 and relation (5.2) we obtain (5.3).

Remark. Relation (5.3) corresponds to that result which was obtained for equivalent single-integral variational problems in Lagrange spaces (cf. [3]).

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