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On an integral inequality for concave functions

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In 1987 A. BEZDEK and K. BEZDEK [2] proved the following interesting proposition:

Theorem A. *Let S be a convex solid of revolution in $\mathbf{R}^3$ with axis of revolution AB. Further, let C be the centroid of S and let C' be the centroid of the 2-dimensional domain obtained by intersecting S with a plane through AB. Then*

$$\frac{1}{2} < \frac{|AC|}{|AC'|} < \frac{3}{2}. \tag{1}$$

As it was shown by the authors double-inequality (1) is a consequence of the following sharp integral inequalities.

Theorem B. *If f is a non-negative concave function defined on* [0, 1] *with* $\sup\limits_{0 \leqq x \leqq 1} f(x) = 1$, *then*

$$\frac{2}{3} \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt} \leqq 1 \tag{2}$$

and

$$\frac{1}{2} \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq 1. \tag{3}$$

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The aim of this paper is to present a short and simple proof for an integral inequality for concave functions which includes the left-hand sides of (2) and (3) as special cases.

Theorem. *Let f be a non-negative, continuous, concave function on $[a, b]$ and let g be a non-negative differentiable function such that the derivative g' is integrable on $[a, b]$. If α and β are real numbers with $\alpha \geqq 0$ and $0 < \beta \leqq 1$, then we have for all $x \in [a, b]$:*

$$(4) \qquad \frac{\alpha+\beta}{\alpha+2\beta} f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt + \frac{\beta}{\alpha+2\beta} \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt \leqq \leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt.$$

Proof. First we note that the function f^{β} is concave on $[a, b]$ (see [6, p. 20]). Further, since every continuous concave function defined on a compact interval can be approximated uniformly by differentiable concave functions (see [6, p. 269]), we may assume that f and f^{β} are differentiable on $[a, b]$. Then we conclude from the mean-value theorem:

$$f^{\beta}(x) \leqq f^{\beta}(t) + \beta(x-t) f^{\beta-1}(t) f'(t) \quad \text{for all } x, t \in [a, b].$$

Multiplication by $g(t) f^{\alpha}(t)$ and integration with respect to t yields:

$$(5) \quad f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt \leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt + \frac{\beta}{\alpha+\beta} \int_a^b (x-t) g(t) (f^{\alpha+\beta}(t))'\,dt.$$

Integration by parts leads to

$$(6) \qquad \int_a^b (x-t) g(t) (f^{\alpha+\beta}(t))'\,dt = (x-b) g(b) f^{\alpha+\beta}(b) - -(x-a) g(a) f^{\alpha+\beta}(a) + \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt \leqq \leqq \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt,$$

and from (5) and (6) we conclude

$$f^{\beta}(x) \int_a^b g(t) f^{\alpha}(t)\,dt \leqq \frac{\alpha+2\beta}{\alpha+\beta} \int_a^b g(t) f^{\alpha+\beta}(t)\,dt - \frac{\beta}{\alpha+\beta} \int_a^b (x-t) g'(t) f^{\alpha+\beta}(t)\,dt$$

which is equivalent to inequality (4).

Remark. Inequality (4) is an extension of a result given in [3].

If we set $g(t)\equiv 1$ and $\alpha=\beta=1$, then we get the following (slightly modified) version of the left-hand side of (2):

$$\frac{2}{3}\max_{0\leqq x\leqq 1} f(x) \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt}. \tag{7}$$

Since the sign of equality holds for $f(x)=x$ we conclude that the constant 2/3 cannot be replaced by a greater number. Furthermore, setting $g(t)=t$ and $\alpha=\beta=1$ in (4) we obtain:

Corollary. *If $f(\not\equiv 0)$ is a non-negative, continuous, concave function on* [0, 1], *then we have for all* $x\in[0, 1]$:

$$\frac{f(x)}{2}+\frac{x}{4}\frac{\int_0^1 f^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt}. \tag{8}$$

Remarks. 1) As an immediate consequence of (8) we get the following form of the left-hand inequality of (3):

$$\frac{1}{2}\max_{0\leqq x\leqq 1} f(x) \leqq \frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt}. \tag{9}$$

Putting $f(x)=1-x$ equality holds in (9); hence the constant 1/2 is best possible.

We note that (7) and (9) are striking companions of Favard's inequality

$$\frac{1}{2}\max_{0\leqq x\leqq 1} f(x) \leqq \int_0^1 f(t)\,dt$$

which is true for all functions f which are non-negative, continuous and concave on [0, 1]; see [1, p. 44] and [4].

2) If f is monotonic, then the two integral ratios given in (2) and (3) can be compared:

Let $f(\not\equiv 0)$ be a non-negative and decreasing function on $[0, 1]$, then

$$\frac{\int_0^1 tf^2(t)\,dt}{\int_0^1 tf(t)\,dt} \leqq \frac{\int_0^1 f^2(t)\,dt}{\int_0^1 f(t)\,dt}.$$

If f is increasing, then the reversed inequality holds; see [5, pp. 302—303].

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