Acta Sci. Math., 56 (1992), 83—88

On the generalized strong de la Vallée Poussin approximation

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Dedicated to Professor Béla Csákány on his 60th birthday

1. Let $\{\varphi_n(x)\}$ be an orthonormal system on a finite interval (a, b). In this paper we shall consider real orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.1}$$

By the Riesz—Fischer theorem the partial sums $s_n(x)$ of any such series converge in the L^2 norm to a square-integrable function $f(x)$.

It is well known that there are many interesting results stating certain summability properties of series (1.1) or providing accurate rate of the approximation for special summation methods both in ordinary and strong sense. Some sample theorems for approximation can be found e.g. in the works [1], [2], [3], [5].

Analysing the theorems being in the above mentioned works we can realize that most of the results concerning any property of ordinary approximation have an analogue in strong sense. In other words, we have the same rate of approximation for strong means as for ordinary ones. But there is a lack in the case of the generalized de la Vallée Poussin summability.

The aim of the present paper is to bring this discrepansy to an end, that is, to show that the analogy also holds for this summability. Namely we shall prove that two theorems of [2] (see Theorems V and VI) can be extended to strong approximation by the same rate, too.

Now we recall the definitions of the generalized ordinary, strong and very strong de la Vallée Poussin summability methods (see [2]).

Let $\lambda := \{\lambda_n\}$ be a non-decreasing sequence of natural numbers for which $\lambda_0 = 1$ and $\lambda_{n+1} \leqq \lambda_n + 1$. Series (1.1) is (V, λ)-summable if

$$V_n(x) := V_n(\lambda;\ x) := \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k(x) \to f(x)$$

Received February 12, 1990.

almost everywhere (a.e.); *strongly* (V,λ)-*summable* if

$$V_n|x| := V_n|\lambda;\ x| := \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_k(x)-f(x)|^2\right\}^{1/2} \to 0$$

a.e.; and *very strongly* (V,λ)-*summable* if for any increasing sequence $\nu:=\{\nu_k\}$ of natural numbers

$$V_n^\nu|x| := V_n|\lambda,\nu;\ x| := \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_{\nu_k}(x)-f(x)|^2\right\}^{1/2} \to 0 \quad \text{a.e.}$$

We also note that if $\lambda_n=n$ then the $V_n(x)$-means reduce to the $(C,1)$-means, if $\lambda_n\equiv 1$ then to the partial sums $s_n(x)$, and if $\lambda_n=\left[\frac{n}{2}\right]$ $(n\geqq 2)$, where $[\beta]$ denotes the integral part of β, then we get the classical de la Vallée Poussin means.

2. Now we can formulate our theorems:

Theorem 1. *Let* $\varrho:=\{\varrho_n\}$ *and* $l:=\{l_n\}$ *be monotone non-decreasing sequences. If the condition*

$$\sum_{n=0}^{\infty} c_n^2\varrho_n^2<\infty \tag{2.1}$$

implies the (V,λ)-*summability of* (1.1) *for any* $\{\varphi_n(x)\}$ *and* $\{c_n\}$ *almost everywhere on a set* E *of positive measure, then the conditions*

$$\sum_{n=0}^{\infty} c_n^2\varrho_n^2 l_n^2<\infty \quad \text{and} \quad l_{\mu_{m+1}}\leqq K l_{\mu_m} \quad \text{with} \quad 1\leqq K<\sqrt{2}, \tag{2.2}$$

where $\mu_0=0$ *and* $\mu_m:=\sum_{k=0}^{m-1}\lambda_{\mu_k}$, *imply that*

$$V_n|\lambda,\nu;\ x| = o_x(l_n^{-1}) \tag{2.3}$$

holds almost everywhere on the set E *for any increasing sequence* $\nu=\{\nu_k\}$ *of positive integers.*

Theorem 2. *If a monotone non-decreasing sequence* $l=\{l_n\}$ *satisfies the conditions*

$$l_{\mu_{m+1}}\leqq K l_{\mu_m} \quad \text{with} \quad 1\leqq K<\sqrt{2}; \quad \text{and} \quad \sum_{k=0}^{m} l_{\mu_k}^2 = O(l_{\mu_m}^2); \tag{2.4}$$

then already the following condition

$$\sum_{n=0}^{\infty} c_n^2 l_n^2<\infty \tag{2.5}$$

implies the validity of (2.3) *almost everywhere in* (a,b) *for any* $\{\varphi_n(x)\}$ *and* $\{\nu_n\}$.

We remind the reader of that these theorems are the strong analogues of Theorems V and VI proved in [2]. Furthermore we recall that the condition

$$\sum_{m=1}^{\infty}\{\sum_{n=\mu_m+1}^{\mu_{m+1}} c_n^2\}\log^2 m<\infty$$

implies the (V,λ)-summability of of (1.1) (see [2], Theorem II).

3. In order to prove our theorems we require some lemmas. In what follows M will denote an absolute constant.

Lemma 1 ([2], Lemma II). *Let $\{p_m\}$ be an increasing sequence of positive integers, let $\{\gamma_m\}$ be a non-decreasing sequence of positive numbers so that*

(3.1) $$\sum_{m=1}^{n}\gamma_{p_m}^2\leqq M\gamma_{p_n}^2,\quad n=1,2,\ldots.$$

If

(3.2) $$\sum_{n=1}^{\infty}c_n^2\gamma_n^2<\infty,$$

then

(3.3) $$s_{p_m}(x)-f(x)=o_x(\gamma_{p_m}^{-1})$$

a.e. in (a,b).

Lemma 2 ([2], Lemma III). *Let $\{p_m\}$ be an increasing sequence of positive integers, $\{u_n\}$ be an arbitrary sequence, furthermore let $\{v_n\}$ be a positive, monotone non-decreasing sequence with the property $v_{p_m+1}=\ldots=v_{p_{m+1}}$ $(m=1,2,\ldots)$. If the p_m-th partial sums of the series $\sum_{n=0}^{\infty}u_nv_n$ converge then the p_m-th partial sums of the series $\sum_{n=1}^{\infty}u_n$ also converge, furthermore if $s=\lim_{m\to\infty}s_{p_m}$, where $s_k:=\sum_{n=1}^{k}u_n$, we also have that*

$$|s_{p_m}-s|=o(v_{p_{m+1}}^{-1}).$$

Lemma 3 ([2], Theorem I). *In order that series (1.1) a.e. on a set E of positive measure should be (V,λ)-summable, it is necessary and sufficient that the partial sums $s_{\mu_m}(x)$ of (1.1) ($\mu_0=1$ and $\mu_m:=\sum_{k=0}^{m-1}\lambda_{\mu_k}$) converge a.e. on E.*

Lemma 4 ([4], Lemma 3). *Let $\delta>0$ and $\{\delta_n\}$ be an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system $\{\varphi_n(x)\}$ the condition*

$$\sum_{n=1}^{\infty}\delta_n\Big(\sum_{k=n}^{\infty}c_k^2\Big)^{\delta}<\infty$$

implies that the partial sums $s_n(x)$ of (1.1) possess a property P, then any subsequence $\{s_{\nu_n}(x)\}$ $(\nu_n<\nu_{n+1})$ of the partial sums of (1.1) also possesses property P.

Finally we need to prove the following new lemma.

Lemma 5. *If a monotone non-decreasing sequence* $l=\{l_n\}$ *satisfies the conditions*

(3.4) $$l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \text{with} \quad 1 \leqq K < \sqrt{2}, \quad m=1,2,\ldots;$$

then condition (2.5) *implies that*

(3.5) $$\{\lambda_{\mu_n}^{-1} \sum_{k=\mu_n-\lambda_{\mu_n}+1}^{\mu_n} |s_k(x)-s_{\mu_n}(x)|^2\}^{1/2} = o_x(l_{\mu_n}^{-1})$$

holds a.e. in (a, b).

Proof. An elementary calculation gives that

(3.6) $$\sum_{m=1}^{\infty} \int_a^b \frac{l_{\mu_m}^2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} |s_k(x)-s_{\mu_m}(x)|^2\, dx =$$
$$= \sum_{m=1}^{\infty} l_{\mu_m}^2 \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} \left(1-\frac{\mu_m+1-k}{\lambda_{\mu_m}}\right) c_k^2 =: \sum\nolimits_1.$$

Let α^+ denote the positive part of α. Using this notion we can estimate $\sum_1$ as follows:

(3.7) $$\sum\nolimits_1 \leqq \sum_{m=1}^{\infty} \sum_{k=\mu_m+1}^{\mu_m} c_k^2 \sum_{n=m}^{\infty} \lambda_{\mu_n}^{-1} l_{\mu_n}^2 (\mu_m-\mu_n+\lambda_{\mu_n})^+.$$

Next we show that

(3.8) $$R_m := \sum_{n=m}^{\infty} \lambda_{\mu_n}^{-1} l_{\mu_n}^2 (\mu_m-\mu_n+\lambda_{\mu_n})^+ = O(l_{\mu_m}^2)$$

holds. On account of the definition of μ_m we have

(3.9) $$R_m = \sum_{n=m}^{\infty} l_{\mu_n}^2 \left(1-\frac{\mu_n-\mu_m}{\lambda_{\mu_n}}\right)^+ = l_{\mu_m}^2 + \sum_{n=m+1}^{\infty} l_{\mu_n}^2 (1-\lambda_{\mu_n}^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k})^+.$$

Putting

$$\Lambda_n^{(m)} := (1-\lambda_{\mu_n}^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k})^+$$

and taking into account that $\lambda_{\mu_{n+1}} \leqq 2\lambda_{\mu_n}$ always holds, thus we get for any $n>m$ that

$$\Lambda_n^{(m)} \leqq (1-(2\lambda_{\mu_{n-1}})^{-1} \sum_{k=m}^{n-1} \lambda_{\mu_k})^+ = \left(\frac{1}{2}-(2\lambda_{\mu_{n-1}})^{-1} \sum_{k=m}^{n-2} \lambda_{\mu_k}\right)^+ \leqq$$
$$\leqq \left(\frac{1}{2}-(4\lambda_{\mu_{n-2}})^{-1} \sum_{k=m}^{n-2} \lambda_{\mu_k}\right)^+ = \left(\frac{1}{4}-(4\lambda_{\mu_{n-2}})^{-1} \sum_{k=m}^{n-3} \lambda_{\mu_k}\right) \leqq \ldots \leqq \left(\frac{1}{2}\right)^{n-m}.$$

Hence, by (3.4) and (3.9), it follows that

$$R_m \leqq l_{\mu_m}^2 + \sum_{n=m+1}^{\infty} l_{\mu_n}^2 \left(\frac{1}{2}\right)^{n-m} \leqq l_{\mu_m}^2 \left(1 + \sum_{n=m+1}^{\infty} \left(\frac{K^2}{2}\right)^{n-m}\right) = O(l_{\mu_m}^2),$$

and this proves (3.8). Consequently, by (3.6), (3.7) and (3.8), using the Beppo Levi theorem, we get that

$$\sum_{m=1}^{\infty} \frac{l_{\mu_m}^2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} |s_k(x) - s_{\mu_m}(x)|^2 < \infty$$

almost everywhere in (a, b), whence (3.5) obviously follows.

4. Proof of Theorem 1. On account of Lemma 4 with $\delta=1$ and $\delta_n := \varrho_n^2 l_n^2 - \varrho_{n-1}^2 l_{n-1}^2$ it is clear that we have to carry the proof only when $\nu_k = k$.

On the other hand a straightforward calculation gives that if $\mu_m < n \leqq \mu_{m+1}$ holds then

$$(V_n|\lambda;\, x|)^2 \leqq (V_{\mu_m}|\lambda;\, x|)^2 + 2(V_{\mu_{m+1}}|\lambda;\, x|)^2;$$

so in order to prove (2.3) it is sufficient to verify that

$$V_{\mu_m}|\lambda;\, x| = o_x(l_{\mu_m}^{-1}) \tag{4.1}$$

holds a.e. on E.

Now we put $\bar{l}_k := l_{\mu_m}$ for $\mu_m < k \leqq \mu_{m+1}$, $m = 0, 1, 2, \ldots$. Then, by (2.2), the series

$$\sum_{n=1}^{\infty} c_n \bar{l}_n \varphi_n(x) \tag{4.2}$$

is (V, λ)-summable a.e. on E; consequently, by Lemma 3, the μ_m-th partial sums of (4.2) also converge a.e. on E. In the next step we use Lemma 2 whence the estimations

$$s_{\mu_m}(x) - f(x) = o_x(\bar{l}_{\mu_{m+1}}^{-1}) = o_x(l_{\mu_m}^{-1}) \tag{4.3}$$

follow a.e. on E.

Since

$$(V_{\mu_m}|\lambda;\, x|)^2 \leqq \frac{2}{\lambda_{\mu_m}} \sum_{k=\mu_m-\lambda_{\mu_m}+1}^{\mu_m} \{|s_k(x) - s_{\mu_m}(x)|^2 + |s_{\mu_m}(x) - f(x)|^2\}, \tag{4.4}$$

so, by Lemma 5 and (4.3), we get (4.1), what completes the proof of Theorem 1.

Proof of Theorem 2. By the same token as in the proof of Theorem 1 we only have to prove estimation (4.1). Now we can use Lemma 1 with $\gamma_m := l_m$ and $p_m := \mu_m$ taking into account conditions (2.4) and (2.5), so we get that

$$s_{\mu_m}(x) - f(x) = o_x(l_{\mu_m}^{-1}) \tag{4.5}$$

holds a.e. in (a, b). By (2.4) and (2.5) we can apply Lemma 5, too; therefore (3.5) and (4.5), regarding (4.4), verify (4.1). Herewith Theorem 2 is also proved.

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