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An additional note on strong approximation by orthogonal series

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1. Let $\{\varphi_n(x)\}$ be an orthogonal system on a finite interval (a, b). In this note we consider real orthogonal series

$$(1.1) \qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

It is well known that the partial sums $s_n(x)$ of any such series converge in the L^2 norm to a function $f(x) \in L^2(a, b)$.

A very general theorem we proved in [5] concerning strong approximation by orthogonal series included, as special cases, many of the results obtained previously by several authors. In addition, our theorem in [5] yielded some new results pertaining to strong approximation by certain Hausdorff and $[J, f]$-means. We refer the reader to Theorems A, B, C, D and E cited in our paper as previously known and to Theorems 2, 3, 2* and 3* as the new results obtained by means of our main theorem.

In order to recall the main theorem and to state the purpose of the present note, we need the following definitions and notations:

Let $\boldsymbol{\alpha} := \{\alpha_k(\omega)\}$, $k=0, 1, \ldots$ denote a sequence of non-negative functions defined for $0 \leqq \omega < \infty$, satisfying

$$(1.2) \qquad \sum_{k=0}^{\infty} \alpha_k(\omega) \equiv 1.$$

We assume that the linear transformation of real sequences $\mathbf{x} := \{x_k\}$ given by

$$A_\omega(\mathbf{x}) := \sum_{k=0}^{\infty} \alpha_k(\omega) x_k, \quad \omega \to \infty,$$

is regular [1, p. 49]. Let $\gamma := \gamma(t)$ denote a non-decreasing positive function defined

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for $0 \leqq t < \infty$ and $\boldsymbol{\mu} := \{\mu_m\}$ $m = 0, 1, \dots$ an increasing sequence of integers with $\mu_0 = 0$ satisfying the following conditions:

There exist positive integers N and h so that

$$\mu_{m+1} \leqq N\mu_m, \quad \gamma(\mu_{m+1}) \leqq N\gamma(\mu_m), \quad \gamma(\mu_{m+h}) \geqq 2\gamma(\mu_m) \tag{1.3}$$

hold for all m.

For $r > 1$, $\omega > 0$ and $m = 0, 1, \dots$ we define

$$\varrho_m(\omega, r) := \left\{\frac{1}{\mu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} (\alpha_k(\omega))^r\right\}^{1/r}. \tag{1.4}$$

In terms of the quantities introduced above we can recall our result in [5]:

Theorem I. *Let $p > 0$ and $g(t)$ a non-decreasing positive function on* $[0, \infty)$. *Suppose that there exist $r > 1$ and a constant $K(r, \boldsymbol{\mu}, \boldsymbol{\gamma})$ such that for every $\omega > 0$*

$$\sum_{m=0}^{\infty} \mu_m \varrho_m(\omega, r)\gamma(\mu_m)^{-p} \leqq K(r, \boldsymbol{\mu}, \boldsymbol{\gamma})\big(g(\omega)/\gamma(\omega)\big)^p. \tag{1.5}$$

If

$$\sum_{n=0}^{\infty} c_n^2 \gamma(n)^2 < \infty, \tag{1.6}$$

then

$$A_\omega(f, p, \mathbf{v}; x) := \Big\{\sum_{k=0}^{\infty} \alpha_k(\omega)|s_{\nu_k}(x) - f(x)|^p\Big\}^{1/p} = O_x\big(g(\omega)/\gamma(\omega)\big) \tag{1.7}$$

almost everywhere (a.e.) in (a, b) for any increasing sequence $\mathbf{v} := \{\nu_k\}$ of positive integers.

If, in addition, for every fixed m,

$$\varrho_m(\omega, r) = o\big((g(\omega)/\gamma(\omega))^p\big), \quad \textit{as} \quad \omega \to \infty, \tag{1.8}$$

then the O_x in (1.7) *can be replaced by o_x.*

We mention that the most important special case of Theorem I is when both (1.5) and (1.8) are satisfied with $g(\omega) \equiv 1$. In this case we get that

$$A_\omega(f, p, \mathbf{v}; x) = o_x\big(\gamma(\omega)^{-1}\big) \tag{1.9}$$

holds a.e. in (a, b).

Next we recall the definition of the generalized ordinary and very strong de la Vallée Poussin summability methods (see [2]) and a theorem proved in [4].

Let $\boldsymbol{\lambda} := \{\lambda_n\}$ be a non-decreasing sequence of natural numbers for which $\lambda_0 = 1$ and $\lambda_{n+1} \leqq \lambda_n + 1$. Series (1.1) is (V, λ)-summable if

$$V_n(\boldsymbol{\lambda}; x) := \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k(x) \to f(x) \quad \text{a.e.};$$

and very strongly (V, λ)-summable if for any increasing sequence $\mathbf{v}=\{\nu_k\}$ of positive integers

$$V_n|\lambda, \mathbf{v}; x| := \left\{\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} |s_{\nu_k}(x)-f(x)|^2\right\}^{1/2} \to 0 \quad \text{a.e.}$$

We also note that if $\lambda_n=n$ then the $V_n(\lambda; x)$-means reduce to the $(C, 1)$-means, if $\lambda_n\equiv 1$ then to the partial sums $s_n(x)$, and if $\lambda_n=\left[\frac{n}{2}\right]$ $(n\geqq 2)$, where $[\beta]$ denotes the integral part of β, then we get the classical de la Vallée Poussin means.

In [4] the first author proved, among others, the following result:

Theorem II. *If a monotone non-decreasing sequence* $\mathbf{l}:=\{l_n\}$ *satisfies the conditions*

$$(1.10)\qquad l_{\mu_{m+1}} \leqq K l_{\mu_m} \quad \textit{with} \quad 1 \leqq K < \sqrt{2}; \quad \textit{and} \quad \sum_{k=0}^{m} l_{\mu_k}^2 = O(l_{\mu_m}^2),$$

where $\mu_0=0$ *and* $\mu_m := \sum_{k=0}^{m-1} \lambda_{\mu_k}$; *then*

$$(1.11)\qquad \sum_{n=0}^{\infty} c_n^2 l_n^2 < \infty$$

implies that

$$(1.12)\qquad V_n|\lambda, \mathbf{v}; x| = o_x(l_n^{-1})$$

holds a.e. in (a, b) *for any* $\{\varphi_n(x)\}$ *and* $\mathbf{v}=\{\nu_n\}$.

In spite of the wide applicability of Theorem I, unfortunately, in the most important special case $g(\omega)\equiv 1$, it cannot be used to estimate the approximation-rate of the partial sums $s_n(x)$ of series (1.1) because then (1.5) does not hold for any $\boldsymbol{\mu}$. Consequently Theorem I does not include the result of Theorem II in the simplest special case when $\lambda_n\equiv 1$.

The aim of the present note is to fill this gap in Theorem I for $0<p\leqq 2$. The corresponding problem for $p>2$ remains open at this time.

In formulating our new result we shall use the notation as above and assume hence forth that the following conditions hold:

$$(1.13)\qquad \gamma(\mu_{m+1}) \leqq N\gamma(\mu_m), \quad g(\mu_{m+1}) \leqq Ng(\mu_m)$$

and

$$(1.14)\qquad \sum_{m=0}^{n} \gamma(\mu_m)^2 \varrho(m) \leqq N\gamma(\mu_n)^2$$

hold for all m and n, where $\varrho(t)$ denotes a non-increasing positive function defined on $[0, \infty)$.

Our theorem reads as follows.

Theorem III. *Suppose that there exists a natural number q such that for all k and m*

$$\alpha_k(n) \leqq N \sum_{i=-q}^{q} \alpha_k(\mu_{m+i}) \quad \textit{with} \quad \mu_m < n < \mu_{m+1} \tag{1.15}$$

and

$$\sum_{i=0}^{\infty} \frac{\gamma(\mu_i)^2}{g(\mu_i)^2} \sum_{j=\mu_m}^{\mu_{m+1}-1} \alpha_j(\mu_i) \leqq N\varrho(m)\gamma(\mu_m)^2 \tag{1.16}$$

hold. Then condition (1.6) *implies that*

$$A_n(f, p, \mathbf{v}; x) = o_x(g(n)/\gamma(n)) \tag{1.17}$$

a.e. in (a, b) *for every* p, $0<p\leqq 2$ *and for every sequence* $\mathbf{v}$.

2. In order to prove our Theorem we need the following lemma.

Lemma [3]. *Let* $\delta>0$ *and* $\{\delta_n\}$ *be an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system the condition*

$$\sum_{n=1}^{\infty} \delta_n \Big(\sum_{k=n}^{\infty} c_k^2\Big)^{\delta} < \infty$$

implies that the sequence $\{s_n(x)\}$ *possesses a property P, then any subsequence* $\{s_{\mu_n}(x)\}$ *also possesses property P.*

3. Proof of Theorem III. By assumptions (1.13) we have for any $\mu_m<l<\mu_{m+1}$ $(m=0, 1, \dots)$ that

$$\frac{g(\mu_m)}{N\gamma(\mu_m)} \leqq \frac{g(l)}{\gamma(l)} \leqq \frac{Ng(\mu_{m+1})}{\gamma(\mu_{m+1})}, \tag{3.1}$$

so, on account of (1.15), it is sufficient to prove (1.17) only for the values μ_n.

First we prove (1.17) in the special case $p=2$ and $\nu_k=k$; and as we have said above, only for the indices μ_n, i.e. we verify that

$$A_{\mu_n}(x) := A_{\mu_n}(f, 2, \{k\}; x) = o_x(g(\mu_n)/\gamma(\mu_n)) \tag{3.2}$$

holds a.e. in (a, b).

Then

$$\int_a^b A_{\mu_n}^2(x)\,dx = \sum_{m=0}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\mu_n) \int_a^b |s_k(x)-f(x)|^2\,dx \leqq \sum_{m=0}^{\infty} \sum_{k=\mu_m}^{\mu_{m+1}-1} \alpha_k(\mu_n) \sum_{i=\mu_m+1}^{\infty} c_i^2.$$

Putting

$$R_{\mu_m}^2 := \sum_{i=\mu_m+1}^{\infty} c_i^2,$$

we get, by (1.16), that

$$\sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\int_a^b A_{\mu_n}^2(x)\,dx \leqq \sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\sum_{m=0}^{\infty}\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\mu_n)R_{\mu_m}^2 = \tag{3.3}$$

$$= \sum_{m=0}^{\infty}R_{\mu_m}^2\sum_{n=0}^{\infty}\frac{\gamma(\mu_n)^2}{g(\mu_n)^2}\sum_{k=\mu_m}^{\mu_{m+1}-1}\alpha_k(\mu_n) \leqq N\sum_{m=0}^{\infty}R_{\mu_m}^2\varrho(m)\gamma(\mu_m)^2 := \sum\nolimits_1 .$$

To estimate $\sum_1$ we use assumptions (1.6), (1.13) and (1.14), and so we have

$$1/N\sum\nolimits_1 \leqq \sum_{m=0}^{\infty}\Big(\sum_{l=m}^{\infty}\sum_{k=\mu_l}^{\mu_{l+1}-1}c_k^2\Big)\varrho(m)\gamma(\mu_m)^2 = \tag{3.4}$$

$$= \sum_{l=0}^{\infty}\Big(\sum_{k=\mu_l}^{\mu_{l+1}-1}c_k^2\Big)\sum_{m=0}^{l}\varrho(m)\gamma(\mu_m)^2 \leqq N\sum_{l=0}^{\infty}\Big(\sum_{k=\mu_l}^{\mu_{l+1}-1}c_k^2\Big)\gamma(\mu_l)^2 \leqq N\sum_{n=0}^{\infty}c_n^2\gamma(n)^2 < \infty.$$

By (3.3) and (3.4), applying Beppo Levi's theorem, we get that

$$\sum_{n=0}^{\infty}\gamma(\mu_n)^2 g(\mu_n)^{-2}A_{\mu_n}^2(x) =$$

$$= \sum_{n=0}^{\infty}\gamma(\mu_n)^2 g(\mu_n)^{-2}\sum_{k=0}^{\infty}\alpha_k(\mu_n)|s_k(x)-f(x)|^2 < \infty$$

a.e. in (a, b). Hence (3.2) obviously follows.

For $0<p<2$

$$A_{\mu_n}(f, p, \{k\};\, x) = o_x\big(g(\mu_n)/\gamma(\mu_n)\big) \tag{3.5}$$

follows from (3.2) using Hölder's inequality and (1.2).

Now, on account of (3.1), relation (3.5) implies

$$A_n(f, p, \{k\};\, x) = o_x\big(g(n)/\gamma(n)\big) \tag{3.6}$$

a.e. in (a, b).

Finally, if we apply the Lemma with property P characterized by (3.6), then (1.7) follows for all p, $0<p\leqq 2$ and all sequences $\mathbf{v}$.

4. Application. We show that Theorem II can be derived from Theorem III. Since in the special case $\lambda_n\equiv 1$, Theorem II represents a statement concerning the partial sums of (1.1), it follows that under the proper conditions Theorem III yields certain results for the rate of approximation achieved by the partial sums, as well.

Now we show that Theorem III in the special case when $\varrho(m)\equiv g(m)\equiv 1$, $\gamma(n)=l_n$ and

$$\alpha_k(n) := \begin{cases} 1/\lambda_n & \text{for } \; n-\lambda_n < k \leqq n, \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

can be applied, with μ_m defined in Theorem II, that is, then (1.6), (1.13), (1.14), (1.15) and (1.16) are fulfilled.

Condition (1.6) holds trivially, (1.13) and (1.14) follow from (1.10).

In order to prove (1.15) we put $q=1$ and $N=2$, i.e. we want to verify that for any k and $\mu_m<n<\mu_{m+1}$

$$\alpha_k(n) \leqq 2[\alpha_k(\mu_{m-1})+\alpha_k(\mu_m)+\alpha(\mu_{m+1})] \tag{4.2}$$

always holds. Since $\lambda_{n+1}-\lambda_n \leqq 1$ for all n, therefore $\mu_m-\lambda_{\mu_m} \leqq n-\lambda_n$, whence, by (4.1),

$$\alpha_k(n) \leqq \alpha_k(\mu_m) \tag{4.3}$$

holds for any $(n-\lambda_n<)k \leqq \mu_m$.

On the other hand, taking into account that $\lambda_{\mu_{m+1}} \leqq \lambda_{\mu_m}+\mu_{m+1}-\mu_m=2\lambda_{\mu_m}$ and $\mu_{m+1}-\lambda_{\mu_{m+1}}=\mu_m+\lambda_{\mu_m}-\lambda_{\mu_{m+1}} \leqq \mu_m$, we get

$$\alpha_k(n) \leqq 2\alpha_k(\mu_{m+1}) \tag{4.4}$$

for any $\mu_m<k(\leqq n)$. Thus (4.3) and (4.4) verify (4.2), and herewith (1.15) is also proved for the entries $\alpha_k(n)$ given in (4.1).

To show (1.16) in the case given above we have to verify that

$$\sum\nolimits_2 := \sum_{i=0}^{\infty} l_{\mu_i}^2 \sum_{j=\mu_m+1}^{\mu_{m+1}} \alpha_j(\mu_i) \leqq N l_{\mu_m}^2 \tag{4.5}$$

holds for every m.

By (4.1) it is clear that if $j>\mu_i$ then $\alpha_j(\mu_i)=0$, therefore

$$\sum\nolimits_2 = \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \sum_{j=\mu_m+1}^{\mu_{m+1}} \alpha_j(\mu_i) = \tag{4.6}$$

$$= \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \lambda_{\mu_i}^{-1}(\mu_{m+1}-\max(\mu_m, (\mu_i-\lambda_{\mu_i})))^+ =: \sum\nolimits_3,$$

where β^+ denotes the positive part of β.

On account of the definition of μ_m we have that

$$\sum\nolimits_3 \leqq \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \lambda_{\mu_i}^{-1}(\mu_{m+1}-\mu_i+\lambda_{\mu_i})^+ = \tag{4.7}$$

$$= \sum_{i=m+1}^{\infty} l_{\mu_i}^2 \left(1-\frac{\mu_i-\mu_{m+1}}{\lambda_{\mu_i}}\right)^+ = l_{\mu_{m+1}}^2 + \sum_{i=m+2}^{\infty} l_{\mu_i}^2 \Big(1-\lambda_{\mu_i}^{-1} \sum_{k=m+1}^{i-1} \lambda_{\mu_k}\Big)^+.$$

Setting

$$\Lambda_i^{(m)} := \Big(1-\lambda_{\mu_i}^{-1} \sum_{k=m+1}^{i-1} \lambda_{\mu_k}\Big)^+$$

and taking into account that $\lambda_{\mu_{k+1}} \leqq 2\lambda_{\mu_k}$ always holds, we have for any $i > m+1$ that

$$\Lambda_i^{(m)} \leqq \Big(1-(2\lambda_{\mu_{i-1}})^{-1}\sum_{k=m+1}^{i-1}\lambda_{\mu_k}\Big)^+ = \Big(\frac{1}{2}-(2\lambda_{\mu_{i-1}})^{-1}\sum_{k=m+1}^{i-2}\lambda_{\mu_k}\Big)^+ \leqq$$

$$\leqq \Big(\frac{1}{2}-(4\lambda_{\mu_{i-2}})^{-1}\sum_{k=m+1}^{i-2}\lambda_{\mu_k}\Big)^+ = \Big(\frac{1}{4}-(4\lambda_{\mu_{i-2}})^{-1}\sum_{k=m+1}^{i-3}\lambda_{\mu_k}\Big)^+ \leqq$$

$$\leqq \Big(\frac{1}{4}-(8\lambda_{\mu_{i-3}})^{-1}\sum_{k=m+1}^{i-3}\lambda_{\mu_k}\Big)^+ \leqq \ldots \leqq \Big(\frac{1}{2}\Big)^{i-m-1}.$$

Hence, by (1.10), (4.6) and (4.7), we obtain that

$$\sum\nolimits_2 \leqq l_{\mu_{m+1}}^2 + 2\sum_{i=m+2}^{\infty} l_{\mu_i}^2\Big(\frac{1}{2}\Big)^{i-m} \leqq l_{\mu_{m+1}}^2 + 2\sum_{i=m+2}^{\infty} l_{\mu_{m+1}}^2\Big(\frac{K^2}{2}\Big)^{i-m} = O(l_{\mu_m}^2),$$

that is, that (4.5) holds. This proves that (1.16) is satisfied, as stated.

It follows that all of the assumptions of Theorem III are fulfilled if the parameters are chosen according to the requirements of Theorem II; therefore we have proved that Theorem III implies Theorem II.

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