On an imbedding theorem

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Introduction. In 1968, P. L ULJANOV [13] gave a sufficient and necessary condition for the imbedding of Hölder class H_p^ω into the space L^q $(1\leqq p<q<\infty)$. The result of Uljanov was generalized later by L. LEINDLER [5], [6]. In this paper we consider an analogous problem for the case of the new modulus $\omega_{\varphi,w}(f,\delta)_p$ introduced by Z. DITZIAN and V. TOTIK [1], namely we give a necessary and sufficient condition for the imbedding of Hölder type class of functions determined by $\omega_{\varphi,w}(f,\delta)_p$ with $w(x)=(1-x)^\alpha(1+x)^\beta$, $\varphi(x)=\sqrt{1-x^2}$ $(\alpha,\beta\geqq 0,\ x\in(-1,1))$ into another class of functions.

An imbedding theorem. Let $1\leqq p<\infty$. Let $u(x)$ be a nonnegative, integrable function on the finite interval (a,b). Denote by $L_u^p(a,b)$ the Banach space of all measurable functions on (a,b) with the norm

$$\|f\|_{L_u^p(a,b)}=\Big\{\int_a^b|f(x)|^p u(x)\Big\}^{1/p}.$$

In the case $u\equiv 1$ we use the notations $L^p(a,b)$, $\|f\|_{L^p(a,b)}$, respectively.

The modulus of a function $f\in L^p(a,b)$ is defined by the formula

$$\omega(f,\delta)_{L^p(a,b)}=\sup_{0<h\leqq\delta}\Big\{\int_a^{b-h}|f(x+h)-f(x)|^p\,dx\Big\}^{1/p},\quad (0\leqq\delta\leqq b-a).$$

Let (we shall use these notations throughout this paper)

$$w(x)=w_{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta,\quad (\alpha,\beta\geqq 0,\ x\in(-1,1));$$

$$\varphi(x)=\sqrt{1-x^2}\quad (x\in(-1,1)).$$

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The weighted modulus of a function f for which $wf\in L^p(-1, 1)$ was introduced by Z. Ditzian and V. Totik as follows:

$$\omega_{\varphi, w}(f, \delta)_p := \sup_{0<h\leqq\delta} \|w\Delta_{\varphi(x)h}f(x)\|_{L^p(-1,1)}$$

where

$$\Delta_{\varphi(x)h}f(x) := \begin{cases} f(x+\varphi(x)h)-f(x) & \text{for } x\colon x+\varphi(x)h\in(0, 1), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\omega(\delta)$ be a modulus of continuity, i.e. $\omega(\delta)$ is an onnegative, increasing continuous function on $[0, 1]$, $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\leqq\omega(\delta_1)+\omega(\delta_2)$ $(0\leqq\delta_1<\delta_1+\delta_2\leqq 1)$. Define the Hölder type class

$$H^{\omega}_{\varphi, w, p} := \{f\colon wf\in L^p(-1, 1),\ \omega_{\varphi, w}(f, \delta)_p = O_f\{\omega(\delta)\}\ (\delta\to 0)\}.$$

We shall prove

Theorem 1. *Let* $1\leqq p<q<\infty$. *Let* $\omega(\delta)$ *be an arbitrary modulus of continuity. Then*

$$H^{\omega}_{\varphi, w, p} \subset L^q_{w^{1/q}\varphi^{(q/p)-1}}(-1, 1) \tag{3}$$

iff

$$\sum_{n=1}^{\infty} n^{(q/p)-2}\omega^q\left(\frac{1}{n}\right) < \infty. \tag{4}$$

For the proof of Theorem 1 we need some lemmas.

For any function $f(x)$ defined on $(-1, 1)$, let $f^*(\Theta):=f(\cos\Theta)$ $(\Theta\in(0, \pi))$. Let $P_n(\alpha, \beta, x)$ be the n-th orthonormal polynomials with respect to the parameters α, β. Then the system

$$\Phi = \{J_n(\alpha, \beta, \theta)\} := \{P_n^*(\alpha, \beta, \theta)[w^*_{\alpha,\beta}(\theta)\varphi^*(\theta)]^{1/2}\}$$

is orthonormal on $(0, \pi)$. Denote by Φ_n the set of all φ-polynomials of degree at most n, i.e. the set of all functions of the form $\sum_{k=0}^{n}\lambda_k J_k(\alpha, \beta, \theta)$ (λ_k are real numbers, $k=0, \ldots, n$).

Lemma 1. *For any* $\varphi_n\in\Phi_n$ $(n=1, 2, \ldots)$ *and* $1\leqq p<q<\infty$, *the inequalities*

$$\|\varphi_n'\|^*_{L^p(0,\pi)} \leqq cn\|\varphi_n\|_{L^p(0,\pi)} \tag{5}$$

and

$$\|\varphi_n\|_{L^p(0,\pi)} \leqq cn^{1/p-1/q}\|\varphi_n\|_{L^p(0,\pi)} \tag{6}$$

hold.

Proof. Combining [3, T. 4] with [8, T. 14] we get (5) and (6).

For $wf\in L^p(-1, 1)$ let

$$E_n(w, f)_p = \inf\|w(f-p_n)\|_{L^p(-1,1)}, \quad p_n\in\pi_n. \tag{7}$$

where π_n denotes the set of all algebraic polynomials of degree at most n $(n=0, 1, \ldots)$.

We define also the best approximation of a function $g \in L^p(0, \pi)$ by Φ-polynomials:

$$E_n^*(g)_p = \inf \|g-\varphi_n\|_{L^p(0,\pi)}, \quad \varphi_n \in \Phi_n. \tag{8}$$

It is clear that

$$E_n(w, f)_p = E_n^*(g_f)_p, \tag{9}$$

where

$$g_f(\theta) := f^*(\theta) w^*(\theta) \sin^{1/p} \theta. \tag{10}$$

Lemma 2 ([11], T. 3). *Let* $1 \leqq p < \infty$. *We have for every* $wf \in L^p(-1, 1)$

$$E_n(w, f)_p \leqq c\omega_{\varphi, w}\left(f, \frac{1}{n}\right)_p \quad (n = 1, 2, \ldots). \tag{11}$$

Lemma 3. *Let* $1 \leqq p < \infty$. *For every* $g \in L^p(0, \pi)$ *the inequality*

$$\omega\left(g, \frac{1}{n}\right)_{L^p(0,\pi)} \leqq cn^{-1} \sum_{k=0}^{n} E_k^*(g)_p \tag{12}$$

holds.

Proof. Using inequality (5) we can prove this Lemma by the same way as that of the inverse theorem for the best trigonometric approximation (see e.g. [7]).

By a result of DITZIAN and TOTIK (see [1], T. 2.1.1.) we have that $\omega_{\varphi, w}(f, \delta)_p$ is equivalent to the K-functional

$$K_{\varphi, w}(f, \delta)_p := \inf_{h \in D^p_{\varphi, w}} \{\|w(f-h)\|_{L^p(-1,1)} + \delta \|w\varphi h'\|_{L^p(-1,1)}\}$$

where $D^p_{\varphi, w}$ denotes the class of all functions g, which are locally absolutely continuous on $(-1, 1)$ and for which wg, $w\varphi g' \in L^p(-1, 1)$.

On the other hand, the other K-functional defined on $L^p(0, \pi)$:

$$K^*(g, \delta)_p := \inf_{h \in D_p} \{\|w^*(\varphi^*)^{1/p}(h-g)\|_{L^p(0,\pi)} + \delta \|w^*(\varphi^*)^{1/p} h'\|_{L^p(0,\pi)}\}$$

where D_p denotes the class of all locally absolutely continuous functions h on $(0, \pi)$ for which $(\varphi^*)^{1/p} w^* h \in L^p(0, \pi)$, is equivalent to the following modulus of continuity

$$\Omega_{A,B}(g, \delta)_p := \sup_{0<h\leqq\delta} \left\{\int_0^B |g(\theta+h)-g(\theta)|^p (w^*(\theta))^p \varphi^*(\theta)\, d\theta\right\}^{1/p} + \tag{13}$$

$$+ \sup_{0<h\leqq\delta} \left\{\int_A^\pi |g(\theta-h)-g(\theta)|^p (w^*(\theta))^p \varphi^*(\theta)\, d\theta\right\}^{1/p},$$

$$(0 < A < B < \pi;\ 0 < \delta < \min(A, \pi-B)).$$

This fact was proved essentially in [11], special cases of which were proved in [9] and [10].

Summing the mentioned statements we have

Lemma 4. *Let* $1<p<\infty$, $0<A<B<\pi$. *Let* $wf\in L^p(-1,1)$ *and*

$$g_f(\theta):=f^*(\theta)w^*(\theta)\sin^{1/p}\theta.$$

Then

$$\omega_{\varphi,w}(f,\delta)_p\sim\Omega_{A,B}(g,\delta)_p\quad(\delta\to 0). \tag{14}$$

After these, let us turn to the

Proof of Theorem 1. a) (4)⇒(3). Let $wf\in L^p(-1,1)$. From (4) it follows by (11), that

$$\sum_{o=1}^{\infty} n^{(q/p)-2}E_n^q(w,f)_p<\infty$$

and so, we have for the function g_f defined by (10)

$$\sum_{n=1}^{\infty} n^{(q/p)-2}E_n^{*q}(g_f)_p<\infty.$$

Hence, by Hardy inequality and (12) we get

$$\sum_{n=1}^{\infty}\omega^q\left(g_f,\frac{1}{n}\right)_{L^p(0,\pi)} n^{(q/p)-2}<\infty$$

which implies by *T.* 1 of [13] that $g_f\in L^q(0,\pi)$, therefore $f\in L^q_{w^q\varphi^{q/p-1}}(-1,1)$.

b) (3)⇒(4). Suppose, that (4) does not hold. Using the method applied in [13], p. 673 one can contruct a function $\varphi_0\in L^p\left[\frac{1}{4},\frac{5}{4}\right]$ satisfying the following conditions

$$\varphi_0(x)=0,\quad x\in[3/4,5/4]; \tag{15}$$

$$\int_{1/4}^{1/4+h}|\varphi_0(x)|^p\,dx\leqq c\omega^p(h); \tag{16}$$

$$\omega(\varphi_{0,\delta})_{L^p(1/4,5/4)}\leqq c\omega(\delta); \tag{17}$$

$$\varphi_0\notin L^q[1/4,5/4]. \tag{18}$$

Let now

$$g_0(\theta):=\begin{cases}\varphi_0(\theta)w^*(\theta)[\varphi^*(\theta)]^{1/p} & \text{for}\quad \theta\in[1/4,5/4],\\ 0\quad \text{for}\quad \theta\in[0,\pi]\setminus[1/4,5/4].\end{cases}$$

We estimate the modulus (13) with $A=3/2$, $B=2$ of the function g_0. By (15), (16) and (17) one can see that

$$\Omega_{3/2,2}(g_0, \delta)_p = O\{\omega(\delta)\} \quad (\delta \to 0).$$

Therefore by (14) we have for the function

$$f_0(x) := g_0(\arccos x) w^{-1}(x) \varphi^{-1/p}(x)$$

$$\omega_{\varphi,w}(f_0, \delta)_p = O\{\omega(\delta)\} \quad (\delta \to 0),$$

which means that $f_0 \in H^{\omega}_{\varphi,w,p}$.

On the other hand by (18) it follows that

$$f_0 \notin L^q_{w^q \varphi^{q/p-1}}(-1, 1).$$

Thus, the necessity of (4) is proved.

Remark 1. The part (3)⇒(4) indeed can be obtained immediately from inequality (6) and T. 1 of [12]. Besides, we have appeared the other proof, because by this method we can prove a generalization of Theorem 1, which will be stated in the following.

For a nonnegative monotonic sequence of numbers $\{\varphi_k\}$, the function

$$\Phi(x) = \sum_{k=1}^{x} k^{(\gamma/p)-2} \varphi_k \quad (\gamma, p \geqq 1)$$

was introduced by Leindler [6]. We denote by $M^{\gamma,p}_{w,\varphi}$ the class of measurable functions f on $(-1, 1)$, for which

$$\int_0^{\pi} g_f^{q+1-(q/p)}(\theta) \Phi(|g_f(\theta)|) \, d\theta < \infty$$

where g_f is defined by (10). Then the following theorem is true.

Theorem 2. *Let* $1 \leqq p \leqq \gamma < \infty$. *Let* $\{\varphi_k\}$ *be a nonnegative monotonic sequence of numbers satisfying* $\varphi_{k^2} \leqq c\varphi_k$ *and in the case* $\gamma > p$, *moreover let*

$$\varphi_k \leqq \varphi_{k+1} \quad (k = 1, 2, \ldots).$$

Then

(19) $$H^{\omega}_{w,\varphi,p} \subset M^{\gamma,p}_{w,\varphi}$$

iff

(20) $$\sum_{n=1}^{\infty} n^{(\gamma/p)-2} \varphi_n \omega^{\gamma}\left(\frac{1}{n}\right) < \infty.$$

Using Lemmas 1—4 we can prove this theorem by the same method as we used to prove Theorem 1, with the modification that the results of Uljanov applied in the proof of Theorem 1 will be replaced by the generalized results of Leindler (see Theorem 3 and its proof in [6]), while the inequality of Hardy used in the proof will be replaced by a generalized inequality (see [4], inequality (1′)).

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