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Relating the normal extension and the regular unitary dilation of a subnormal tuple of contractions

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In this paper we deal with only bounded linear operators on complex infinite dimensional separable Hilbert spaces. If $S=(S_1, \ldots, S_n)$ is a tuple of operators on a Hilbert space $\mathscr{H}$, then for any n-tuple $k=(k_1, \ldots, k_n)$ of integers k_i, S^k denotes $S_1^{k_1}S_2^{k_2}\ldots S_n^{k_n}$, where $S_i^{k_i}$ is to be interpreted as $S_i^{*(-k_i)}$ if k_i is negative. If a Hilbert space $\mathscr{H}$ is contained in some Hilbert space $\mathscr{K}$, then $P(\mathscr{K}, \mathscr{H})$ will denote the projection of $\mathscr{K}$ onto $\mathscr{H}$. If for a tuple $S=(S_1, \ldots, S_n)$ of n commuting operators on $\mathscr{H}$, there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a tuple $M=(M_1, \ldots, M_n)$ of n commuting operators on $\mathscr{K}$ such that $S^kx=P(\mathscr{K}, \mathscr{H})M^kx$ for any x in $\mathscr{H}$ and any n-tuple k of non-negative integers, then S on $\mathscr{H}$ is said to *dilate to M on* $\mathscr{K}$; if moreover $\mathscr{H}$ is invariant for each M_i, then S *on* $\mathscr{H}$ is said to *extend to M on* $\mathscr{K}$. If S on $\mathscr{H}$ dilates to M on $\mathscr{K}$ and each M_i is unitary, then M *on* $\mathscr{K}$ is said to be a *unitary dilation of S on* $\mathscr{H}$. If S on $\mathscr{H}$ extends to M on $\mathscr{K}$ and each M_i is normal, then M *on* $\mathscr{K}$ is said to be a *normal extension of S on* $\mathscr{H}$, and S is said to be *subnormal.* Among all the normal extensions of a subnormal tuple S, there is a *minimal* one which is unique up to unitary equivalence (see [4]). In particular, if N on $\mathscr{K}$ is the minimal normal extension of S on $\mathscr{H}$, then $\mathscr{K}=\bigvee(N^k\mathscr{H}$: k is a tuple of non-positive integers), where $\bigvee$ denotes the closed linear span in the norm $\|\cdot\|_{\mathscr{K}}$ of $\mathscr{K}$.

For our purposes, a special type of unitary dilation, known in the literature as *regular unitary dilation* (or *Sz.-Nagy—Brehmer dilation*) (see [3], [7]) is important. For any n-tuple $k=(k_1, \ldots, k_n)$ of integers, define $k+=(\max(k_1, 0)), \ldots, \max(k_n, 0))$ and $k-=(\min(k_1, 0), \ldots, \min(k_n, 0))$. If for a tuple S of n commuting operators on $\mathscr{H}$, there exist a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a tuple U of n commuting unitaries on $\mathscr{K}$ such that $S^{k-}S^{k+}x=P(\mathscr{K}, \mathscr{H})U^{k-}U^{k+}x$ for any x in $\mathscr{H}$ and any n-tuple k of integers, then U *on* $\mathscr{K}$ is said to be a *regular unitary dilation* of S on $\mathscr{H}$; U is *minimal* if $\mathscr{K}=\bigvee\{U^k\mathscr{H}: k$ is an n-tuple of integers$\}$.

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In what follows, we will use the symbols $\bar{D}^n$, T^n and m_n to denote the closed unit polydisk in $\mathbf{C}^n$, the unit polycircle in $\mathbf{C}^n$ and the normalized product arc-length measure on T^n respectively. The spectral measure of a normal or unitary tuple M will be denoted by $\mu(M)$. In case $n=1$, it is well known (see [2], [5]) that if N on $\mathcal{K}$ is the minimal normal extension of a contraction S on $\mathcal{H}$, then $\mu(N)|T^1$ is absolutely continuous with respect to m_1, provided S is pure; that is, there does not exist a non-trivial closed reducing subspace $\mathcal{H}'$ of $\mathcal{H}$ such that $S|\mathcal{H}'$ is normal. (An examination of the proof in [5] and Theorem 6.4 in Chapter II of [7] actually reveals that "$S|\mathcal{H}'$ is normal" can be replaced by "$S|\mathcal{H}'$ is unitary".) A contranction S on a Hilbert space $\mathcal{H}$ is said to be $C_{0.}$ (see [7]) if $\|S^n h\|_{\mathcal{H}} \to 0$ as $n \to \infty$ for any h in $\mathcal{H}$. It is obvious from Theorem 3.2 in Chapter I of [7] that a $C_{0.}$ contraction does not have a non-trivial unitary part. At this stage, the reader may refer to the statement of Theorem 1 below and the question raised at the end of the paper.

Lemma 1. *If S is a subnormal tuple of contractions on $\mathcal{H}$, then S has a regular unitary dilation.*

Proof. This follows from Theorem 4.1 of [1] and from the observation made in the proof of Corollary to Theorem 3.1 of [1].

Lemma 2. *If U on $\mathcal{K}$ is a minimal regular unitary dilation of S on $\mathcal{H}$ and each S_i is a $C_{0.}$ contraction, then $\|\mu(U)(\cdot)x\|^2_{\mathcal{K}}$ is absolutely continuous with respect to m_n for any x in $\mathcal{K}$.*

Proof. Let U on $\mathcal{K}$ be a minimal regular unitary dilation of S on $\mathcal{H}$. Define operators D_i $(i=0, 1, \ldots, n)$ from $\mathcal{H}$ to $\mathcal{K}$ as follows: $D_0=I$ ($Ix=x$ for any x in $\mathcal{H}$), $D_{i+1}=D_i-U^*_{i+1}DT_{i+1}$, $(i=0, \ldots, n-1)$. Let A be the closed linear span of $D_n\mathcal{H}$ in $\mathcal{K}$. It follows from Theorem 1 of [3] that U^kA and U^lA are orthogonal to each other with respect to the inner product $\langle .,. \rangle$ of $\mathcal{K}$ for any two distinct integer n-tuples k and i, and

$$\mathcal{K} = \vee\{U^m A\colon\ m \text{ is an } n\text{-tuple of integers}\}.$$

Let $\xi=(\xi_1, \ldots, \xi_n)$ denote a generic point of T^n. For any a in A and any n-tuple k of integers, we have

$$\int_{T^n} \xi_1^{k_1}\xi_2^{k_2}\ldots\xi_n^{k_n}\, d\|\mu(U)(\xi)a\|^2_{\mathcal{K}} = \langle U^k a, a\rangle =$$

$$= \begin{cases} \|a\|^2_{\mathcal{K}} & \text{if} \quad k_i = 0, \quad \text{for each } i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the trigonometric polynomials are dense in $C(T^n)$, the space of continuous functions with the supremum norm, it follows that

$$\|\mu(U)(.)a\|_{\mathscr{K}}^2 = \|a\|_{\mathscr{K}}^2 m_n(.).$$

From our observations above and utilizing the fact that $\mu(U)$ commutes with all U^m, it is easy to deduce that $\|\mu(U)(.)x\|_{\mathscr{K}}^2$ is absolutely continuous with respect to m_n for any x in $\mathscr{K}$.

Theorem 1. *Let S be a subnormal tuple of $C_{0.}$ contractions on $\mathscr{H}$. If N on $\mathscr{K}$ is the minimal normal extension of S, then $\mu(N)/T^n$ is absolutely continuous with respect to m_n.*

Proof. Let S on $\mathscr{H}$ be a subnormal tuple of $C_{0.}$ contractions and N on $\mathscr{K}$ be its minimal normal extension. By Lemma 1, S has a regular unitary dilation U on some Hilbert space $\mathscr{K}'$. Define

$$\mathscr{L} = \vee\{U^k\mathscr{H}\colon k \text{ is an } n\text{-tuple of integers}\},$$

$\vee$ denoting the closed linear span in the norm of $\mathscr{K}'$, and let $W_i = U_i/\mathscr{L}$ $(i=1, \ldots, n)$. Then W on $\mathscr{L}$ is a minimal regular unitary dilation of S on $\mathscr{H}$.

Now for any h in $\mathscr{H}$ and any n-variable complex polynomial q, we have

$$\|q(N)h\|_{\mathscr{K}}^2 = \int_{D^n} |q(y)|^2 \, d\|\mu(N)(y)h\|_{\mathscr{K}}^2$$

and

$$\|q(W)h\|_{\mathscr{L}}^2 = \int_{T^n} |q(\xi)|^2 \, d\|\mu(W)(\xi)h\|_{\mathscr{L}}^2.$$

Since

$$\|q(N)h\|_{\mathscr{K}}^2 = \|q(S)h\|_{\mathscr{H}}^2 = \|P(\mathscr{L},\mathscr{H})q(W)h\|_{\mathscr{H}}^2 \leqq \|q(W)h\|_{\mathscr{L}}^2,$$

it follows in particular that

$$\int_{T^n} |q(\xi)|^2 \, d\|\mu(N)(\xi)h\|_{\mathscr{K}}^2 \leqq \int_{T^n} |q(\xi)|^2 \, d\|\mu(W)(\xi)h\|_{\mathscr{L}}^2. \tag{1}$$

It is known that the unit polydisk algebra, as restricted to T^n, is an approximating in modulus algebra (see [6]); that is, any positive continuous function on T^n can be approximated uniformaly on T^n by the modulii of polynomials. It follows from (1) that if f is any positive continuous function on T^n, then

$$\int_{T^n} f(\xi) \, d\|\mu(N)(\xi)h\|_{\mathscr{K}}^2 \leqq \int_{T^n} f(\xi) \, d\|\mu(W)(\xi)h\|_{\mathscr{L}}^2. \tag{2}$$

It is clear from (2) that $\|(\mu(N)|T^n)(.)h\|_{\mathscr{K}}^2$ is absolutely continuous with respect to $\|\mu(W)(.)h\|_{\mathscr{L}}^2$ for any h in $\mathscr{H}$. Next appeal to Lemma 2 to deduce that $\|(\mu(N)|T^n)(.)h\|_{\mathscr{K}}^2$ is absolutely continuous with respect to m_n for any h in $\mathscr{H}$. The desired conclusion now follows by using the minimality of N.

Question. If S is a subnormal tuple of contractions on $\mathscr{H}$ and if there is no non-trivial closed subspace $\mathscr{H}'$ of $\mathscr{H}$ which is reducing for each S_i and on which each S_i is unitary, is it true that $\mu(N)|T^n$ is absolutely continuous with respect to m_n, where N is the minimal normal extension of S?

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