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A note on compact weighted composition operators on $L^p(\mu)$

JOR-TING CHAN

1. Introduction

Let (X, Σ, μ) be a complete σ-finite measure space and let T be a measurable transformation from X into itself, by which we mean $T^{-1}A\in\Sigma$ for every $A\in\Sigma$. Define the composition operator C_T by $C_T=f\circ T$ for every Σ-measurable function f on X. In order that functions which agree almost everywhere are mapped to functions with the same property, we require the measure $\mu\circ T^{-1}$ to be absolutely continuous with respect to μ. For a fixed Σ-measurable function Θ on X define the multiplication operator by $M_\Theta f=\Theta\cdot f$. The composite $M_\Theta\circ C_T$ is called a weighted composition operator. In this note we shall give a necessary and sufficient condition under which $M_\Theta\circ C_T$ is a compact operator on $L^p(\mu)$ $(1\leq p<\infty)$. The case $p=2$ has been investigated by SINGH and DHARMADHIKARI in [3] under the assumption that C_T is a bounded operator on $L^2(\mu)$ with dense range. But as pointed out by CAMPBELL and JAMISON [1], one of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator on $L^p(\mu)$. To quote their example, let T be the map $T(x)=x^2$ on $[0, 1]$. Then C_T does not define a mapping on $L^1[0, 1]$. However with $\Theta(x)\equiv x$, $M_\Theta\circ C_T$ is bounded operator on $L^1[0, 1]$. A necessary and sufficient condition on Θ and T under which $M_\Theta\circ C_T$ is a bounded operator on $L^p(\mu)$ is given in [2]. Before stating their result, some preliminaries are in order.

Let $T^{-1}\Sigma$ denote the relative completion of the σ-algebra generated by $\{T^{-1}A\colon A\in\Sigma\}$. If f is a non-negative Σ-measurable function on X, then there exists a unique (a.e.) $T^{-1}\Sigma$-measurable function $E(f)$, called the conditional expectation of f with respect to $T^{-1}\Sigma$, such that

$$\int_A f\,d\mu=\int_A E(f)\,d\mu \quad \text{for all} \quad A\in T^{-1}\Sigma.$$

We shall also need the following facts:

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(1) f is $T^{-1}\Sigma$-measurable if and only if $f\equiv g\circ T$ for some Σ-measurable function g.

(2) $E(f\cdot g\circ T)=E(f)\cdot g\circ T$ whenever the conditional expectations are well-defined.

In view of (1) above we write $E(f)\circ T^{-1}$ to denote a function g for which $E(f)\equiv g\circ T$.

Now let h be the Radon—Nikodym derivative $\dfrac{d\mu\circ T^{-1}}{d\mu}$. Then $M_\Theta\circ C_T$ is a bounded operator on $L^p(\mu)$ if and only if $h\cdot E(|\Theta|^p)\circ T^{-1}$ is μ-essentially bounded and in this case the operator norm of $M_\Theta\circ C_T$ equals $\|h\cdot E(|\Theta|^p)\circ T^{-1}\|_\infty^{1/p}$ [2, Proposition 2.1]. We shall include the proof for easy reference. For any $f\in L^p(\mu)$, we have

$$\int_X |\theta\cdot f\circ T|^p\,d\mu = \int_X |\theta|^p\cdot|f|^p\circ T\,d\mu = \int_X E(|\theta|^p\cdot|f|^p\circ T)\,d\mu =$$

$$= \int_X E(|\theta|^p)\cdot|f|^p\circ T\,d\mu = \int_X E(|\theta|^p)\circ T^{-1}\cdot|f|^p\,d\mu\circ T^{-1} =$$

$$= \int_X E(|\theta|^p)\circ T^{-1}\cdot|f|^p\cdot h\,d\mu = \int_X (h\cdot E(|\theta|^p)\circ T^{-1})|f|^p\,d\mu.$$

The assertion follows immediately from the equations. We also note that as far as the condition is concerned, $E(|\Theta|^p)\circ T^{-1}$ does not depend on any particular choice of g for which $E(|\Theta|^p)\equiv g\circ T$. For a thorough discussion of what appeared above, please consult [1] and [2].

2. The results

Theorem 2.1. *The weighted composition operator* $M_\Theta\circ C_T$ *is a compact operator on* $L^p(\mu)$ *if and only if for any* $\varepsilon>0$, *the set* $\{h\cdot E(|\Theta|^p)\circ T^{-1}\geqq\varepsilon\}$ *consists of finitely many atoms.*

Proof. ($\Rightarrow$) Assume the contrary. Then for some $\varepsilon>0$, the set

$$\{h\cdot E(|\Theta|^p)\circ T^{-1}\geqq\varepsilon\}$$

either contains a nonatomic subset or has infinitely many atoms. In both cases we can find a sequence of pairwise disjoint measurable subsets $\{A_n\}$ with $0<\mu(A_n)<\infty$ for every n. Define $f_n=\mu(A_n)^{-1/p}\chi_{A_n}$. Then $\|f_n\|=1$ and $\|M_\Theta\circ C_T f_n\|^p= =\mu(A_n)^{-1}\int_X h\cdot E(|\theta|^p)\circ T^{-1}\cdot\chi_{A_n}\,d\mu\geqq\varepsilon$. When $n\neq m$, the functions $M_\Theta\circ C_T f_n$ and

$M_\Theta \circ C_T f_m$ have disjoint supports and hence $\|M_\Theta \circ C_T f_n - M_\Theta \circ C_T f_n\|^p > 2\varepsilon$. Therefore $M_\Theta \circ C_T$ is not compact.

($\Leftarrow$) Let $\varepsilon > 0$ and let $A = \{h \cdot E(|\theta|^p) \circ T^{-1} \geqq \varepsilon\}$. Put $\Theta' = \Theta \chi_{T^{-1}A}$. Then under the hypothesis that A consists finitely many atoms, $M_{\Theta'} \circ C_T$ is a finite rank operator. For every $f \in L^p(\mu)$,

$$\|M_\theta \circ C_T(f) - M_{\theta'} \circ C_T(f)\|^p = \int_X |\theta \chi_{X \setminus T^{-1}A}|^p \cdot |f|^p \circ T \, d\mu =$$

$$= \int_X |\theta|^p \cdot (\chi_{X \setminus A} \circ T) \cdot (|f|^p \circ T) \, d\mu = \int_X h E(|\theta|^p) \circ T^{-1} \cdot |f|^p \chi_{X \setminus A} \, d\mu \leqq \varepsilon \|f\|^p.$$

So $M_\Theta \circ C_T$ is the limit of some finite rank operators and is therefore compact.

Corollary 2.2. *If X is nonatomic, then a weight composition operator is not compact unless it is the zero operator.*

In [3, Theorem 3.6] Singh and Dharmadhikari assert that if C_T is a bounded operator on $L^2(\mu)$ with dense range, then $M_\Theta \circ C_T$ is compact if and only if $\Theta \equiv 0$ on the set $\{h \circ T \neq 0\}$. A closer look at their proof reveals that the latter condition is equivalent to $|\Theta|^2 \cdot h \circ T \equiv 0$ a.e. But then $(M_\Theta \circ G) \circ (M_\Theta \circ G)^* \equiv M_{|\Theta|^2 \cdot h \circ T} \equiv 0$ gives $M_\Theta \circ C_T \equiv 0$.

If X is $\mathbf{N}$, the set of natural numbers and if μ is the counting measure on $\mathbf{N}$, we denote as usual $L^p(\mu)$ by l^p. Suppose that C_T does not define a bounded operator on l^p, then in contrast to [3, Theorem 3.3], the condition $\lim_{n \to \infty} \Theta(n) \equiv 0$ does not imply that $M_\Theta \circ G$ is compact.

Example 2.3. On l^1, consider the mapping

$$(x_1, x_2, x_3, \ldots) \mapsto \left(x_1, \frac{1}{2} x_2, \frac{1}{2} x_2, \frac{1}{3} x_3, \frac{1}{3} x_3, \frac{1}{3} x_3, \ldots\right).$$

This mapping can be realized as a weighted composition operator with $T(n) \equiv k$ and $\Theta(n) \equiv \frac{1}{k}$ whenever $\frac{(k-1)k}{2} < n \leqq \frac{k(k+1)}{2}$. A simple computation shows $h(n) \equiv n$ and $E(\Theta) \circ T^{-1}(n) \equiv \frac{1}{n}$. An appeal to Theorem 2.1 yields $M_\Theta \circ C_T$ is not compact. Actually this fact can be established by a direct argument. Let $\{e_n\}$ be the canonical basis in l^1. Then clearly $\{M_\Theta \circ C_T(e_n)\}$ does not have any norm-convergent subsequence.

References

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DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
KENT RIDGE
SINGAPORE 0511