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ϱ-dilations and hypo-Dirichlet algebras

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1. Introduction. Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let H be a complex Hilbert space and $L(H)$ the algebra of all bounded linear operators on H. $I=I_H$ is the identity operator in H. An algebra homomorphism $f\to T_f$ of A in $L(H)$, which satisfies

$$T_1 = I \quad \text{and} \quad \|T_f\| \leqq \|f\|$$

is called a representation of A on H. A representation $\varphi\to U_\varphi$ of $C(X)$ on a Hilbert space K is called a ϱ-dilation of the representation $f\to T_f$ of A if H is a Hilbert subspace of K and

$$T_f = \varrho P U_f|H \quad (f\in A_\tau)$$

where P is the orthogonal projection of K onto H, A_τ is the kernel of a nonzero complex homomorphism τ of A, and $0<\varrho<\infty$.

If the uniform closure of $A+\bar{A}$, that is, $[A+\bar{A}]$ has finite codimension in $C(X)$ then A is called a hypo-Dirichlet algebra and it is called a Dirichlet algebra when $[A+\bar{A}]=C(X)$. If A is a Dirichlet algebra on X and $f\to T_f$ a representation of A on H, then there exists a 1-dilation (cf. [7], [5]). It is known that only two hypo-Dirichlet (non-Dirichlet) algebras have 1-dilations [1], [9]. R. G. DOUGLAS and V. I. PAULSEN [4, Corollary 2.3] showed that an operator representation of a hypo-Dirichlet algebra is similar to an operator representation which has a 1-dilation.

In this paper, using their method we show that many natural hypo-Dirichlet algebras have ϱ-dilations. Then it follows that their representations are similar to those which have 1-dilations. A well known theorem of T. ANDO [2] shows that the

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bidisc algebra has 1-dilation. The theory of spectral sets is concerned with determing when a particular set Y in $\mathbf{C}$ is spectral for an operator T and if it is, deciding whether or not T possesses a normal 1-dilation whose spectrum is contained in ∂Y. Our main theorem shows that T possesses a normal ϱ-dilation. The other applications are related with ϱ-contractions (cf. [8]).

2. Main theorem. Let A be a hypo-Dirichlet algebra with $\dim C(X)/[A+\bar{A}]= =n<\infty$. Fix τ a non-zero complex homomorphism of A and let N_τ be the set of all representing measures of τ. Then $\dim N_\tau=n$ and there exists a core measure m of N_τ (cf. [6, p. 106]). Hence if $v\in N_\tau$ then $v=hdm$ and $h\in L^\infty(m)$ where $L^\infty(m)$ denotes the usual Lebesgue space. Thus N_τ can be considered as a subset of $L^\infty(m)$. In this paper we put a natural condition on N_τ: $N_\tau\subset C(X)$. Many important hypo-Dirichlet algebras satisfy it.

Theorem. *Let A be a hypo-Dirichlet algebra and let τ be a nonzero complex homomorphism with $N_\tau\subset C(X)$. Then a representation of A has a ϱ-dilation with respect to τ.*

Proof. Let $\{u_j\}_{j=1}^n$ be a normalized orthogonal basis in the real linear span of $(N_\tau-N_\tau)$ (r.l.s. of $(N_\tau-N_\tau)$) with respect to the inner product in $L^2(m)$ where m is a core measure of N_τ. Then for $1\leqq j\leqq n$

$$u_j = \sum_{l=1}^{n} \alpha_l^{(j)} (h_l^{(j)} - k_l^{(j)})$$

where each $\alpha_l^{(j)}$ is a real constant and $h_l^{(j)}, k_l^{(j)}\in N_\tau$. Put for $v\in C(X)$

$$\Phi(v) = v - \sum_{j=1}^{n} \Big(\int v u_j\, dm\Big) u_j + s(v)$$

where

$$s(v) = \sum_{j=1}^{n} \Big(\sum_{l=1}^{n} |\alpha_l^{(j)}| \int v (h_l^{(j)} + k_l^{(j)})\, dm\Big) \|u_j\|_\infty .$$

Then Φ is a positive map from $C(X)$ to $[A+\bar{A}]$, $\Phi(1)=1+s(1)$ and

$$s(1) = 2 \sum_{j=1}^{n} \Big(\sum_{l=1}^{n} |\alpha_l^{(j)}|\Big) \|u_j\|_\infty < \infty. \tag{1}$$

In fact, since $N_\tau\subset C(X)$,

$$[A+\bar{A}] \oplus [N_\tau - N_\tau] = C(X)$$

where $\oplus$ denotes the orthogonal direct sum of $L^2(m)$. Hence if $v\in C(X)$ then

$$v = v_1 + v_2$$

where $v_1 \in [A+\bar{A}]$ and $v_2 \in [N_\tau - N_\tau]$, consequently

$$\Phi(v) = v_1 + s(v)$$

and therefore $\Phi(v) \in [A+\bar{A}]$, and the positivity and the finiteness of $\Phi(1)$ are clear. If $f \in A_\tau$ then

$$\Phi(f) = f \tag{2}$$

because $s(f)=0$. This is different from Lemma 2.1 in [4].

If we extend T to $\tilde{T}$: $[A+\bar{A}] \to L(H)$ by $\tilde{T}_{f+\bar{g}} = T_f + T_g^*$, then $\tilde{T}$ is positive by [3, p. 152—153]. Thus $\Phi(1)^{-1}\tilde{T}\circ\Phi$: $C(X) \to L(H)$ is positive and $\Phi(1)^{-1}\tilde{T}\circ\Phi(1) = I_H$. By the dilation theorem of M. A. Naimark (cf. [10, Theorem 7.5]) there exists a Hilbert space K, an orthogonal projection P: $H \to K$ and a multiplicative linear map $\varphi \to U_\varphi$ of $C(X)$ in $L(K)$, which satisfies $U_1 = I_K$, $\|U_\varphi\| \leqq \|\varphi\|$, $\varphi \in C(X)$ and

$$\tilde{T}\circ\Phi(\varphi) = \Phi(1) P U_\varphi | H.$$

By (2), if $f \in A_\tau$,

$$T_f = \Phi(1) P U_f | H.$$

Corollary. *Suppose* $\dim N_\tau = 1$ *in Theorem, then*

$$\varrho = \inf\left\{\frac{2\|h-k\|_\infty}{\int |h-k|^2\, dm}: h, k \in N_\tau\right\} + 1.$$

Proof. By (1) in the proof of Theorem with $n=1$

$$\Phi(1) = 2|\alpha_1^{(1)}|\, \|u_1\|_\infty + 1$$

where

$$u_1 = \alpha_1^{(1)}(h_1^{(1)} - k_1^{(1)}), \quad h_1^{(1)}, k_1^{(1)} \in N_\tau$$

and

$$|\alpha_1^{(1)}|^2 \int |h_1^{(1)} - k_1^{(1)}|^2\, dm = 1.$$

This implies the corollary.

We concentrated in unital contractive homomorphism but our technique can be used for unital contractions.

3. Concrete examples. In this section we will calculate ϱ of ϱ-dilation in few concrete examples or apply Theorem to them.

(1) Let n be a positive integer and Y_i $(1 \leqq i \leqq n)$ disjoint compact subsets of $\mathbf{C}$ with non-empty interior Y_i^0. Suppose $R(Y_i)|X_i$ is a Dirichlet algebra on X_i where $R(Y_i)$ denotes the uniform closure of the set of the rational functions with poles off Y_i and X_i is the boundary of Y_i. Put $X = \bigcup_{i=1}^{n} X_i$ and $Y = \bigcup_{i=1}^{n} Y_i$, then X is the boundary of Y and $R(Y)|X$ is a Dirichlet algebra on X. Put

$$A = \{f \in R(Y)|X: f(x_i) = f(x_1) \text{ for } i > 1\}$$

where $x_i \in Y_i^0$ $(1 \leqq i \leqq n)$. A is a uniform algebra on X and if $n>1$ then A is not a Dirichlet algebra but a hypo-Dirichlet algebra.

A representation of A has a ϱ-dilation with $\varrho=n$.

Proof. Let $\tau(f)=f(x_1)$ then τ is a nonzero complex homomorphism. Put u_i be a characteristic function of X_i $(1 \leqq i \leqq n)$ and let D be the commutative C^*-algebra generated by $\{u_i \colon 1 \leqq i \leqq n\}$. Then $A_\tau D \subset A_\tau$, $A_\tau + \bar{A}_\tau + D$ is uniformly dense in $C(X)$ and $\dim D=n$. Let m_i be a harmonic measure of x_i $(1 \leqq i \leqq n)$ and $m=\sum_{i=1}^{n} m_i/n$ then m is a representing measure of τ. In the proof of Theorem, put

$$\Phi(v) = v - \sum_{j=1}^{n} \frac{1}{m(X_j)} \int_{X_j} v\, dm\, u_j + s(v) \quad (v \in C(X))$$

and

$$s(v) = \sum_{j=1}^{n} \frac{1}{m(X_j)} \int_{X_j} v\, dm.$$

Then Φ is a positive map from $C(X)$ to $[A+\bar{A}]$, and if $f \in A_\tau$ then $\Phi(f)=f$ and $\Phi(1)=n$. This can be shown as in the proof of Theorem because

$$[A_\tau + \bar{A}_\tau] \oplus D = C(X)$$

and $DA_\tau \subset A_\tau$. Thus a representation of A has a ϱ-dilation with $\varrho=\Phi(1)=n$.

If $\dim D=n$ then A is one kind of hypo-Dirichlet algebras of finite codimension $n-1$. By a theorem of R. G. Douglas and V. I. Paulsen [4] the completely bounded norm of the representation T of A, $\|T\|_{cb} \leqq 2n-1$ but our result implies $\|T\|_{cb} \leqq$ $\leqq n-1$.

(2) Let $\mathscr{A}$ be the disc algebra on the circle Γ and

$$A = \{f \in \mathscr{A} \colon f'(0) = \dots = f^{(n)}(0)\}$$

where $f^{(j)}(0)$ denotes the j-derivative at the origin. Then A is a hypo-Dirichlet algebra on $X=\Gamma$ and $\dim C(\Gamma)/[A+\bar{A}]=2n$.

A representation of A has a ϱ-dilation with $\varrho=8n+1$.

Proof. $d\theta/2\pi$ is the core measure of N_τ where $\tau(f)=f(0)$. Then

$$\text{r.l.s.}(N_\tau - N_\tau) = \text{r.l.s.}(\cos\theta,\ \cos 2\theta,\ \dots,\ \cos n\theta;\ \sin\theta,\ \sin 2\theta,\ \dots,\ \sin n\theta).$$

In the proof of Theorem, put for $v \in C(\Gamma)$

$$\Phi(v) = v - 2\sum_{j=1}^{n} \left\{ \left(\frac{1}{2\pi} \int v \sin j\theta\, d\theta \right) \sin j\theta + \left(\frac{1}{2\pi} \int v \cos j\theta\, d\theta \right) \cos j\theta \right\} + s(v)$$

and

$$s(v) = 2\sum_{j=1}^{n} \left\{ \frac{1}{2\pi} \int v(2-\sin j\theta)\, dm + \frac{1}{2\pi} \int v(2-\cos j\theta)\, d\theta \right\}.$$

Then Φ is a positive map from $C(\Gamma)$ to $[A+\bar{A}]$, if $f\in A_\tau$ then $\Phi(f)=f$ and $\Phi(1)=$ $=8n+1$. In fact, since

$$\Phi(v) = v+2\sum_{j=1}^{n}\left\{\left(\frac{1}{2\pi}\int v\,d\theta\right)(1-\sin j\theta)+\left(\frac{1}{2\pi}\int v(1-\sin j\theta)\,d\theta\right)(1+\sin j\theta)+\right.$$

$$\left.+\left(\frac{1}{2\pi}\int v\,d\theta\right)(1-\cos j\theta)+\left(\frac{1}{2\pi}\int v(1-\cos j\theta)\,d\theta\right)(1+\cos j\theta)\right\},$$

Φ is positive. The other statements are clear. Thus a representation of A has a ϱ-dilation with $\varrho=\Phi(1)=8n+1$.

(3) Let $a_1, \dots, a_n$ be distinct points in the open unit disc and

$$A = \{f\in\mathscr{A}: f(a_j) = f(0),\ j = 1, \dots, n\}.$$

Then A is a hypo-Dirichlet algebra on $X=\Gamma$ and $\dim C(\Gamma)/[A+\bar{A}]=2n$. $d\theta/2\pi$ is the core measure of N_τ where $\tau(f)=f(0)$. Then $N_\tau\subset C(\Gamma)$ and hence we can apply Theorem to this hypo-Dirichlet algebra.

(4) Let Y be a compact subset of $\mathbf{C}$ and let $R(Y)$ be the uniform closure of the set of rational functions in $C(Y)$. Suppose the complement Y^C of Y has a finite number n of components and Y^0 is a nonempty connected set. Let $A=R(Y)|X$ where X is the boundary of Y and τ a nonzero complex homomorphism defined by the evaluation at a point t in Y^0. Then A is a hypo-Dirichlet algebra on X and $\dim C(X)/[A+\bar{A}]=n$. If m is a harmonic measure for t then m is a core measure in N_τ and $N_\tau\subset C(X)$. Hence we apply Theorem to this hypo-Dirichlet algebra and hence a representation of A has a ϱ-dilation.

In the four examples we concentrated in unital contractive homomorphisms our technique can be used for unital contractions.

(5) Let

$$A = \{f\in\mathscr{A}: f(0) = f(1)\}.$$

Then A is a hypo-Dirichlet algebra on $X=\Gamma$ and $\dim C(\Gamma)/[A+\bar{A}]=1$. $(d\theta/2\pi+d\delta_1)/2$ is the core measure of N_τ where $\tau(f)=f(0)=f(1)$ and δ_1 is a dirac measure at 1. Then N_τ can not be embedded in $C(\Gamma)$ and hence we can not Theorem to this hypo-Dirichlet algebra. However the author [9] showed previously by the different method that a representation of A has a 1-dilation.

4. Normal ϱ-dilation. Results in this section are corollaries of Theorem and Examples (2)—(4).

Corollary 1. *If $T\in L(H)$ and*

$$\|f(T)\| \leq \sup_{|z|\leq 1}|f(z)|$$

for all analytic polynomials f with $f'(0)=\dots=f^{(n)}(0)$, *then there exists a Hilbert space* $K\supseteqq H$ *and a unitary operator U on K such that*

$$f(T) = (8n+1)Pf(U)|K$$

for all analytic polynomials with $f(0)=f'(0)=\dots=f^{(n)}(0)=0$, *where P is the orthogonal projection from K to H.*

Proof. Put $T_f=f(T)$ for each analytic polynomials f with $f'(0)=\dots=f^{(n)}(0)$, then $f\to T_f$ extends to a representation of A in Example 2. Thus the representation of A has a ϱ-dilation with $\varrho=8n+1$ and the corollary follows.

Corollary 2. *Let* $\{a_j\}_{j=1}^n$ *be in the open unit disc. If* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{|z|\leqq 1} |f(z)|$$

for all analytic polynomials f with $f(0)=f(a_1)=\dots=f(a_n)$, *then there exists a Hilbert space* $K\supseteqq H$ *and a unitary operator U on K such that*

$$f(T) = \varrho Pf(U)|K$$

for all analytic polynomials with $f(0)=f(a_1)=\dots=f(a_n)=0$, *where P is the orthogonal projection from K to H.*

Proof. It can be shown that this is a corollary of Example 3 as in the proof of Corollary 1.

Corollary 3. *Let Y be a compact subset of* **C** *in Example* 4. *If Y contains the spectrum* $\sigma(T)$ *of* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{z\in Y} |f(z)|$$

for all f in $R(Y)$ *then there exists a Hilbert space* $K\supseteqq H$ *and a normal operator N on K with* $\sigma(N)\subseteqq\partial Y$ *such that*

$$f(T) = \varrho Pf(N)|H$$

for all f in $R(Y)$ *with* $\tau(f)=0$, *where P is the orthogonal projection from K to H.*

Proof. It can be shown that this is a corollary of Example 4 as in the proof of Corollary 1.

J. Agler [1] proved $\varrho=1$ when $n=1$. R. G. Douglas and V. I. Paulsen [4] showed that there exists an invertible operator S on H such that $S^{-1}TS$ has a normal dilation.

Corollary 4. *If* $T\in L(H)$ *and*

$$\|f(T)\| \leqq \sup_{|z|\leqq 1} |f(z)|$$

for all analytic polynomials f with $f(0)=f(1)$, *then there exists a Hilbert space* $K \supseteq H$ *and a unitary operator U on K such that*

$$f(T) = Pf(U)|H$$

for all analytic polynomials f with $f(0)=f(1)$, *where P is the orthogonal projection from K to H. In particular, for all* $n \geqq 1$

$$T^n - PU^n H = T - PU|H.$$

Proof. It can be shown that this is a corollary of Example 5 as in the proof of Corollary 1.

In Corollary 4, T is a polynomially bounded operator. We could not answer the following question which is a special case of Problem 6 of Halmos: Is T similar to a contraction?

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