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$\varkappa$-products of slender modules

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Throughout, modules will be right unital over an arbitrary, but fixed ring R.

Let $\varkappa$ be an infinite cardinal, and H_j ($j\in J$) a set of R-modules. By their *$\varkappa$-product* $\Pi^{<\varkappa}\{H_j|j\in J\}$ is meant the submodule of the direct product $H=\Pi\{H_j|j\in J\}$ consisting of all the elements $h=(h_j)$ whose support supp $h=\{j\in J|h_j\neq 0\}$ has cardinality $<\varkappa$. (We shall write supp $K=\cup\{\text{supp } h|h\in K\}$ for a subset K of H.) An R-module A is called *slender* if for R-modules $e_iR\cong R$ ($i<\omega$) and for any R-homomorphism

$$\varphi\colon \Pi\{e_iR|i<\omega\}\to A$$

we have $\varphi e_i=0$ for almost all i. Slender modules behave in many respects like slender abelian groups; cf. Dimitrić [2]. Slender modules need not be torsion-free, not even over commutative domains [3, p. 77].

In this note, our purpose is to investigate properties of $\varkappa$-products of slender modules over arbitrary rings. We shall concentrate on the problem of homomorphisms η of a product $\Pi\{G_i|i\in I\}$ of R-modules G_i with non-measurable index set I into the $\varkappa$-product $\Pi^{<\varkappa}\{H_j|j\in J\}$ of slender modules H_j. A generalization of a well-known theorem by Łoś [3, p. 52] guarantees that, for each $j\in J$, only finitely many ηG_i have nonzero projections in H_j, but virtually nothing is known about the global behavior of such a homomorphism η.

We study both the kernel and the image of η. Easy examples show that meaningful results on the image of η can only be obtained if the modules G_i are not too large: more precisely, if they can be generated by fewer elements than the cofinality cof $\varkappa$ of $\varkappa$. We shall prove the following theorem which also generalizes the well-known result that direct sums of slender modules are slender [3, p. 77].

Theorem. *Let I be a non-measurable index set and $\varkappa$ an infinite cardinal. Assume G_i ($i\in I$) are R-modules each of which can be generated by strictly less than $\lambda=\text{cof}\,\varkappa$*

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elements, and H_j $(j\in J)$ *are slender R-modules. Given an R-homomorphism*

(1) $$\eta\colon G=\prod\{G_i\mid i\in I\}\to H=\prod^{<\varkappa}\{H_j\mid j\in J\},$$

define

(2) $$X=\{i\in I\mid \eta G_i\neq 0\}\quad\text{and}\quad Y=\bigcup\{\operatorname{supp}\eta G_i\mid i\in I\}.$$

Then we have:

(A) $|X|<\varkappa$;

(B) $\Pi\{G_i\mid i\in I\setminus X\}\leqq\ker\eta$;

(C) $|Y|<\varkappa$;

(D) $\operatorname{Im}\eta\leqq\Pi\{H_j\mid j\in Y\}\leqq H$.

For $\varkappa$ regular, conclusions (A) and (B) are actually proved without requiring the G_i to be less than $\operatorname{cof}\varkappa$ generated. The proof requires a more sophisticated argument if $\varkappa$ is a singular cardinal. We break down the proof into several lemmas and propositions dealing with portions of the Theorem.

For recent work on products of slender modules, and for applications of $\varkappa$-products in ring and module theory, see [3], [7], [8] and [1], as well as the literature quoted there.

1. Preliminaries

The symbols $\varkappa,\lambda$ will denote infinite cardinals (or ordinals); $\varkappa^+$ denotes the successor cardinal of $\varkappa$, and $|X|$ stands for the cardinality of a set X. For an R-module G, gen G means the minimum cardinality of generating sets of G. Here a cardinal $\varkappa$ is *measurable* if there exists a non-principal ultrafilter on $\varkappa$ which is closed under countable intersections. For the set theoretical concepts and results needed here, we refer to Jech [5; p. 27—28; p. 52].

By making use of [3, p. 52], the proof of the main theorem of Łoś on slenderness ([4; p. 161, Theorem 94.4]) can be modified so as to hold for slender R-modules, rather than for abelian groups.

1.1. J. Łoś Theorem. *Let I be a non-measurable index set and A a slender R-module. For any R-homomorphism* $\varphi\colon\Pi\{G_i|i\in I\}\to A$ *where the* G_i *are arbitrary R-modules we have:*

(i) $\varphi G_i=0$ *for almost all* $i\in I$; *and*

(ii) *if* $\varphi G_i=0$ *for all* $i\in I$, *then* $\varphi=0$.

It is straightforward to check that (1.1) continues to hold if the direct product ΠG_i is replaced by the $\varkappa$-product $\Pi^{<\varkappa}G_i$ with uncountable non-measurable $\varkappa$.

Using coordinate-wise arguments, we can at once derive the following corollary.

1.2. Corollary. *Let*

$$\eta: G = \Pi\{G_i \mid i \in I\} \to H = \Pi\{H_j \mid j \in J\} \tag{3}$$

be an R-homomorphism where the G_i are R-modules, all H_j are slender, and the index set I is non-measurable. If $\eta G_i = 0$ for each $i \in I$, then $\eta = 0$.

From now on we assume that the index sets are infinite.

In order to compare homomorphisms into products with those into $\varkappa$-products, we include the following result.

1.3. Proposition. *Assume the hypotheses of* (1.2). *Then the subset* $X = \{i \in I \mid \eta G_i \neq 0\}$ *of I satisfies:*

(i) $|X| \leqq |J|$, *and*

(ii) η *vanishes on* $\Pi\{G_i \mid i \in I \setminus X\}$.

Proof. Let $\varrho_j: H \to H_j$ be the jth coordinate projection. By the definition of slenderness, for each $j \in J$, the set

$$f(j) = \{i \in I \mid \varrho_j \eta G_i \neq 0\} \tag{4}$$

is finite. Evidently, $\cup\{f(j) \mid j \in J\} = X$ whence (i) is obvious. Since $\eta G_i = 0$ for all $i \in I \setminus X$, (ii) follows immediately from (1.2).

To facilitate proofs, we state here a lemma the proof of which is an easy exercise in set theory.

1.4. Lemma. *Let I be a set of infinite cardinality $\varkappa$ and $\bar{\varkappa}$ a cardinal $< \varkappa$. Suppose that $\{F_j \mid j \in J\}$ is a set of finite subsets of I such that, for each $i \in I$, the cardinality of $\{j \in J \mid i \in F_j\}$ does not exceed $\bar{\varkappa}$. Assume that $|J| = \varkappa$, which holds in particular if $\cup\{F_j \mid j \in J\} = I$. Then there is a subset $S \subset J$ such that*

(a) $|S| = \varkappa$;

(b) *the sets* F_j $(j \in S)$ *are pairwise disjoint.*

Note. If $\varkappa$ is regular, and in particular, weakly inaccessible, then it suffices to assume $|\{j \in J \mid i \in F_j\}| < \varkappa$ for each $i \in I$ to obtain (a)—(b).

2. Maps into $\varkappa$-products, regular $\varkappa$

In this section, we assume $\varkappa$ is a regular cardinal. Our first concern is the kernel of homomorphisms of products of modules into the $\varkappa$-products of slender modules. The following theorem gives fairly complete information about the kernel. (The restriction on gen G_i is not required for regular cardinals $\varkappa$.)

2.1. Proposition. *Let G_i $(i\in I)$ be R-modules, H_j $(j\in J)$ slender R-modules, and $|I|$ a non-measurable cardinal. Let* (1) *be an R-homomorphism where $\varkappa$ is a regular cardinal. Then (A) and (B) of Theorem hold.*

Proof. As (B) is a consequence of (1.2), only (A) requires a verification.

By way of contradiction, suppose that (A) is false. Without loss of generality, we may then assume that $X=I$ has cardinality $\varkappa$ (in particular, $\varkappa$ is non-measurable) and $G_i=g_iR$ are nonzero cyclic R-modules.

As in the proof of (1.3), we form the sets $f(j)$ [cf. (4)] which are finite for each $j\in J$. Setting $Y=\bigcup\{\text{supp}\,\eta g_i|i\in I\}$, we have $I=\bigcup\{f(j)|j\in Y\}$ because of $X=I$. The finiteness of the $f(j)$ and $|I|=\varkappa$ imply $|Y|=\varkappa$. Since $\varrho_j\eta g_i=0$ for $j\in J\setminus Y$ and every $i\in I$, we may assume $Y=J$ and $f(j)\neq\emptyset$ for each $j\in J$.

The next step in our proof is to select a subset S of J such that the finite subsets $f(j)$ $(j\in S)$ are pairwise disjoint and $|S|=\varkappa$. This can be done with the aid of (1.4) (where $\bar{\varkappa}$ is the immediate predecessor of $\varkappa$ if such an ordinal exists; otherwise no such $\bar{\varkappa}$ is needed).

For each $j\in S$, set $C_j=\oplus\{G_i|i\in f(j)\}\neq 0$. Manifestly, $G^*=\Pi\{C_j|j\in S\}$ is a summand of G and $H^*=\Pi^{<\varkappa}\{H_j|j\in S\}$ is a summand of H. The restriction of η to G^* followed by the projection $H\to H^*$ yields a map $\eta^*\colon G^*\to H^*$ such that, for each $j\in S$, $0\neq\eta C_j\leqq H_j$. For every $j\in S$, pick a $c_j\in C_j$ satisfying $\eta^*c_j\neq 0$, and let

$$c=(\dots,c_j,\dots)\in G^*\quad(j\in S).$$

In view of $\varrho_j\eta^*C_k=0$ for all $j\neq k$ in S, the slenderness of H_j implies $\varrho_j\eta^*(c-c_j)=0$ for every $j\in S$ (recall the non-measurability of $\varkappa$). Consequently, $\varrho_j\eta^*c=\varrho_j\eta^*c_j\neq 0$ for all $j\in S$, contradicting the fact that the support of η^*c must have cardinality $<\varkappa$.

We turn our attention to the question as to when the image of η in (1) has to be contained in the $\varkappa$-product of a smaller subset of the H_j.

It is readily seen that some sort of restriction on the G_i is necessary in order to obtain such a conclusion. In fact, if one of the G_i's is the direct sum $\bigoplus_{j\in J}H_j$ and η acts on each H_j as the identity map, then the $\varkappa$-product of the H_j over the entire index set J is needed to accomodate $\text{Im}\,\eta$. This example also shows that it won't be of any help to assume the slenderness of the G_i's. Cardinality restrictions on the G_i seem to be inevitable.

Accordingly, let us assume gen $G_i<\varkappa$ for each $i\in I$. If $\varkappa$ is a regular cardinal, then $|\text{supp}\,\eta G_i|<\varkappa$. Keeping this in mind, we prove:

2.2. Proposition. *Let G_i $(i\in I)$ be R-modules with $\text{gen}\,G_i<\varkappa$ where I is non-measurable. If* (1) *is an R-homomorphism with H_j slender and $\varkappa$ regular, then both* (C) *and* (D) *of Theorem hold true.*

Proof. (2.1) shows that η acts non-trivially only on a subproduct $\Pi\{G_i|i\in X\}$ where $|X|<\varkappa$. By the regularity of $\varkappa$, likewise $Y=\bigcup\{\operatorname{supp}\eta G_i|i\in X\}$ has cardinality less than $\varkappa$. Assertion (D) is an obvious consequence of (C).

3. Maps into $\varkappa$-products, singular $\varkappa$

In this section, $\varkappa$ denotes a singular cardinal.

Let us start with a weak version of (2.1). Viewing (3) as a map into the $\varkappa^+$-product of the H_j, we derive:

3.1. Corollary. *Under the hypotheses of* (2.1), *but assuming $\varkappa$ is singular, we have:*

(i') $|X|\leqq\varkappa$, *and*

(ii') η *vanishes on* $\Pi\{G_i|i\in I\setminus X\}$.

We shall improve on (3.1) by limiting the sizes of the generating systems of the G_i.

The analogue of (2.2) fails if $\varkappa$ is singular, even if I is restricted to have cardinality $\lambda=\operatorname{cof}\varkappa$ — as is shown by the following example.

Let $\{J_\alpha|\alpha<\lambda\}$ be a set of pairwise disjoint subsets of J such that $|J_\alpha|<\varkappa$ for all $\alpha<\lambda$ and $\sup|J_\alpha|=\varkappa$. Let $G_\alpha=\oplus\{H_j|j\in J_\alpha\}$, $G=\Pi\{G_\alpha|\alpha<\lambda\}$ and $\eta\colon G\to H==\Pi^{<\varkappa}\{H_j|j\in J\}$ be induced by the identity maps on the H_j. This η exists (Im η is already in the λ^+-product of the H_j) and provides a counterexample.

Our best bet is cutting down the sizes of G_i to below λ. This enables us to obtain reasonably strong results. The point of departure is the following.

3.2. Lemma. *Let $\varkappa$ be a singular cardinal, I a non-measurable index set of cardinality $\leqq\lambda=\operatorname{cof}\varkappa$, and H_j $(j\in J)$ a family of slender modules. If $\operatorname{gen}G_i<\lambda$ for all the R-modules G_i, then for any R-homomorphism* (1) *conclusions* (C) *and* (D) *of Theorem hold true.*

Before entering into the proof of (3.2), we prove two auxiliary lemmas. In the next lemma, $\varkappa$ can be any infinite cardinal.

3.3. Lemma. *Suppose μ is a non-measurable ordinal $<\varkappa$ and*

$$\eta\colon G=\Pi\{G_\alpha|\alpha<\mu\}\to H=\Pi^{<\varkappa}\{H_j|j\in J\}$$

is an R-homomorphism where $G_\alpha=g_\alpha R$ are non-zero cyclic modules and H_j are slender. If $\varkappa_0$ is a cardinal number satisfying

$$\mu\leqq\varkappa_0<|\operatorname{supp}\eta g_0|,$$

then there exist a subset Y of J and an ordinal $\beta<\mu$ such that

(a) $Y \subset \operatorname{supp} \eta g_0$;
(b) $|Y| > \varkappa_0$;
(c) $Y \cap \operatorname{supp} \eta g_\alpha = \emptyset$ *for all* $\alpha \geqq \beta$.

Proof. Let ϱ_j denote the jth coordinate projection $H \to H_j$. Define a function $\psi\colon J \to \mu$ by letting $\psi(j)$ be the smallest ordinal $\gamma < \mu$ such that

$$\varrho_j \eta \Pi \{g_\alpha R | \gamma \leqq \alpha < \mu\} = 0.$$

Owing to the slenderness of H_j, such a $\psi(j)$ exists, so ψ is well-defined. For $\alpha < \mu$, we set

$$(5) \qquad Y_\alpha = \{j \in \operatorname{supp} \eta g_0 \,|\, \psi(j) = \alpha\}.$$

Visibly, the Y_α are pairwise disjoint and their union for $\alpha < \mu$ is exactly $\operatorname{supp} \eta g_0$. Consequently,

$$\varkappa_0 < |\operatorname{supp} \eta g_0| = \sum_{\alpha < \mu} |Y_\alpha| = \max \{\mu, \sup_{\alpha < \mu} |Y_\alpha|\}.$$

Hence $\mu \leqq \varkappa_0$ implies $\varkappa_0 < \sup |Y_\alpha|$ which means that $|Y_\beta| > \varkappa_0$ for a suitable ordinal $\beta < \mu$. This β and $Y = Y_\beta$ are as desired.

The next lemma is more technical.

3.4. Lemma. *Assume $\varkappa$ is a singular cardinal, $\lambda = \operatorname{cof} \varkappa$ is non-measurable, G_α are non-zero cyclic and H_j are slender. If there are cardinals $\varkappa_\alpha (\alpha < \lambda)$ satisfying*
(a) $\varkappa_\alpha < |\operatorname{supp} \eta g_\alpha|$ *for* $\alpha < \lambda$;
(b) $\lambda \leqq \varkappa_0 < \varkappa_1 < \ldots < \varkappa_\alpha < \ldots \quad (\alpha < \lambda)$;
(c) $\sup \varkappa_\alpha = \varkappa$,
then there exist subsets J_α of J and ordinals $\mu(\alpha) < \lambda$ for all $\alpha < \lambda$ such that
(i) $\mu(0) < \mu(1) < \ldots < \mu(\alpha) < \ldots$;
(ii) $J_\alpha \subset \operatorname{supp} \eta g_{\mu(\alpha)}$;
(iii) $|J_\alpha| > \varkappa_\alpha$ *for* $\alpha < \lambda$;
(iv) *the sets $J_0, J_1, \ldots, J_\alpha, \ldots$ are pairwise disjoint.*

Proof. We take advantage of the function $\psi\colon J \to \lambda$ defined in the proof of (3.3), and in addition to (i)—(iv) we also require that J_α be of the form Y_β as in (5). More precisely, we impose an additional condition:
(v) $J_\alpha = \{j \in \operatorname{supp} \eta g_{\mu(\alpha)} | \psi(j) = \beta(\alpha)$ for some $\beta(\alpha) < \lambda\}$.
Right away we note that (v) implies
(vi) $\varrho_j \eta g_\delta = 0$ holds for every $j \in J_0 \cup \ldots \cup J_\alpha$ and for every $\delta > \operatorname{supp} \{\mu(\alpha), \beta(0), \ldots \ldots, \beta(\alpha)\}$.

The J_α and $\mu(\alpha)$ will be constructed by transfinite induction. To start off, put $\mu(0) = 0$. Application of (3.3) yields a subset $Y \subset J$ and an ordinal $\beta(0) < \lambda$ satisfying (a)—(c) of (3.3) as well as (v). Define $J_0 = Y_{\beta(0)}$. Then for $\alpha = 0$, all of (i)—(v) hold.

Let $\gamma<\lambda$, and suppose that $\mu(\alpha)<\lambda$ and $J_\alpha\subset J$ have been selected for all $\alpha<\gamma$ satisfying conditions (i)—(v) for indices $<\gamma$. Define $\mu(\gamma)$ to be any ordinal $<\lambda$ exceeding $\mu(\alpha)$ and $\beta(\alpha)$ for all $\alpha<\gamma$; since $\lambda=\operatorname{cof}\varkappa$, such a $\mu(\gamma)$ does exist. Apply (3.3) to $g_{\mu(\gamma)}$ playing the role af g_0 and $\varkappa_\gamma$ the role of $\varkappa_0$, in order to obtain a set $Y=\{j\in\operatorname{supp}\eta g_{\mu(\gamma)}|\psi(j)=\delta\}$ for some $\delta<\lambda$ and an ordinal $\beta(\gamma)$ as stipulated by (3.3) (a)—(c). Setting $J_\gamma=Y$ and $\beta(\gamma)=\delta$ (see the proof of (3.3)), conditions (i)—(iii) and (v) will clearly hold for all indices $\leqq\gamma$. To convince ourselves that $J_\gamma\cap J_\alpha=\emptyset$ for every $\alpha<\gamma$, it suffices to note that (vi) implies $\varrho_j\eta e_{\mu(\gamma)}=0$ for every $j\in J_\alpha$. This completes the proof of (3.4).

Proof of (3.2). Since $\operatorname{gen} G_i<\lambda$ implies $|\operatorname{supp}\eta G_i|<\varkappa$, the assertion follows at once whenever $|I|<\lambda$. So let us assume $|I|=\lambda$ in which case we can think of I as consisting of the ordinals $<\lambda$.

From (1.2) we infer that $Y=\bigcup\{\operatorname{supp}\eta G_\alpha|\alpha<\lambda\}$ is the smallest subset of J with the property $\eta G\leqq\Pi^{<\varkappa}\{H_j|j\in Y\}$. Hence $|Y|\leqq\varkappa$ is immediate. By way of contradiction assume that $|Y|=\varkappa$.

Passing to a summand of G, we may assume that the cardinal numbers $|\operatorname{supp}\eta G_\alpha|$ are all different and $>\lambda$. Reindexing, we obtain an ascending chain

$$\lambda<|\operatorname{supp}\eta G_0|<|\operatorname{supp}\eta G_1|<\ldots<|\operatorname{supp}\eta G_\alpha|<\ldots\quad(\alpha<\lambda)$$

whose supremum is $\varkappa$. Since $\operatorname{gen} G_{\alpha+1}<\lambda$ and $|\operatorname{supp}\eta G_{\alpha+1}|>|\operatorname{sup}\eta G_\alpha|$, there must be a generator $g_\alpha\in G_{\alpha+1}$ whose image ηg_α has support of cardinality strictly $>|\operatorname{supp}\eta G_\alpha|$. Setting $\varkappa_\alpha=|\operatorname{supp}\eta G_\alpha|$, we obtain an ascending chain of cardinals,

$$\lambda=\varkappa_0<\varkappa_1<\ldots<\varkappa_\alpha<\ldots\quad(\alpha<\lambda)$$

with $\sup\varkappa_\alpha=\varkappa$, and with $\varkappa_\alpha<|\operatorname{supp}\eta g_\alpha|$ for all $\alpha<\lambda$.

Restricting η to the submodule $\bar{G}=\Pi\{g_\alpha R|a<\lambda\}$, (3.4) yields the existence of subsets $J_\alpha\subset J$ and ordinals $\mu(\alpha)<\lambda$ satisfying (i)—(iv) of (3.4). Define an element $\bar{g}=(\bar{g}_\alpha)_{\alpha<\lambda}\in\bar{G}$ as follows. Let $\bar{g}_\alpha=g_{\mu(\beta)}$ if $\alpha=\mu(\beta)$ for some $\beta<\lambda$, and let $\bar{g}_\alpha=0$ otherwise. From the definition of the $\mu(\beta)$ it is clear that $\varrho_j\eta(\bar{g}-g_{\mu(\alpha)})=0$ for all $j\in J_\alpha$. Hence $\varrho_j\eta\bar{g}=\varrho_j\eta g_{\mu(\beta)}\neq0$ for $j\in J_\alpha$, and we conclude that

$$|\operatorname{supp}\eta\bar{g}|\geqq\bigcup\{J_\alpha|\alpha<\lambda\}=\sup\varkappa_\alpha=\varkappa.$$

This contradiction completes the proof of (3.2).

We still need the following lemma.

3.5. Lemma. *Let $\varkappa$ be a non-measurable singular cardinal, and I an index set of cardinality $\varkappa$. If G_i are R-modules with* $\operatorname{gen} G_i<\lambda=\operatorname{cof}\varkappa$, *and if* (1) *is an R-homomorphism with H_j slender, then there exists a cardinal $\bar{\varkappa}<\varkappa$ such that, for each $i\in I$, the set* $\operatorname{supp}\eta G_i$ *has cardinality $\leqq\bar{\varkappa}$.*

Proof. First observe that $\text{gen}\, G_i < \lambda$ and $|\text{supp}\, \eta g_i| < \varkappa$ for each $g_i \in G_i$ imply that $|\text{supp}\, \eta G_i| < \varkappa$. Denying the existence of a $\bar{\varkappa}$ of the indicated type means that we can select a subset $\{G_\alpha | \alpha < \lambda\}$ of $\{G_i | i \in I\}$ such that the cardinalities $\varkappa_\alpha = |\text{supp}\, \eta G_\alpha|$ form an increasing chain (with increasing α) whose supremum is $\varkappa$. The restriction of η to $G' = \Pi\{G_\alpha | \alpha < \lambda\}$ is a homomorphism satisfying the hypotheses of (3.2). Therefore, we conclude that $\eta G'$ has a support of cardinality $< \varkappa$, in contradiction to $\bigcup |\text{supp}\, \eta G_\alpha| = \bigcup \varkappa_\alpha = \varkappa$.

Proof of Theorem. (2.1) and (2.2) take care of the case in which $\varkappa$ is a regular cardinal. So assume $\varkappa$ is singular.

(A) Without loss of generality, we may assume $|I| \geqq \varkappa$; otherwise there is nothing to prove. It suffices to verify (A) for $|I| = \varkappa$. By way of contradiction, assume $|X| = \varkappa$. We apply (1.4) to the set $\{f(j) | j \in J\}$ defined in (2) to obtain a subset $S \subset J$ of cardinality $\varkappa$ with $f(j)$ $(j \in S)$ pairwise disjoint; (3.5) assures the existence of a cardinal $\bar{\varkappa}$ needed in (1.4). Consider the following element of G: $\bar{g} = (g_i)$ where $g_i \in G_i$ with $\eta g_i \neq 0$ if $i \in \bigcup\{f(j) | j \in S\}$ and $g_i = 0$ otherwise. An argument similar to the one used at the end of the proof of (2.1) leads us to the conclusion that $\eta \bar{g}$ must have a support of cardinality $\varkappa$ — a contradiction.

(B) follows from (A) in view of (1.2).

(C) Because of (3.5), we have $|\text{supp}\, \eta G_i| \leqq \bar{\varkappa} < \varkappa$ for each i. This, together with (A), implies $|Y| < \varkappa$.

(D) is an immediate consequence of (C).

4. Embedding of μ-products in $\varkappa$-products

The case when the map η in (1) is a monomorphism deserves particular attention. In the following two corollaries, no restriction on gen G_i is needed.

4.1. Corollary. *Let $G_i \neq 0$ $(i \in I)$ and H_j $(j \in J)$ be R-modules, $|I|$ and $\varkappa$ non-measurable cardinals. If the H_j are slender and if, for some cardinal μ, there is a monomorphism*

$$\eta\colon G = \Pi^{<\mu}\{G_i | i \in I\} \hookrightarrow H = \Pi^{<\varkappa}\{H_j | j \in J\},$$

then either $|I| < \varkappa$ or $\mu \leqq \varkappa$.

Proof. If $|I| \geqq \varkappa$ and $\mu > \varkappa$, then G contains a submodule which is the product of $\varkappa$ cyclic submodules $g_i R$ with $\eta g_i \neq 0$. This is impossible in view of (A), (B) in Theorem.

The next result is an immediate consequence of the preceding one. It generalizes a result on products and direct sums of slender groups, due to Łoś [6, p. 271].

4.2. Corollary. *Let both* G_i $(i\in I)$ *and* H_j $(j\in J)$ *be families of non-zero slender modules. If* $|I|$, $|J|$ *are non-measurable cardinals, and if* $\varkappa\leqq|J|$, $\mu\leqq|I|$, *then* $\Pi^{<\varkappa}\{G_i|i\in I\}\cong\Pi^{<\varkappa}\{H_j|j\in J\}$ *implies* $\varkappa=\mu$.

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