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Central pattern functions

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To Professor Béla Csákány on his 60th birthday

A finite algebra $\mathfrak{A}$ with base set A is called *functionally complete* if every (finitary) operation on A is an algebraic function of $\mathfrak{A}$ (in GRÄTZER's sense [3]). WERNER [8] proved that every finite algebra $\langle A; t\rangle$ where t is the ternary discriminator function on A is functionally complete. FRIED and PIXLEY [2] showed that (in the case $|A|>2$) the algebra $\langle A; d\rangle$ with d the dual discriminator function on A is also functionally complete. The ternary discriminator and the dual discriminator are the most familiar examples of *pattern functions*. B. CSÁKÁNY [1] proved that for $|A|>2$ every finite algebra $\langle A; f\rangle$ where f is a non-trivial pattern function on A is functionally complete. B. Csákány suggested the following generalization of pattern function (see [6]). Consider an n-ary relation $\varrho\subseteqq(A^n)$ on A. Two k-tuples $\langle x_1, \ldots, x_k\rangle, \langle y_1, \ldots, y_k\rangle\in A^k$ are of the same pattern with respect to ϱ if for $i_1, \ldots, i_n\in\{1, \ldots, k\}$, $\langle x_{i_1}, \ldots, x_{i_n}\rangle\in\varrho$ and $\langle y_{i_1}, \ldots, y_{i_n}\rangle\in\varrho$ mutually imply each other. An operation f: $A^k\to A$ is a *ϱ-pattern function* if $f(x_1, \ldots, x_k)$ always equals some x_i, $i\in\{1, \ldots, k\}$, where i depends on the ϱ-pattern of $\langle x_1, \ldots, x_k\rangle$ only. The ϱ-pattern functions with ϱ the equality relation are the (usual) pattern functions.

The aim of this paper is to prove a functional completeness theorem on ϱ-pattern functions with ϱ central, which is analogous to the theorems mentioned above.

An n-ary relation ϱ on A is called *central* [5], if $\varrho\neq A^n$ and there exists a nonvoid proper subset C of A such that

(1) $\langle a_1, \ldots, a_n\rangle\in\varrho$ whenever at least one $a_j\in C$ $(1\leqq j\leqq n)$;

(2) $\langle a_1, \ldots, a_n\rangle\in\varrho$ implies $\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)}\rangle\in\varrho$ for every permutation σ of the indices $1, \ldots, n$;

(3) $\langle a_1, \ldots, a_n\rangle\in\varrho$ if $a_i=a_j$ for some $i\neq j$ $(1\leqq i, j\leqq n)$. Note that every unary relation C distinct from $\emptyset$ and A is central.

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Let ε be an equivalence and ϱ an arbitrary n-ary relation on A. If for $a_1, \ldots, a_n$ $b_1, \ldots, b_n \in A$, $(a_1, \ldots, a_n) \in \varrho$ and $(a_1, b_1) \in \varepsilon, \ldots, (a_n, b_n) \in \varepsilon$ together imply $(b_1, \ldots$ $\ldots, b_n) \in \varrho$, then ε is said to be *compatible* with ϱ. We say that ϱ is *simple*, if no non-trivial equivalence on A is compatible with ϱ. An operation f on A is said to *preserve* ϱ if ϱ is a subalgebra of the nth direct power of the algebra $\langle A; f \rangle$.

We will use the following version of ROSENBERG's completeness theorem (see [5]).

A finite algebra $\langle A; f \rangle$ with a single fundamental operation f is functionally complete iff

(a) f is a monotonic with respect to no bounded partial order on A,

(*b*) f preserves no non-trivial equivalence on A,

(c) f preserves no binary central relation on A,

(d) f is surjective and essentially at least binary,

(e) f preserves no quaternary relation.

$\theta = \{\langle a_0, a_1, a_2, a_3 \rangle \in A^4 | a_0 + a_1 = a_2 + a_3\}$ where $\langle A; + \rangle$ is an elementary abelian p-group (p is prime number).

Let A be a finite set. For $k \geqq 2$ and for arbitrary $(k-1)$-ary, resp. l-ary $(1 \leqq \leqq l \leqq k-1)$ relations τ and θ on A we define the k-ary τ-pattern functions f_k^τ, g_k^τ resp. the l-ary θ-pattern functions h_k^θ on A as follows

$$f_k^\tau(x_1, \ldots, x_k) = \begin{cases} x_k, & \text{if } (x_1, \ldots, x_{k-1}) \in \tau \\ x_1 & \text{otherwise,} \end{cases}$$

$$g_k^\tau(x_1, \ldots, x_k) = \begin{cases} x_1, & \text{if } (x_1, \ldots, x_{k-1}) \in \tau \\ x_k & \text{otherwise,} \end{cases}$$

$$h_k^\theta(x_1, \ldots, x_k) = \begin{cases} x_k, & \text{if } (x_{i_1}, \ldots, x_{i_l}) \in \theta \quad \text{for some} \quad 1 \leqq i_1 < \ldots < i_l \leqq k, \\ x_1 & \text{otherwise.} \end{cases}$$

If τ and θ are the equality relation on A, then f_3^τ is the ternary discriminator, g_3^τ is the dual discriminator and h_k^θ is a near projection.

Theorem. *Let τ and θ be arbitrary central relations on an at least three element finite set A. The algebras $\langle A; f \rangle$ with $f = f_k^\tau$ or g_k^τ are functionally complete if and only if τ is simple. The algebras $\langle A; h_k^\theta \rangle$ are not functionally complete.*

Remark 1. If $|A| = 2$, then τ and θ are unary. In this case f_k^τ and g_k^τ are monotone on $A (= \{0, 1\})$, and h_k^θ is a projection; therefore $\langle A; f \rangle$ with $f = f_k^\tau, g_k^\tau$, or h_k^θ is not functionally complete.

For the proof of Theorem 1 we need the following lemma.

Lemma. *Let τ be a relation and f an arbitrary τ-pattern function on A. If τ is not simple, then $\langle A; f \rangle$ is not functionally complete.*

Proof. If τ is not simple, then there exists an nontrivial equivalence ε on A which is compatible with τ. Clearly, ε is a congruence of $\langle A;f\rangle$. Hence $\langle A;f\rangle$ is not functionally complete.

Remark 2. If an at least binary arbitrary central relation τ on A has at least two central elements, then τ is not simple. In this case Lemma implies that, for an arbitrary τ-pattern function f, the algebra $\langle A;f\rangle$ is not functionally complete.

Proof of Theorem. First we prove that the algebras $\langle A; h_k^\theta\rangle$ are not functionally complete. If the centre of θ has at least two elements, this follows from Remark 2. If the centre of θ consists of a single element c, then the equivalence of A with blocks $\{c\}$ and $A\setminus\{c\}$ is an non-trivial congruence of $\langle A; h_k^\theta\rangle$. Therefore $\langle A; h_k^\theta\rangle$ is not functionally complete.

It remains to show that the algebras $\langle A;f\rangle$ with $f=f_k^\tau$ or g_k^τ and τ simple are functionally complete. Rosenberg's criterion will be used. Clearly, (d) is true for f_k^τ and g_k^τ. Furthermore, they depend on all of their variables and $f_k^\tau(x_1, \dots, x_k)$, $g_k^\tau(x_1, \dots, x_k)\in\{x_1, \dots, x_k\}$ for $x_1, \dots, x_k\in A$. Then, by Lemma 1 in [7], (e) also holds for them. Thus it is enough to prove that neither f_k^τ nor g_k^τ does preserve the relations ϱ in (a), (b), (c). Therefore we have to present a $k\times 2$ matrix with entries in A such that all rows belong to ϱ, but the row of column values does not belong to ϱ.

(a) Let $\leqq$ be a bounded partial order on A with least element 0 and greatest element 1 $(0,1\in A)$. In view of Remark 2, we can suppose that c is a unique central element of τ. We will use the following matrices to show that none of the functions f_k^τ, g_k^τ does preserve $\leqq$

$$
\begin{array}{cc}
h & h\\
t_1 & 1\\
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
t_{k-2} & 1\\
0 & 0\\
\hline
h & 0
\end{array}
\quad
\begin{array}{cc}
t_1 & 1\\
0 & h\\
t_2 & t_2\\
\cdot & \cdot\\
\cdot & \cdot\\
t_{k-2} & t_{k-2}\\
0 & 0\\
\hline
t_1 & 0
\end{array}
\quad
\begin{array}{cc}
0 & h\\
t_1 & t_1\\
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
t_{k-2} & t_{k-2}\\
1 & 1\\
\hline
1 & h
\end{array}
\quad
\begin{array}{cc}
h & h\\
t_1 & 1\\
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
t_{k-2} & 1\\
1 & h\\
\hline
1 & h
\end{array}
\quad
\begin{array}{cc}
0 & h\\
t_1 & 1\\
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
t_{k-2} & 1\\
1 & 1\\
\hline
1 & h
\end{array}
\quad
\begin{array}{cc}
1 & 1\\
0 & t_1\\
\cdot & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
0 & t_{k-2}\\
h & h\\
\hline
1 & h
\end{array}
\;.
$$

Let h always denote an element of A distinct from 0 and 1. Consider the operation f_k^τ, and first suppose $c=1$. Since h is not a central, there exist $t_1, \dots, t_{k-2}(\in A)$ for which $(h, t_1, \dots, t_{k-2})\notin\tau$. Then the first matrix shows that f_k^τ does not preserve $\leqq$. Next suppose $c=h$. Since 0 is not central, there exist $t_1, \dots, t_{k-2}(\in A)$ for which $(t_1, 0, t_2, \dots, t_{k-2})\notin\tau$, and the second matrix applies. Finally, if $c=0$, then h is not a central, and there exist $t_1, \dots, t_{k-2}(\in A)$ with $(h, t_1, \dots, t_{k-2})\notin\tau$, and now the third matrix does the job. Now consider the operation g_k^τ, and first suppose $c=1$.

Since h is not central, there exist $t_1, \ldots, t_{k-2}(\in A)$ with $(h, t_1, \ldots, t_{k-2})\notin\tau$. Then the fourth matrix shows that g_k^τ does not preserve $\leqq$. If $c=h$, then 0 is not central, and there exist $t_1, \ldots, t_{k-2}(\in A)$ with $(0, t_1, \ldots, t_{k-2})\notin\tau$, and the fifth matrix is used. Finally, suppose $c=0$, then 1 is not central, and there exist $t_1, \ldots, t_{k-2}(\in A)$ with $(1, t_1, \ldots, t_{k-2})\notin\tau$. In this case using the sixth matrix we also get that g_k^τ does not preserve $\leqq$.

(b) Let ε be an arbitrary non-trivial equivalence on A. We prove that the operations f_k^τ and g_k do not preserve ε. Since τ is simple, there exist elements $a_1, \ldots, a_{k-1}$, $b_1, \ldots, b_{k-1}(\in A)$ with $(a_1, \ldots, a_{k-1})\in\tau, (a_1, b_1)\in\varepsilon, \ldots, (a_{k-1}, b_{k-1})\in\varepsilon, (b_1, \ldots, b_{k-1})\notin\tau$. Let $(t, b_1)\notin\varepsilon$, then $(a_1, t)\notin\varepsilon$ holds as well, and the matrix

$$\begin{array}{cc} a_1 & b_1 \\ \vdots & \vdots \\ a_{k-1} & b_{k-1} \\ t & t \\ \hline t & b_1 \\ a_1 & t \end{array}$$

shows that none of f_k^τ and g_k^τ do not preserve ε.

(c) Let ϱ be a binary central relation with centre C_ϱ. Let c be a unique central element of τ. To show that f_k^τ and g_k^τ do not preserve ϱ we use the following matrices

$$\begin{array}{cc} b & b \\ t_1 & c \\ \vdots & \vdots \\ t_{k-2} & c \\ a & a \\ \hline b & a \\ \text{or} & \\ a & b \end{array} \qquad \begin{array}{cc} d & d \\ t_1 & l \\ c & c \\ \hline d & c \\ \text{or} & \\ c & d \end{array} \qquad \begin{array}{cc} d & d \\ t_1 & l \\ \vdots & \vdots \\ t_{k-2} & l \\ c & c \\ \hline d & c \\ \text{or} & \\ c & d. \end{array}$$

Now we have two cases.

(1) If $c\in C_\varrho$, then let $(a, b)\notin\varrho$. We can choose elements $t_1, \ldots, t_{k-2}$ with $(b, t_1, \ldots, t_{k-2})\notin\tau$. Considering the first matrix we get that f_k^τ and g_k^τ do not preserve ϱ.

(2) If $c\notin C_\varrho$, then let d and l such that $(c, d)\notin\varrho$, and $l\in C_\varrho$. For $k=3$, if $(d, l)\in\tau$ then let t_1 such that $(d, t_1)\notin\tau$, and if $(d, l)\notin\tau$ then let $t_1=d$. From the second matrix we get that f_k^τ and g_k^τ do not preserve ϱ. Finally, if $k\geqq 4$, there are elements $t_1, \ldots, t_{k-2}$ with $(d, t_1, \ldots, t_{k-2})\notin\tau$ and the third matrix works.

Remark 3. Let A be a finite set, $|A|\geqq 3$. For an arbitrary relation ϱ on A

we define the following k-ary ϱ-pattern function on A

$$t_k^{\varrho}(x_1, x_2, \ldots, x_k) = \begin{cases} x_k, & \text{if } \; x_1 \varrho x_2 \varrho \ldots \varrho x_{k-1} \\ x_1 & \text{otherwise,} \end{cases}$$

$$s_k^{\varrho}(x_1, x_2, \ldots, x_k) = \begin{cases} x_1, & \text{if } \; x_1 \varrho x_2 \varrho \ldots \varrho x_{k-1} \\ x_k & \text{otherwise.} \end{cases}$$

We saw in [7] that $\langle A; f \rangle$ with $f = t_k^{\varrho}$ of $f = s_k^{\varrho}$ are functionally complete, if $k \geqq 3$, and ϱ is an arbitrary permutation on A or $\varrho = \delta \cup \delta^{-1}$ with an arbitrary permutation δ on A. If ϱ is an arbitrary central relation on A, then

$$t_k^{\varrho}(x_1, x_2, \ldots, x_2, x_3) = f_3^{\varrho}(x_1, x_2, x_3),$$

and

$$s_k^{\varrho}(x_1, x_2, \ldots, x_2, x_3) = g_3^{\varrho}(x_1, x_2, x_3).$$

Hence, using the Theorem, the following result follows.

$\langle A; f \rangle$ with $f = t_k^{\varrho}$ or $f = s_k^{\varrho}$ functionally complete if and only if ϱ is an arbitrary simple central relation.

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