Acta Sci. Math., 56 (1992), 229—236

Estimation of generalized moments of additive functions over the set of shifted primes

K.-H. INDLEKOFER and I. KÁTAI

1. Introduction. A function φ: $[0,\infty)\to[0,\infty)$ is called subadditive, if it is monotonically increasing, $\varphi(x)\to\infty$ as $x\to\infty$, and the condition

$$\varphi(x+y) \leqq c_1(\varphi(x)+\varphi(y)) \quad \text{for} \quad x, y \geqq 1 \tag{1.1}$$

holds with a suitable constant $c_1>0$.

It is clear that the functions $\log(1+x)$, x^{Γ} $(\Gamma>0)$ are subadditive. On the other hand (1.1) implies that $\varphi(x)=0(x^c)$ $(x\to0)$ with some constant c.

We are interested in giving necessary and sufficient conditions for an additive function f for which

$$(P(x)=)\, P(x):=\sum_{p\leqq x}\varphi(|f(p+1)-\alpha(x)|)\ll \operatorname{li} x \tag{1.2}$$

holds true with a suitable function $\alpha(x)$. Here, and in what follows p runs over the set $\mathcal{P}$ of primes.

For the sake of simplicity we extend the domain of φ to the whole real line by $\varphi(-x):=\varphi(x)$. Then

$$\varphi(x+y) \leqq c_2+c_3(\varphi(x)+\varphi(y)) \tag{1.3}$$

obviously holds for $x, y\in\mathbf{R}$, where c_2, c_3 are suitable positive constants.

For an arbitrary additive function f let

$$A_f(x):=\sum_{\substack{p\leqq x\\ |f(p)|<1}}\frac{f(p)}{p} \tag{1.4}$$

Received January 3, 1991.

and (E_f), (E_f^*) denote the conditions:

$$(E_f) \qquad \sum_{\substack{p\in\mathcal{P}\\ |f(p)|<1}} \frac{f(p)}{p} \text{ is convergent (finite)}$$

$$(E_f^*) \qquad \sum_{\substack{|f(p)|\geqq 1\\ p\in\mathcal{P}}} 1/p < \infty.$$

An additive function f is said to be finitely distributed if (E_{f^2}), (E_f^*) hold true. Let $\pi(x, k, l)$ denote the number of the primes $q \leqq x$ for which $q \equiv l \pmod k$.

Theorem 1. *Let φ be subadditive. Assume that f is an additive function, for which there exists a real-valued function $\alpha(x)$ such that* (1.2) *holds. Then f can be written as $f = \lambda \log + h$, where h is finitely distributed, $\lambda \in \mathbf{R}$, furthermore*

$$(1.5)_1 \qquad \sum_{|h(p^m)|\geqq 1} \frac{\varphi(h(p^m))}{p^m} < \infty$$

and

$$(1.6) \qquad \sum_{\substack{|h(q^m)|\geqq 1\\ q^m\geqq x^{3/4}}} \varphi(h(q^m))\pi(x, q^m, -1) = 0(\mathrm{li}\, x).$$

We have $\alpha(x) = \alpha^*(x) + 0(1)$, *where*

$$(1.7) \qquad \alpha^*(x) := \lambda \log x + A_h(x).$$

In contrary, let h be a finitely distributed additive function for which $(1.5)_1$ *and* (1.6) *hold and let* $f = \lambda \log + h$, $\lambda \in \mathbf{R}$. *Then* (1.2) *holds.*

Remarks. A. Assume that

$$(1.5)_\Gamma \qquad \sum_{|h(p^m)|\geqq 1} \frac{\varphi^\Gamma(h(p^m))}{p^m} < \infty$$

holds with a suitable $\Gamma > 1$. Then $(1.5)_1$ and (1.6) are satisfied.

B. It is not known whether condition (1.6) could be omitted or not. Let $P^*(n)$ denote the largest prime power divisor of n. Assume that

$$(1.8) \qquad \limsup_p \frac{(\log P^*(p+1))\log\log P^*(p+1)}{\log(p+1)} = \infty.$$

Then the condition (1.6) cannot be omitted in the Theorem, i.e. there exists such a finitely distributed h for which $(1.5)_1$ holds, but (1.6) does not hold.

Proof of Remark B. According to (1.8), there exists a sequence $p_1 < p_2 < \dots$ of primes, $Q_1 < Q_2 < \dots$ of prime powers, such that $p_i + 1 = a_i Q_i$, and

$$l_i := \frac{\log(p_i+1)}{a_i \cdot i^2} \to \infty.$$

Let now $h \geqq 0$ be defined on the set of prime powers q^m such that $h(q^m)=0$ if $q^m \in \{Q_i\}_{i\in\mathbf{N}}$, and $\varphi(h(Q_i))=Q_i/i^2$. Then $(1.5)_1$ holds, while

$$\sum_{p_i^{3/4}<q^m\leqq p_i} \varphi(h(q^m))\pi(p_i, q^m, -1) \geqq \varphi(h(Q_i)) = \frac{Q_i}{i^2} = \frac{p_i+1}{a_i \cdot i^2} = l_i \frac{p_i+1}{\log(p_i+1)}.$$

Thus (1.6) does not hold.

The theorem and Remark A will be proved in sections 3 and 4.

2. Lemmata. The main result of the proof of our theorem is a recent deep result of A. HILDEBRAND ([1], Theorem 4), which we state now as

Lemma 1. *There exist positive absolute constants δ and c such that if $x \geqq 2$, and f is a realvalued additive function satisfying*

$$\max_{a\in\mathbf{R}} \#\{p \leqq x: f(p+1)\in[a, a+1]\} \geqq (1-\delta)\pi(x), \tag{2.1}$$

then

$$\min_{|\lambda|\leqq c} \sum_{p\leqq x} \frac{1}{p} \min(1, |f(p)-\lambda \log p|^2) \leqq c. \tag{2.2}$$

Remark. Assume that (2.1) holds for an unbounded sequence x_ν of x. Then, for each x_ν there exists a $\lambda_\nu(=\lambda)$ for which (2.2) holds ture, $|\lambda_\nu| \leqq c$. Set λ be a limit point of the sequence $\{\lambda_\nu\}$. Then, from (2.2)

$$\sum_p \frac{1}{p} \min(1, |f(p)-\lambda \log p|^2) < \infty,$$

which implies that $h(n):=f(n)-\lambda \log n$ is finitely distributed. Another important tool is the following

Lemma 2. *Let α be a real number satisfying $0 \leqq \alpha < 2$. Then we have, for every $x \geqq 2$ and every additive f,*

$$\sum_{p\leqq x} |f(p+1)-E(x)|^\alpha \ll \frac{x}{\log x} B^\alpha(x),$$

where

$$E(x) = \sum_{p^m\leqq x} \frac{f(p^m)}{p^m}, \quad B(x) = \Big(\sum_{p^m\leqq x} \frac{|f(p^m)|^2}{p^m}\Big)^{1/2},$$

and the implied constant depends only on α.

Remark. This analogue of the Turán—Kubilius inequality was established by P. D. T. A. ELLIOTT for strongly additive functions (see [2], Lemma 4.18], the general case can be proved in the same way.

The following assertion due to ELLIOTT ([2], Lemma 4.19).

Lemma 3. *Let m be a non-negative integer, and δ a real number,* $0<\delta\leqq 1/2$. *Then there is a number c, depending upon n but not δ, so that the inequality*

$$(*)\qquad \sum_{x^{1-\delta}<Q\leqq x} p^{n-1}\pi(x, Q-1)^n \leqq \delta c\left(\frac{x}{\log x}\right)^n$$

holds for all sufficiently large values of x. Here Q runs over all prime powers.

Elliott proved this inequality letting Q to run over the set of primes only. $(*)$ can be proved in the same way.

Lemma 4. *The number of solutions of the equation* $p+1=aq$ *in prime variables* $p, q<x$ *is less than* $\dfrac{cx}{l(a)\log^2 x}$ *uniformly as* $1\leqq a\leqq\sqrt{x}$.

Lemma 5 (Titchmarsh inequality). *We have*

$$\pi(x, k, l) < \frac{cx}{\varphi(k)\log x/k} \quad \textit{if} \quad 1\leqq k<x, \quad x\geqq 2.$$

For the proof of Lemma 4 and 5 see Halberstam—Richert [3].

Lemma 6. *Let g be a strongly multiplicative function such that* $0\leqq g(p)\leqq c$ *holds for every prime p. Then*

$$\sum_{p\leqq x} g(p+1) \ll \pi(x)\exp\left(\sum_{p\leqq x}\frac{g(p)-1}{p}\right).$$

For the proof see [4], Lemma 1.

3. Proof of the Theorem. *Necessity.* Assume that (1.2) holds. Then the condition (2.1) of Lemma 1 is satisfied for every large x, consequently $f=\lambda\log+h$, where h is a finitely distributed function, $\lambda\in\mathbf{R}$. Let $\alpha_1(x)=\alpha(x)-\lambda\log x$. Since $h(n)-\alpha_1(x)=f(n)+\lambda\log\dfrac{x}{n}-\alpha(n)$, by (1.3), (1.2), and by

$$\sum_{p\leqq x}\varphi\left(\lambda\log\frac{x}{p+1}\right)\ll \operatorname{li} x,$$

we get

$$(3.1)\qquad \sum_{p\leqq x}\varphi\big(h(p+1)-\alpha_1(x)\big)\ll \operatorname{li} x.$$

Let h_1 be strongly additive defined for primes q such that

$$h_1(q)=\begin{cases} h(q) & \text{if } |h(q)|<1,\\ 0 & \text{otherwise,}\end{cases}$$

and let h_2 be defined so that $h_2(n):=h(n)-h_1(n)$.

Since $\varphi(x)\ll|x|^c<c_1e^{|x|}$, and (E_{h^*}), (E_h^*) hold, we have

$$\sum_{p\leq x}\varphi\big(h_1(p+1)-A(x)\big)\ll e^{-A(x)}\sum_{p\leq x}e^{h_1(p+1)}+e^{A(x)}\sum_{p\leq x}e^{-h_1(p+1)}.$$

By Lemma 6, the right hand side is bounded by

$$\ll(\mathrm{li}\,x)\exp\left(\sum_{q\leq x}\frac{e^{h_1(q)}-1-h_1(q)}{q}\right)+$$

$$(\mathrm{li}\,x)\exp\left(\sum_{q\leq x}\frac{e^{-h_1(q)}-1+h_1(q)}{q}\right)\ll\mathrm{li}\,x.$$

Thus

(3.2) $$\sum_{p\leq x}\varphi\big(h_2(p+1)-A(x)\big)\ll\mathrm{li}\,x,$$

whence

(3.3) $$\sum_{p\leq x}\varphi\big(h_2(p+1)-\alpha_2(x)\big)\ll\mathrm{li}\,x,$$

$$\alpha_2(x)=\alpha_1(x)-A(x)$$

immediately follows.

We shall prove that $\alpha_2(x)$ is bounded.

Let y be a large positive number. Let $h_2^{(1)}$, $h_2^{(2)}$ be additive functions, $h_2(n)=h_2^{(1)}(n)+h_2^{(2)}(n)$, and for prime powers q^m let

$$h_2^{(1)}(q^m)=\begin{cases}h(q^m) & \text{if}\quad q^m\leq y\\ 0 & \text{otherwise.}\end{cases}$$

Since $h_2^{(1)}(n)$ is bounded, therefore from (3.3) we have

(3.4) $$\sum_{p\leq x}\varphi\big(h_2^{(2)}(p+1)-\alpha_2(x)\big)\ll\mathrm{li}\,x.$$

Let Q_y denote the set of all prime powers $q^m\geq y$ for which either $m\geq 2$ or $|h(q)|\geq 1$ holds. By using Lemma 4 and 5 and the Eratosthenian sieve one can get easily that there exists at least $cx/\log x$ prime p up to x, such that $q^m\|p+1$ implies that $q^m\notin Q_y$. For such a prime p we have $h_2^{(2)}(p+1)=0$. Consequently, (3.4) can be held only if $\alpha_2(x)=0(1)$.

Let $q^m\in Q_y$ and S_qm be the set of those primes $p\leq x$ for which $p+1=q^m v$, where v is square free, $(v,q)=1$, and v does not contain any prime factor $R>2$ for which $|h(R)|>1$. It is clear that $h_2^{(2)}(p+1)=h(q^m)$. By using the above argument and the prime number theorem for residue classes one get readily that

$$\#(S_qm)>\frac{c}{\varphi(q^m)}\frac{x}{\log x}$$

uniformly as $q^m\leq\log\log x$, say. Hence, by (3.4), and by $\alpha_2(x)=0(1)$, we get $(1.5)_1$ immediately, and even that $\alpha(x)=\lambda\log x+A(x)+0(1)$.

Let $\mathcal{M}$ denote the set of those integers D, all exact prime-power factors of which belong to Q_1. Let us write $p+1$ in the form $p+1=Dv$, where v is square-free and does not contain any prime factors for which $h_2(q)\neq 0$, and $D\in\mathcal{M}$. This representation is unique. It is clear that $h_2(p+1)=h(D)$. Consequently

$$\text{li}\, x \gg \sum_{p\leq x} \varphi\big(h_2(p+1)\big) = \sum_{D\in\mathcal{M}} \varphi\big(h_2(D)\big)\pi(x, D, -1), \tag{3.5}$$

and (1.6) holds true.

The necessity of the conditions is proved.

Sufficiency. Assume that $(1.5)_1$, (1.6), (1.7) are satisfied, where h is a finitely distributed function. We shall prove that (1.2) holds, if $f=\lambda\log+h$, $\lambda\in\mathbf{R}$. By using the subadditive property of φ, it is enough to prove it for $\lambda=0$, i.e. if $f=h$ is finitely distributed. We keep the notations $h_1, h_2, \mathcal{M}$.

It is enough to prove that

$$\sum_{p\leq x} \varphi\big(h_1(p+1)-A(x)\big) \ll \text{li}\, x, \tag{3.6}$$

and that

$$\sum_{p\leq x} \varphi\big(h_2(p+1)\big) \ll \text{li}\, x. \tag{3.7}$$

The first inequality was deduced from Lemma 6 earlier. It remains to prove only (3.7). We have

$$A := \sum_{p\leq x} \varphi\big(h_2(p+1)\big) = \sum_{\substack{D\in\mathcal{M}\\ D<x}} \varphi\big(h_2(D)\big)\pi(x, D, -1) = \sum\nolimits_1 + \sum\nolimits_2,$$

where in $\sum_1$ we sum over $D\leq x^{1-\delta}$, and in $\sum_2$ over the others. Here δ is a constant, $0<\delta<0, 1$. By Lemma 5,

$$\sum\nolimits_1 \ll \text{li}\, x \sum_{D\in\mathcal{M}} \frac{\varphi\big(h_2(D)\big)}{l(D)}.$$

Let us consider $\sum_2$. We split the sum $\sum_2=\sum_2'+\sum_2''$, where in $\sum_2'$ we sum over those $D>x^{1-\delta}$ which can be written as $D=D_1D_2$, where $(D_1, D_2)=1$ and $D_i<x^{1-\delta}$ $(i=1, 2)$. Since $\varphi\big(h_2(D)\big)\ll\varphi\big(h_2(D_1)\big)+\varphi\big(h_2(D_2)\big)$, we can use Lemma 5 again,

$$\sum\nolimits_2' \ll \text{li}\, x \sum_{D_1\in\mathcal{M}} \frac{\varphi\big(h_2(D_1)\big)}{l(D_1)} + \text{li}\, x \sum_{D_2\in\mathcal{M}} \frac{\varphi\big(h_2(D_2)\big)}{l(D_2)}.$$

If D is considered in $\sum_2''$, then D has the form $D=D_1\cdot D_2$, where $D_1>x^{1-\delta}$ and D_1 is a prime or a prime power, $D_1=q^m$. Thus

$$\sum\nolimits_2'' \ll \sum_{D_2<\sqrt{x}} \varphi\big(h_2(D_2)\big)\pi(x, D_2, -1) + \sum_{\substack{q^m\in Q_1\\ q^m>x^{1-\delta}}} \varphi\big(h_2(q^m)\big)\pi(x, q^m, -1).$$

Collecting our inequalities, and taking into account (1.6), we have

$$A \ll \operatorname{li} x \sum_{D\in\mathcal{M}} \frac{\varphi\left(h_2(D)\right)}{l(D)} + O(\operatorname{li} x).$$

Finally we prove that the sum on the right hand side is convergent.

Indeed, iterating (1.1), we get that

$$\varphi\left(h_2(D)\right) \leqq \sum_{q^m \| D} c^{\omega(D/q^m)} \cdot \varphi\left(h_2(q^m)\right),$$

where $\omega(n)$ is the number of distinct prime divisors of n. Thus we have

$$\sum_{D\in\mathcal{M}} \frac{\varphi\left(h_2(D)\right)}{l(D)} \leqq \sum_{D} \sum_{q^m \| D} \frac{\varphi\left(h_2(q^m)\right)}{l(D)} \cdot \frac{c^{\omega(D/q^m)}}{l(D/q^m)} \leqq$$

$$\leqq \left(\sum_{q^m\in Q_1} \frac{\varphi\left(h_2(q^m)\right)}{l(q^m)}\right)\left(\sum_{H\in\mathcal{K}} \frac{c^{\omega(H)}}{l(H)}\right).$$

$(1.5)_1$ implies the convergence of the first sum. The second sum is convergent as well.

The sufficiency part is proved.

4. Proof of Remark A. To estimate

$$S = \sum_{\substack{|h(q^m)|\geqq 1 \\ x^{3/4}\leqq q^m < x}} \varphi\left(h(q^m)\right) \pi(x, q, -1),$$

we apply Lemma 3, namely that

$$\sum_{x^{3/4} < q^m < x} \pi(x, q^m, -1)^{n+1} q^n < c_n (\operatorname{li} x)^{n+1}$$

holds for every integer $n \geqq 1$. Let n be so large that $\alpha_1 := 1 + 1/n \leqq \Gamma$, $\gamma = \dfrac{n}{n+1}$, β be defined from $\dfrac{1}{\alpha_1} + \dfrac{1}{\beta} = 1$. Then, by Hölder's inequality,

$$S = \sum_{q^m > x^{3/4}} \frac{\varphi\left(h(q^m)\right)}{q^{m\gamma}} \pi(x, q^m, -1) q^{\gamma m} \ll$$

$$\ll \left(\sum_{|h(q^m)|\geqq 1} \frac{\varphi\left(h(q^m)\right)^{\alpha_1}}{q^{\gamma\alpha_1}}\right)^{1/\alpha_1} \left(\sum (\pi(x, q^m, -1) q^{m\gamma})^{\beta}\right)^{1/\beta}.$$

By Lemma 3, and by $(1.5)_\Gamma$ we get $S = O(\operatorname{li} x)$. This finishes the proof.

References

[1] A. HILDEBRAND, Additive and multiplicative functions on shifted primes, *Proc. London Math. Soc.*, **59** (1989), 209—232.

[2] P. D. T. A. ELLIOTT, *Probabilistic number theory*. I, Springer (New York, 1979).

[3] H. HALBERSTAM and H. RICHERT, *Sieve methods*, Academic Press (London, 1974).

[4] K.-H. INDLEKOFER and I. KÁTAI, Momente additiver Funktionen auf der Folge $\{p+1\}$, *Lit. Math. Sbornik*, **28** (1988), 669—679.

(K.-H. I.)
FACHBEREICH MATHEMATIK-INFORMATIK
UNIVERSITÄT-GH PADERBORN
WARBURGER STR. 100
D—4790 PADERBORN

(I. K.)
EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, H—1117
BOGDÁNFY ÚT 10/B.