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On a theorem of Kátai-Wirsing

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1. Introduction. An arithmetic function $f(n)$ is said to be additive if $(m,n)=1$ implies that

$$f(mn) = f(m)+f(n)$$

and it is completely additive if the above equality holds for all positive integers m and n. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of complex-valued additive and completely additive functions, respectively.

The problem concerning the characterization of $\log n$ as an additive arithmetic function was studied by several authors. The first such characterization is apparently that of P. ERDŐS [3]. He proved in 1946 that if a real valued additive function f satisfies the condition

$$f(n+1)-f(n) \to 0 \quad \text{as} \quad n \to \infty,$$

then $f(n)$ is a constant multiple of $\log n$. Later I. KÁTAI [4] and E. WIRSING [6] improving this result, proved that a function $f\in\mathcal{A}$ satisfying

$$\sum_{n\leq x} |f(n+1)-f(n)| = o(x) \quad \text{as} \quad x \to \infty$$

must be of the form $f=U\log$ for some complex constant U.

On the other hand, solving a conjecture of Kátai, P. D. T. A. ELLIOTT [1] showed that if a real function f is additive and satisfies the condition

$$f(An+B)-f(an+b) \to C \quad \text{as} \quad n \to \infty \tag{1}$$

for some integers $A>0, B, a>0, b$ with $Ab-aB\neq 0$ and for a real constant C, then $f(n)=U\log n$ holds for all positive integers n which are prime to $Aa(Ab-aB)$. In his proof Elliott relaxed the condition (1) to

$$\sum_{n\leq x} |f(An+B)-f(an+b)|^2 = o(x)$$

for the case $A\neq a$.

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Our purpose in this paper is to give a complete characterization of those functions $f, g \in \mathcal{A}$ for which the relation

$$\sum_{n \leq x} |g(an+b)-f(n)-d| = o(x) \tag{2}$$

holds for some fixed positive a, b and for a complex constant d.

We shall prove the following

Theorem 1. *Assume that $f, g \in \mathcal{A}$ satisfy* (2) *for some fixed positive integers a, b and for a complex constant d. Then there are a complex constant U and functions $F \in \mathcal{A}$, $G \in \mathcal{A}$ such that*

$$f(n) = U \log n + F(n)$$

$$g(n) = U \log n + G(n)$$

and

$$G(an+b)-F(n)-d+U \log a = 0$$

hold for all positive integers n.

Theorem 2. *Assume that $f \in \mathcal{A}$ satisfies the condition*

$$\sum_{n \leq x} |f(An+B)-f(Cn)-D| = o(x) \tag{3}$$

for some positive integers A, B, C and for a complex constant D. Then there are a complex constant U and a function $F \in \mathcal{A}$ such that

$$f(n) = U \log n + F(n)$$

and

$$F(n) = F[(n, BCC_A)]$$

hold for all positive integers n, where C_A denotes the product of all prime divisors of C which are prime to A.

We note that our theorems can be derived from a recent result due to P. D. T. A. ELLIOTT [2], which was obtained with analytic methods. Here we shall prove our results by using elementary methods, which were used in [5].

2. Auxiliary results. In this section we assume that a function $f \in \mathcal{A}$ satisfies (3), i.e.

$$\sum_{n \leq x} |f(An+B)-f(Cn)-D| = o(x)$$

holds for some positive integers A, B, C and for a complex constant D.

Let C_A denote the product of all prime divisors of C which are prime to A. For an arbitrary positive integer n, let $E(n)=E_B(n)$ be the product of all prime power factors of B composed from the prime divisors of n, i.e. $E(n)|B$, $(E(n), B/E(n))=1$ and every prime divisor of $E(n)$ is a divisor of n.

Lemma 1. *For every fixed positive integer k and Q we have*

$$f(BCC_AQ^k) = kf(BCC_AQ)-(k-1)f(BCC_A), \tag{4}$$

furthermore

$$f(ACC_A^2E(C)) = 2f(CC_AE(C))-f(E(C))+D. \tag{5}$$

Proof. For each positive integer Q we define the sequence

$$R = R(AC_AQ) = \{R_k\}_{k=1}^{\infty}$$

by the initial term $R_1=1$ and by the formula

$$R_k = R_k(AC_AQ) = 1+AC_AQ+ \dots +(AC_AQ)^{k-1} \tag{6}$$

for all integers $k \geqq 2$. Moreover, let

$$T_k(n, Q) = (AC_AQ)^k E(CQ)n+BR_k(AC_AQ). \tag{7}$$

By using (6) and (7), we have

$$T_{k+1}(n, Q) = AC_AQT_k(n, Q)+B \tag{8}$$

and

$$(CC_AQE(CQ), T_k(n, Q)/E(CQ)) = 1 \tag{9}$$

for all integers $k \geqq 1$. Thus, using (3), (7), (8), (9) and the additivity of f, we have

$$\sum_{n\leqq x} |f(T_1(n, Q))-f(CC_AQE(CQ)n)-D| = o(x)$$

and

$$\sum_{n\leqq x} |f(T_k(n, Q))-f(T_{k-1}(n, Q))-H(Q)| = o(x)$$

for all integers $k \geqq 2$, where

$$H(Q):=f(CC_AQE(CQ))-f(E(CQ))+D.$$

These imply that

$$\sum_{n\leqq x} |f(T_k(n, Q))-f(CC_AQE(CQ)n)-(k-1)H(Q)-D| = o(x) \tag{10}$$

holds for every integer $k \geqq 1$.

We shall deduce from (10) that

$$f(A^{k-1}CC_A^kQ^kPE(CQ)) = (k-1)H(Q)+f(CC_AQPE(CQ)) \tag{11}$$

holds for every positive integer k, Q and P.

Let k, Q and P be positive integers. Considering

$$n:= PR_k(AC_AQ)\{APCQR_k(AC_AQ)m+1\} \tag{12}$$

and taking into account (10), it is easily seen that (11) holds if k, Q and P satisfy the relation

(13) $$\big(P, R_k(AC_AQ)\big) = \big(PE(CQ)+B, R_k(AC_AQ)\big) = 1.$$

It is obvious that (13) is satisfied in the following cases:

$$P=1,\ Q=2B;\quad P=1,\ Q=2pB,$$

where p is a prime. Thus, we get from (11) that

$$f(p^k) = kf(p) \quad \text{if} \quad (p, 2ABC) = 1.$$

This with the additivity of f shows that

(14) $$f(nm) = f(n)+f(m) \quad \text{if} \quad (n, m, 2ABC) = 1.$$

Thus, by using (10), (12) and (14), we see that (11) also holds if we relax the condition (13) to

(15) $$\big(P, R_k(AC_AQ), 2B\big) = \big(PE(CQ)+B, R_k(AC_AQ), 2\big) = 1.$$

Assume that $(2, ABC)=1$ and k is an odd positive integer. In this case one can check that (15) holds for $P=Q=1$ and $P=1, Q=2$. Thus, we get from (11) that

(16) $$f(2^k) = kf(2) \quad \text{for all odd positive integers } k.$$

On the other hand, (15) also holds for $P=2^\nu$, $Q=2$ and $k=2$, where $\nu \geqq 0$ is an integer. From (11) we have

(17) $$f(ACC_A^2 2^{\nu+2}E(C)) = H(2)+f\big(CC_A 2^{\nu+1}E(C)\big).$$

Thus, we get from (17) that

$$f(2^k) = kf(2)+(k-1)\big(H(1)+f(CC_AE(C))-f(ACC_A^2E(C))\big)$$

holds for every positive integer k, which with (16) shows that

$$f(2^k) = kf(2) \quad (k = 1, 2, \ldots).$$

This with (14) implies that

(18) $$f(nm) = f(n)+f(m) \quad \text{if} \quad (n, m, ABC) = 1.$$

Similarly as above, by using (10), (12) and (18) we also have (11) if k, Q and P satisfy

(19) $$(P, R_k(AC_AQ), B) = 1.$$

Finally, let $P=P_1 \cdot P_2$, where $(P_1, P_2)=(P_1, AC_AQ)=1$ and every prime divisor of P_2 is a divisor of AC_AQ. We have

$$\big(P_2, R_k(AC_AQ), B\big) = 1,$$

therefore by (11) and (19) it follows that

$$f(A^{k-1}CC_A^kQ^kP_2E(CQ)) = (k-1)H(Q)+f(CC_AP_2E(CQ)).$$

Since $(P_1, AC_AQP_2)=1$, by using the additivity of f, we get

$$f(A^{k-1}CC_A^kQ^kPE(CQ)) = f(A^{k-1}CC_AQ^kP_2E(CQ))+f(P_1) =$$
$$= (k-1)H(Q)+f(CC_AQPE(CQ)),$$

which proves (11).

Applying (11) in the case $Q=1$, we obtain that

$$f(A^{k-1}CC_A^kPE(C)) = (k-1)H(1)+f(CC_APE(C))$$

holds for every positive integer k and P, consequenly

(20) $$f(A^{k-1}CC_A^kQ^kE(CQ)) = (k-1)H(1)+f(CC_AQ^kE(CQ)).$$

On the other hand, (11) with $P=1$ implies

$$f(A^{k-1}CC_A^kQ^kE(CQ)) = (k-1)H(Q)+f(CC_AQE(CQ)),$$

which with (20) gives

$$f(CC_AQ^kE(CQ)) = (k-1)(H(Q)-H(1))+f(CC_AQE(CQ)).$$

This, using the fact $(E(CQ), B/E(CQ))=1$ and the additivity of f, shows that

$$f(BCC_AQ^k) = kf(BCC_AQ)-(k-1)f(BCC_A).$$

So, we have proved Lemma 1, because (5) follows from (11) in the case $k=2$ and $P=Q=1$.

Lemma 2. *Let A, B be positive integers and D be a complex constant. If $f\in\mathcal{A}^*$ satisfies the condition*

(21) $$\sum_{n\leqq x} |f(An+B)-f(n)-D| = o(x) \quad \text{as} \quad x\to\infty,$$

then there is a complex constant U such that

$$f(n) = U\log n \quad (n = 1, 2, 3, \ldots).$$

Proof. We first note that, by using (5) of Lemma 1 and the fact $f\in\mathcal{A}^*$, (21) implies

(22) $$f(A) = D.$$

If $A=1$, then our assertion follows from the theorem of I. Kátai—E. Wirsing mentioned in Section 1. In the following we assume that $A\geqq 2$ and

(23) $$\sum_{n\leqq x} |f(An+B)-f(An)| = o(x).$$

Let I_f denote those pairs (k, r) of positive integers for which

$$\sum_{n\leqq x} |f(kn+r)-f(kn)| = o(x).$$

Since $(A, B)\in I_f$ and $f\in\mathcal{A}^*$, we have $(A, 1)\in I_f$, furthermore if $(k_0, 1)\in I_f$, then $(k, 1)\in I_f$ for all integers $k\geqq k_0$, because

$$f\big((k+1)n+1\big)-f\big((k+1)n\big) = \{f(kn+1)-f(kn)\}-$$
$$-\big\{f\big[k\big((k+1)n+1\big)+1\big]-f\big[k\big((k+1)n+1\big)\big]\big\}.$$

Thus, we have $(k, 1)\in I_f$ for every integer $k\geqq A$.

We shall prove that if $(h+1, 1)\in I_f$ and integers k, r satisfy

$$0<r<k/h \quad\text{and}\quad (k, r)=1, \tag{24}$$

then $(k, r)\in I_f$. We prove this by using induction on r. For $r=1$ our assertion is true, because $1<k/h$ implies $k>h$. Assume that for every integer k, r satisfying (24) and $r<R$ we have $(k, r)\in I_f$. Let K be an integer such that

$$0<R<K/h \quad\text{and}\quad (K, R)=1. \tag{25}$$

Let k and r be positive integers which satisfy

$$Rk = Kr+1 \quad\text{and}\quad k<K,\ r<R. \tag{26}$$

It is easily seen by (25) and (26) that $(k, r)=1$, furthermore

$$Kr < Kr+1 = Rk < Kk/h,$$

which implies that $r<k/h$. Thus, k, r satisfy (24), and so $(k, r)\in I_f$.

On the other hand, we have

$$f(Kn+R)-f(Kn) = \{f[K(kn+r)+1]-f[K(kn+r)]\}+\{f(kn+r)-f(kn)\},$$

therefore, by using the fact $(K, 1)\in I_f$ and $(k, r)\in I_f$, we have $(K, R)\in I_f$. Thus we have proved (24).

We now deduce from (23) that $(2, 1)\in I_f$. To see this enough show that

$$(h+1, 1)\in I_f \quad\text{with}\quad h+1>2 \quad\text{implies}\quad (h, 1)\in I_f. \tag{27}$$

Assume that $(h+1, 1)\in I_f$ and $h+1>2$. Let

$$S(x) = \sum_{n\leqq x} |f(hn+1)-f(hn)|.$$

For each integer d with $0\leqq d\leqq h-1$ we can choose positive integers $K=K(d)$ and $R=R(d)$ such that

$$(hd+1)K = h^2R+1. \tag{28}$$

We have

(29)

$$S(x) = \sum_{n\le x} |f(hn+1)-f(hn)| = \sum_{d=0}^{h-1} \sum_{hm+d\le x} \left|f(h^2m+hd+1)-f\big(h(hm+d)\big)\right| =$$

$$= \sum_{d=0}^{h-1} \sum_{hm+d\le x} \Big|f\big(h^2(Km+R)+1\big)-f\big(h^2(Km+R)\big)+$$

$$+f\big(K(hm+d)+hR-Kd\big)-f\big(K(hm+d)\big)\Big|$$

and so $S(x)=o(x)$ if $hR-Kd=0$, because $(h+1, 1)\in I_f$ and $h+1>2$ implies $(h^2, 1)\in I_f$. If $hR-Kd\ne 0$, then we get from (28) that

$$0 < hR-Kd = (K-1)/h < K/h$$

and

$$(K, hR-Kd) = (K, hR) = 1.$$

Thus, $k:=K$ and $r:=hR-Kd$ satisfy the condition (24), and so $(K, hR-Kd)\in I_f$. By using this and the fact $(h^2, 1)\in I_f$, we also get from (29) that $S(x)=o(x)$. This shows that $(h, 1)\in I_f$, consequently $(2, 1)\in I_f$.

Assume now that

(30) $$A(x) = \sum_{n\le x} |f(2n+1)-f(2n)| = o(x).$$

Let q be a fixed prime. As we have proved above, from (30) we have $(q, r)\in I_f$ if $0<r<q$ (see (24)). Let

$$T(x) := \sum_{n\le x} f(n).$$

Then, we have

$$\sum_{\substack{n\le x\\ n\equiv 0 \bmod q}} f(n) = \sum_{m\le x/q} \{f(q)+f(m)\} = \left[\frac{x}{q}\right] f(q)+T\left(\frac{x}{q}\right).$$

Let r be an integer for which $0<r<q$. Then $(q, r)\in I_f$, and so

$$\sum_{\substack{n\le x\\ n\equiv r \bmod q}} f(n) = \sum_{qm+r\le x} \{f(qm+r)-f(qm)\} + \sum_{qm+r\le x} f(qm) = o(x)+\left[\frac{x}{q}\right] f(q)+T\left(\frac{x}{q}\right).$$

These imply that

$$T(x) = q\left[\frac{x}{q}\right] f(q)+qT\left(\frac{x}{q}\right)+o(x) = xf(q)+qT\left(\frac{x}{q}\right)+o(x)$$

as $x\to\infty$, from which we get

$$\frac{f(q)}{\log q} = \lim_{x\to\infty} \frac{T(x)}{x\log x} =: U.$$

From this and using $f\in\mathcal{A}^*$ the proof of Lemma 2 is finished.

3. Proof of Theorem 2. Assume that $f \in \mathcal{A}$ satisfies the condition (3). Then from Lemma 1 we get that

$$f(BCC_A Q^k) = kf(BCC_A Q) - (k-1) f(BCC_A) \tag{31}$$

holds for every positive integer k and Q, where C_A denotes the product of all prime divisors of C which are prime to A.

For each prime p let $e = e(p)$ be a non-negative integer for which $p^e \| BCC_A$. Then for all integers $\beta \geqq e$ we deduce from (31) that

$$f(p^{\beta+1}) - f(p^\beta) = f(p^{e+1}) - f(p^e). \tag{32}$$

Now we write

$$f(n) = f_1(n) + F(n),$$

where f_1 is a completely additive function defined as follows:

$$f_1(p) := f(p^{e+1}) - f(p^e), \quad e = e(p). \tag{33}$$

Then, from (32) and (33) it follows that

$$F(p^{\beta+1}) = F(p^\beta),$$

which implies

$$F(p^k) = F[(p^k, BCC_A)] \quad (k = 0, 1, 2, \ldots).$$

Thus, we have

$$F(n) = F[(n, BCC_A)] \quad (n = 1, 2, 3, \ldots). \tag{34}$$

We shall prove that $f_1 = U \log$ for some constant U.

We note that by (3) we have

$$\sum_{n \leqq x} |f(ABC_A n + B) - f(BCC_A n) - D| = o(x). \tag{35}$$

By using $f = f_1 + F$ and (34) we get that

$$f(ABC_A n + B) - f(BCC_A n) - D = f_1(ABC_A n + B) - f_1(BCC_A n) + F(ABC_A n + B) -$$
$$- F(BCC_A n) - D = f_1(ABC_A n + B) - f_1(n) - \{f_1(BCC_A) - F(B) + F(BCC_A) + D\}$$

and so, by (35) and Lemma 2, there is a complex constant U such that $f_1 = U \log$. This completes the proof of Theorem 2.

4. Proof of Theorem 1. Assume that $f, g \in \mathcal{A}$ satisfy the condition (2), i.e.

$$\sum_{n \leqq x} \left| g(an+b) - f(n) - d \right| = o(x), \tag{36}$$

where a and b are positive integers and d is a complex constant.

For each positive integer N we have

$$(abN+1, a(abN+1)n+b) = 1$$

and

$$(abN+1)(a(abN+1)n+b) = a\left[(abN+1)^2 n+b^2 N\right]+b$$

for every positive integer n. Thus, by using the additivity of f, we get

$$f[(abN+1)^2 n+b^2 N]-f[(abN+1)n]-g(abN+1) =$$
$$= -\{g\,[(abN+1)(a(abN+1)n+b)]-f[(abN+1)^2 n+b^2 N]-d\}+$$
$$+\{g[a(abN+1)n+b]-f[(abN+1)n]-d\},$$

which with (36) implies that

$$\sum_{n\le x} |f[(abN+1)^2 n+b^2 N]-f[(abN+1)n]-g(abN+1)| = o(x). \tag{37}$$

Applying Lemma 1 with $A=(abN+1)^2$, $B=b^2N$ and $C=(abN+1)$ it follows from (37) that

$$f[b^2(abN+1)NQ^k] = kf[b^2(abN+1)NQ]-(k-1)f[b^2(abN+1)N] \tag{38}$$

holds for every positive integer k and Q. Since (38) holds for each fixed positive integer N, so (38) also holds for every positive integer N.

For each prime p, let N_p be the smallest positive integer for which $p\nmid abN_p+1$. It is obvious that $N_p\in\{1, 2\}$ for all primes p. We apply (38) with $Q=p$ and $N=N_p$ to get

$$f(b^2N_pp^k) = kf(b^2N_pp)-(k-1)f(b^2N_p). \tag{39}$$

Similiarly, as in the proof of Theorem 2, we can deduce from (39) that there are functions $f_1\in\mathcal{A}^*$ and $F\in\mathcal{A}$ such that

$$f = f_1+F \tag{40}$$

and

$$F(p^k) = F[(p^k, b^2N_p)] \quad (k = 0, 1, 2 \dots), \tag{41}$$

where p is a prime number. Since $N_p\in\{1, 2\}$, one can check from (41) and the fact $(b, N_2)=1$ that

$$F(n) = F[(n, b^2)]+F[(n, N_2)] \quad (n = 1, 2, 3, \dots). \tag{42}$$

By using (40) and (42), we have

$$f[(abN+1)^2N_2m+b^2N]-f[(abN+1)N_2m]-g(abN+1) = \tag{43}$$
$$= f_1[(abN+1)^2N_2m+b^2N]-f_1(m)-D,$$

where

$$D:=g(abN+1)+f_1[(abN+1)N_2]-F[(abN+1)^2N_2m+b^2N]+F[(abN+1)N_2m]=$$
$$=g(abN+1)+f_1[(abN+1)N_2]-\{F[(m,b^2)]+F[(N,N_2)]\}+\{F[(m,b^2)]+F(N_2)\}=$$
$$=g(abN+1)+f_1((abN+1)N_2)+F[(ab+1,N_2)].$$

Applying (37) with $n=N_2m$, by (43) we have

$$\sum_{n\leqq x}|f_1[(abN+1)^2N_2m+b^2N]-f_1(m)-D|=o(x),$$

which, by using Lemma 2, implies

(44) $$f_1=U\log \quad \text{for some constant } U$$

and

$$g(abN+1)+F[(abN+1,N_2)]=f_1(abN+1)=U\log(abN+1).$$

The last equality holds for every positive integer N, consequently

$$g(m)+F[(m,N_2)]=U\log m$$

holds for all positive integers m which are prime to ab. Let

(45) $$G(m):=g(m)-U\log m \quad (m=1,2,3,\ldots).$$

Then, we have

(46) $$G(m)=0 \quad \text{if} \quad (m,2ab)=1.$$

Finally, we shall prove that

$$G(an+b)-F(n)-d+U\log a=0 \quad (n=1,2,3,\ldots),$$

which with (40), (44), (45) gives the proof of Theorem 1.

Since

$$G(an+b)-F(n)-d+U\log a=$$
$$=\{g(an+b)-f(n)-d\}-\{U\log(an+b)-U\log n-U\log a\}$$

we obtain from (36) that

(47) $$\sum_{n\leqq x}|G(an+b)-F(n)-d+U\log a|=o(x).$$

Let r be an arbitrary integer for which $0\leqq r<2b^2$. Then we get from (42) and (47) that

$$F(2b^2m+r)=F(r) \quad (m=1,2,\ldots)$$

and

(48) $$\sum_{m\leqq x}|G(2ab^2m+ar+b)-F(r)-d+U\log a|=o(x).$$

Let M be a positive integer. By (46), we have $G(2ab^2t+1)=0$ $(t=1,2,\ldots)$, consequently

$$(49)\qquad G(2ab^2M+ar+b)-F(r)-d+U\log a =$$
$$= G(2ab^2M+ar+b)+G(2ab^2t+1)-F(r)-d+U\log a =$$
$$= G\big[2ab^2\big((2ab^2M+ar+b)t+M\big)+ar+b\big]-F(r)-d+U\log a$$

holds for every positive integer t. Thus, we get from (48) and (49) that

$$\sum_{t\leqq x} |G(2ab^2M+ar+b)-F(r)-d+U\log a| = o(x),$$

which implies

$$(50)\qquad G(2ab^2M+ar+b)-F(r)-d+U\log a = 0$$

for each positive integer M, i.e. (50) holds for every positive integer M. Since r is an arbitrary integer for which $0\leqq r<2b^2$, and (50) holds for every positive integer M, we have

$$G(an+b)-F(n)-d+U\log a = 0 \quad (n = 1, 2, \ldots).$$

This completes the proof of Theorem 1.

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