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Moufang Lie loops and homogeneous spaces

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0. Introduction

The classical model of the Moufang Lie loops is the multiplication on S^7 defined by Cayley numbers of norm one. This multiplication is in a close relation with the spherical geometry of S^7, which is a symmetric Riemannian space of constant curvature. This connection has been generalised by O. Loos in [9] to any Moufang loop by proving that the modified local multiplication $(x, y)\to x^{1/2}\cdot y\cdot x^{1/2}$ gives the (local) geodesic loop multiplication of a symmetric space (cf. [6]). The analogous correspondance gives a differential geometric machinery for the investigation of analytic Bol and Moufang loops (cf. [2], [4], [10]). For group multiplications, one has a 1-parameter family of modified local loop multiplications $x \underset{(\sigma)}{\circ} y=x^{\sigma}\cdot y\cdot x^{1-\sigma}$ $(\sigma\in\mathbf{R})$ investigated by M. A. Akivis [1], that are geodesic loops.

This paper is devoted to study geodesic loops of reductive homogeneous spaces associated with Moufang loops. Such a geodesic loop is gotten from the modified multiplication $x \underset{(\sigma)}{\circ} y=x^{\sigma}\cdot y\cdot x^{1-\sigma}$ for each $\sigma\in\mathbf{R}$ in the case of the groups. For Moufang loops one obtains geodesic loop of reductive homogeneous space only for $\sigma=\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$. For $\sigma=\frac{1}{2}$, the geodesic loop of the symmetric space was investigated by O. Loos in [9].

For $\sigma=\frac{2}{3}$, we give a description of the corresponding reductive space structure in this paper. An analogues description can be obtained for $\sigma=\frac{1}{3}$ using the right multiplication instead of the left one. Our method to describe this reductive space structure is to represent the original loop multiplication on the Moufang loop by a geodesic loop multiplication of an invariant connection on a reductive

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homogeneous space. Then we deform this latter geodesic loop multiplication as $x\underset{(\sigma)}{\circ}y=x^{\sigma}\cdot y\cdot x^{1-\sigma}$ $\left(\sigma=\frac{2}{3}\right)$ and prove that this gives the geodesic loop multiplication of the canonical connection of the reductive homogeneous space.

1. The local loops $x\underset{(\sigma)}{\circ}y=x^{\sigma}\cdot y\cdot x^{1-\sigma}$ $(\sigma\in\mathbf{R})$ on a group

Let $\mathcal{G}$ be a connected Lie group and let us consider the action of the group $\mathcal{K}=\mathcal{G}\times\mathcal{G}$ on $\mathcal{G}$ given by $((g_1,g_2),g)\in(\mathcal{G}\times\mathcal{G})\times\mathcal{G}\to g_1\cdot g\cdot g_2^{-1}\in\mathcal{G}$. The isotropy subgroup of this action is the diagonal $\mathcal{H}=\Delta(\mathcal{G}\times\mathcal{G})=\{(g,g);g\in\mathcal{G}\}$. We denote by $\mathfrak{m}^{\sigma}$ the transversal subspace in the Lie algebra $\mathcal{T}_{(e,e)}(\mathcal{G}\times\mathcal{G})=\mathfrak{g}+\mathfrak{g}$ of $\mathcal{K}=\mathcal{G}\times\mathcal{G}$ to the diagonal $\Delta(\mathfrak{g}+\mathfrak{g})$ defined by $\mathfrak{m}^{\sigma}=\{(\sigma X,(\sigma-1)X), X\in\mathcal{G}\}$, where $\mathfrak{g}=\mathcal{T}_e\mathcal{G}$ and $\sigma\in\mathbf{R}$. It is clear that $\mathrm{Ad}_{\mathcal{H}}\mathfrak{m}^{\sigma}\subset\mathfrak{m}^{\sigma}$, hence the subspace $\mathfrak{m}^{\sigma}$ is a reductive complement of $\Delta(\mathfrak{g}+\mathfrak{g})$ in $\mathfrak{g}+\mathfrak{g}$. Let ∇^{σ} denote the canonical connection of the reductive homogeneous space $\mathcal{K}/\mathcal{H}$ given by $\mathfrak{m}^{\sigma}$.

Theorem 1.1. *The geodesic loop multiplication of the canonical connection ∇^{σ} of the reductive homogeneous space $\mathcal{K}/\mathcal{H}$ defined by*

$$\exp_x^{\sigma}\circ\tau_{e,x}^{\sigma}\circ(\exp_e^{\sigma})^{-1}y,\quad x,y\in\mathcal{G}$$

can be expressed in a normal neighbourhood of $e\in\mathcal{G}$ as

$$\exp_x^{\sigma}\circ\tau_{e,x}^{\sigma}\circ(\exp_e^{\sigma})^{-1}y=x\underset{(\sigma)}{\circ}y=x^{\sigma}\cdot y\cdot x^{1-\sigma},$$

where $\exp_x^{\sigma}$ denotes the exponential map at $x\in\mathcal{G}$ and $\tau_{e,x}^{\sigma}$ is the parallel translation $\mathcal{T}_e\mathcal{G}\to\mathcal{T}_x\mathcal{G}$ along the geodesic through e and x with respect to the connection ∇^{σ}.

Proof. If $(\sigma X,(1-\sigma)X)\in\mathfrak{m}^{\sigma}$ then the orbit of the 1—parameter subgroup $\exp t(\sigma X,(1-\sigma)X)$ through $e\in\mathcal{G}$ is

$$\exp t(\sigma X,(1-\sigma)X)e=(\exp t\sigma X,\exp t(\sigma-1)X)e=$$
$$=\exp t\sigma X\cdot e\cdot\exp t(1-\sigma)X=\exp tX.$$

Using Proposition 2.4 in [7, Chap. X.] we obtain the parallel translation $\tau_{e,x}^{\sigma}$ in the form $\tau_{e,\exp tX}^{\sigma}=\mathcal{T}_e\lambda_{\exp t\sigma X}\circ\mathcal{T}_e\varrho_{\exp t(1-\sigma)X}$, where λ_x and ϱ_x denote the left and the right multiplication maps on $\mathcal{G}$, respectively. Since the mapping $\lambda_{\exp t\sigma X}\circ\varrho_{\exp t(1-\sigma)X}$ is an affine transformation of the connection ∇^{σ}, the geodesics of this connection have the form

$$\lambda_{\exp t\sigma X}\circ\varrho_{\exp t(1-\sigma)X}\exp sY=\exp t\sigma X\cdot\exp sY\cdot\exp t(1-\sigma)X=$$
$$=\exp X\cdot\exp s(\mathrm{Ad}_{\exp t(\sigma-1)X}Y).$$

It follows that the geodesics of the connections ∇^σ $(\sigma\in\mathbf{R})$ are independent of the parameter σ. Hence the geodesic loop multiplication of the connection ∇^σ satisfies the equations

$$\exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}\circ(\exp^\sigma_{\exp Y})^{-1}\exp Y = \exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}Y =$$

$$= \exp^1_{\exp X}\circ\mathcal{T}_e\lambda_{\exp\sigma X}\circ\mathcal{T}_e\varrho_{\exp(1-\sigma)X}Y = \exp^1_{\exp X}\circ\mathcal{T}_e\lambda_{\exp X}\circ\mathrm{Ad}_{\exp(\sigma-1)X}Y.$$

Since the group multiplication coincides with the geodesic loop multiplication of its left canonical connection ∇^1 we obtain

$$\exp^\sigma_{\exp X}\circ\tau^\sigma_{e,\exp X}\circ(\exp^\sigma_{\exp Y})^{-1}\exp Y = \exp X\cdot\exp\circ\mathrm{Ad}_{\exp(\sigma-1)X}Y =$$

$$= \exp\sigma X\cdot\exp Y\cdot\exp(1-\sigma)X,$$

that proves the theorem.

2. The left canonical connection of a loop

Let $\mathcal{L}$ be a smooth loop with identity element $e\in\mathcal{L}$. We define the translated loop multiplications centered at $a\in\mathcal{L}$ by the formula

$$x\underset{a}{\cdot}y := x\cdot a\backslash y, \quad (a\in\mathcal{L})$$

where $x\cdot y = x\underset{e}{\cdot}y$ and $x\cdot x\backslash y = y$. This loop is isotopic to the original loop multiplication and has $a\in\mathcal{L}$ as identity element.

Let λ_x denote the left multiplication map of the loop $\mathcal{L}$ and $\mathcal{T}_e\lambda_x\colon \mathcal{T}_e\mathcal{L}\to\mathcal{T}_x\mathcal{L}$ be its tangent map.

Definition 2.1. The *left canonical connection* ∇ of the loop $\mathcal{L}$ is defined by the parallel vector fields

$$X(x) := \mathcal{T}_e\lambda_x X(e).$$

Since these vector fields are globally defined, the connection ∇ is obviously flat.

Proposition 2.2. *The left canonical connection of the translated loop multiplication* $x\underset{a}{\cdot}y$ $(a\in\mathcal{L})$ *on* $\mathcal{L}$ *coincides with the left canonical connection* ∇ *of the original loop.*

Proof. The assertion follows from the definition, because the left multiplication map of the translated loop multiplication $x\underset{a}{\cdot}y$ is $\lambda_x\lambda_a^{-1}$.

Proposition 2.3. *The covariant derivative* $(\nabla_Z T)(X, Y)$ *of the torsion tensor field* $T(X, Y)$ *of the connection* ∇ *is*

$$(\nabla_Z T)(X, Y) = \langle Z, Y, X\rangle - \langle Z, X, Y\rangle,$$

where $\langle X, Y, Z\rangle_a$ *is the associator of the translated loop multiplication* $x \underset{a}{\cdot} y$.

Proof. We denote by $\tau(x)$ the mapping $\mathcal{T}_e\lambda_x\colon \mathcal{T}_e\mathcal{L}\to\mathcal{T}_x\mathcal{L}$. Using the covariant constant vector fields $X(x)=\tau(x)X_e$ and $Y(x)=\tau(x)Y_e$, the torsion tensor field takes the form

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = -[X, Y] = -\dot{\tau}(x)Y_e(X(x)) + \dot{\tau}(x)X_e(Y(x)) =$$
$$= -\dot{\tau}(x)Y_e(\tau(x)X_e) + \dot{\tau}(x)X_e(\tau(x)Y_e),$$

where $\dot{\tau}(Y)$ denotes the derivative of the mapping τ by the variable x in the direction Y. It follows

$$(\nabla_Z T)(X, Y)_e = Z[\tau^{-1}(x)T(X, Y)]_e = -\dot{\tau}(e)T(X, Y)_e(Z_e) + Z[T(X, Y)]_e =$$
$$= -\dot{\tau}(e)[-\dot{\tau}(x)Y_e(\tau(x)X_e) + \dot{\tau}(x)X_e(\tau(x)Y_e)](Z) - \ddot{\tau}(e)Y_e(X_e, Z_e) -$$
$$-\dot{\tau}(e)Y_e(\dot{\tau}(e)X_e(Z_e)) + \ddot{\tau}(e)X_e(Y_e, Z_e) + \dot{\tau}(e)X_e(\dot{\tau}(e)Y_e(Z_e)).$$

We consider now a local coordinate system defined on a neighbourhood of the identity e on which the loop multiplication is of the form

$$x\cdot y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + \{\text{higher order terms}\},$$

where q is a bilinear, r and s are trilinear maps on the coordinate vector space. Then we can write

$$T(X, Y)_e = -q(X, Y) + q(Y, X),$$
$$(\nabla_Z T)(X, Y)_e = q(Z, q(X, Y)) - q(Z, q(Y, X)) - r(X, Z, Y) - r(Z, X, Y) -$$
$$-q(q(Z, X), Y) + r(Y, Z, X) + r(X, Z, Y) + q(q(Z, Y), X).$$

By the Theorem IX. 6.6. in [5], the commutator and the associator of the loop have the forms

$$[X, Y] = q(X, Y) - q(Y, X)$$

and

$$\langle X, Y, Z\rangle = q(q(X, Y), Z) - q(X, q(Y, Z)) + r(X, Y, Z) +$$
$$+ r(Y, X, Z) - s(X, Y, Z) - s(X, Z, Y),$$

respectively. Hence we obtain

$$(\nabla_Z T)(X, Y)_e = \langle Z, Y, X\rangle - \langle Z, X, Y\rangle,$$

which proves the assertion at $e\in\mathcal{L}$. For $a\neq e$ we consider in the same way the loop multiplication $x \underset{a}{\cdot} y$ instead of $x\cdot y$ to prove the assertion.

3. Alternative family of loops

Definition 3.1. The family of loop multiplications $x \underset{a}{\cdot} y$ defined on $\mathscr{L}$ is called *alternative* if the identities

$$x \underset{a}{\cdot} (x \underset{a}{\cdot} y) = (x \underset{a}{\cdot} x) \underset{a}{\cdot} y, \quad x \underset{a}{\cdot} (y \underset{a}{\cdot} y) = (x \underset{a}{\cdot} y) \underset{a}{\cdot} y, \quad x \underset{a}{\cdot} (y \underset{a}{\cdot} x) = (x \underset{a}{\cdot} y) \underset{a}{\cdot} x,$$

are satisfied for all $x, y, a \in \mathscr{L}$.

Proposition 3.2. *If the family of loop multiplications $x \underset{a}{\cdot} y$ defined on $\mathscr{L}$ is alternative then the torsion tensor field of its canonical connection ∇ satisfies*

$$(\nabla_Z T)(X, Y) = \frac{1}{3}\{T(T(X, Y), Z) + T(T(Y, Z), X) + T(T(Z, X), Y)\}.$$

Proof. By the assumption of the alternativity of the loop system the associator is

$$\langle X, Y, Z \rangle = \frac{1}{6}\{[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]\}$$

(cf. Remark IX. 6.18. in [5]). Since $T(X, Y) = -[X, Y]$, the assertion follows from Proposition 1.3.

Proposition 3.3. *The connection $\mathring{\nabla}$ defined by $\mathring{\nabla}_X Y = \nabla_X Y + \frac{1}{3} T(X, Y)$ is complete. Its torsion and curvature tensor fields $\mathring{T}(X, Y)$, $\mathring{R}(X, Y)Z$ satisfy*

$$\mathring{\nabla}_Z \mathring{T} = 0, \quad \mathring{\nabla}_Z \mathring{R} = 0,$$

i.e. the manifold $\mathscr{L}$ with the connection $\mathring{\nabla}$ is a locally reductive homogeneous space.

Proof. Since the connections ∇ and $\mathring{\nabla}$ have the same geodesics $\mathring{\nabla}$ is complete. The relations $\mathring{\nabla}_Z \mathring{T} = 0, \mathring{\nabla}_Z \mathring{R} = 0$ follows by standard calculations from Proposition 3.2.

Theorem 3.4. *Let $\mathscr{L}$ be a connected and simply connected manifold equipped with an alternative family of loop multiplications $x \underset{a}{\cdot} y$. Then $\mathscr{L}$ can be represented as a global reductive homogeneous space $\mathscr{L} = \mathscr{G}/\mathscr{H}$, where the Lie algebras of the Lie groups $\mathscr{G}$, $\mathscr{H}$ satisfy $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $\mathrm{Ad}_{\mathscr{H}} \mathfrak{m} \subset \mathfrak{m}$. If $\mathring{\nabla}$ is the canonical connection of the homogeneous space $\mathscr{G}/\mathscr{H}$ and $\mathring{T}$ is its torsion tensor field, then the left canonical connection ∇ of the family of loop multiplications takes the form*

$$\nabla_X Y = \mathring{\nabla}_X Y - \mathring{T}(X, Y).$$

Proof. The assertion follows from the preceding propositions because a complete, connected and simply connected locally reductive homogeneous space is a global one (cf. Chap. X. Theorem 2.8. in [7]).

4. Geodesic loops of the left canonical connection

Proposition 4.1. *Let $\mathscr{L}$ be a Moufang loop and let ∇ denote its left canonical connection. If $x(t)$ is a geodesic through the point $a=x(0)$ with respect to ∇ then $x(t)$ is a one-parameter subgroup of the translated multiplication $x \underset{a}{\cdot} y$, and the parallel translation along $x(t)$ coincides with the map $\tau(x(t))\circ\tau(a)^{-1}$, where $\tau(x)=\mathscr{T}_e\lambda_x$. Consequently the loop multiplication can be written in the form*

$$x \underset{a}{\cdot} y = \exp_x\circ\tau(x)\circ\tau(a)^{-1}\circ\exp_a^{-1} y$$

in a neighbourhood of a.

Proof. Since the Moufang loops are alternative and their isotops are also Moufang loops, the family of loop multiplications $x \underset{a}{\cdot} y$ consists of diassociative loops (cf. Chap. VI. in [3]). The geodesic loop multiplication $x \underset{a}{\cdot} y=\exp_x\circ\tau(x)\circ$ $\circ\tau(a)^{-1}\circ\exp_a^{-1} y$ of the left canonical connection of the diassociative loops coincides with the original multiplication because the geodesics of the left canonical connection of the diassociative loops are the curves $x\cdot\exp tY$ $(x\in\mathscr{L}, Y\in\mathscr{T}_e\mathscr{L})$, where $\exp tY$ is the 1-parameter subgroup of $\mathscr{L}$ tangent to $Y\in\mathscr{T}_e\mathscr{L}$. Hence the assertion follows from the results in the Section 1.

Proposition 4.2. *Let $x \underset{a}{\circ} y$ denote the local loop multiplication with identity element a defined by*

$$x \underset{a}{\circ} y = x^{2/3} \underset{a}{\cdot} (y \underset{a}{\cdot} x^{1/3})$$

on a Moufang loop $\mathscr{L}$. If $x(t)$ is a geodesic with respect to the left canonical connection ∇ through $x(0)=a$, then it is a one-parameter subgroup of the loop with multiplication $x \underset{a}{\circ} y$. The parallel translation along the one-parameter subgroup $x(t)$ with respect to the connection $\mathring{\nabla}_X Y=\nabla_X Y+\frac{1}{3}T(X, Y)$ coincides with the map $\mathscr{T}_a\mathring{\lambda}_a(x(t))$, where $\mathring{\lambda}_a$ denotes the left multiplication map $\mathring{\lambda}_a(x)y=x \underset{a}{\circ} y$. Consequently,

$$x \underset{a}{\circ} y = \exp_x\circ\mathscr{T}_a\mathring{\lambda}_a\circ\exp_a^{-1} y$$

in a neighbourhood of a.

Proof. We know, that the loop multiplications $x \underset{a}{\cdot} y$ $(a\in\mathscr{L})$ have 1-parameter subgroups in each direction $X\in\mathscr{T}_a\mathscr{L}$, that are geodesics of the connection ∇. Since $a \underset{e}{\cdot} \exp_e tY$ is a 1-parameter subgroup of the multiplication $x \underset{a}{\cdot} y$, it can be written in the form

$$a \underset{e}{\cdot} \exp_e tY = \exp_a t\mathscr{T}_e\lambda_e(a)Y.$$

Similarly, $x^{-1} \underset{e}{\cdot} (\exp_e tY \underset{e}{\cdot} x)$ is a 1-parameter subgroup of the multiplication $x \underset{e}{\cdot} y$, hence it can be written in the form $\exp_e t\mathcal{T}_e\lambda_e(x)^{-1}\circ\mathcal{T}_e\varrho_e(x)Y$. Thus we have

$$x\underset{e}{\circ}\exp_e tY = x\underset{e}{\cdot}(x^{-1/3}\underset{e}{\cdot}(\exp_e tY\underset{e}{\cdot}x^{1/3})) = x\underset{e}{\cdot}(\exp_e t\mathcal{T}_e\lambda_e(x^{1/3})^{-1}\circ\mathcal{T}_e\varrho_e(x^{1/3})Y) =$$
$$= \exp_x t\mathcal{T}_e\lambda_e(x^{2/3})\circ\mathcal{T}_e\varrho_e(x^{1/3})Y$$

and so

$$x\underset{e}{\circ}y = \exp_x\mathcal{T}_e\lambda_e(x^{2/3})\circ\mathcal{T}_e\varrho_e(x^{1/3})\exp^{-1}y$$

follows.

We prove now that the mapping $\mathcal{T}_e\lambda_e(x^{2/3})\circ\mathcal{T}_e\varrho_e(x^{1/3})$ is the parallel translation of the connection $\mathring{\nabla}$ along the geodesic segment $\exp_e tX$ $(0\leq t\leq 1)$, where $\exp_e X=x$. First, we note that the 1-parameter groups of the multiplications $x\underset{a}{\cdot}y$ and $x\underset{a}{\circ}y$ coincide for all $a\in\mathcal{L}$. Let $Y(t)$ denote the vector field

$$Y(t) = \mathcal{T}_e\lambda_e\left(\exp_e\frac{2}{3}tX\right)\circ\mathcal{T}_e\varrho_e\left(\exp_e\frac{1}{3}tX\right)Y_0$$

along $\exp_e tX$, where $Y_0\in\mathcal{T}_e\mathcal{L}$. If $x_0=\exp_e t_0X$, $y_0=\exp_e\left(\frac{2}{3}t_0X\right)\underset{e}{\cdot}y\underset{e}{\cdot}\exp_e\left(\frac{1}{3}t_0X\right)$ and $X(t_0)=\mathcal{T}_e\lambda_e(x_0)X$ then we can write

$$= \exp_e\frac{2}{3}(t-t_0)X\underset{e}{\cdot}y_0\underset{e}{\cdot}\exp_e\frac{1}{3}(t-t_0)X = \exp_e\frac{2}{3}tX\underset{e}{\cdot}y\underset{e}{\cdot}\exp_e\frac{1}{3}tX.$$

Now

$$Y(t) = \mathcal{T}_{x_0}\lambda_{x_0}\left(\exp_{x_0}\frac{2}{3}(t-t_0)X(t_0)\right)\circ\mathcal{T}_{x_0}\varrho_{x_0}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)Y(t_0)$$

follows, and then

$$(\nabla_X Y)(t_0) = \frac{d}{dt}\{\mathcal{T}_{x_0}\lambda_{x_0}^{-1}(\exp_{x_0}(t-t_0)X(t_0))Y(t)\}_{t_0} =$$
$$= \frac{d}{dt}\left\{\mathcal{T}_{x_0}\lambda_{x_0}^{-1}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)\circ\mathcal{T}_{x_0}\varrho_{x_0}\left(\exp_{x_0}\frac{1}{3}(t-t_0)X(t_0)\right)Y(t_0)\right\}_{t_0}.$$

We introduce a coordinate system around x_0 in which the multiplication $x\underset{x_0}{\cdot}y$ is of the form

$$x\underset{x_0}{\cdot}y = x+y+q(x,y)+r(x,x,y)+s(x,y,y)+\{\text{higher order terms}\}.$$

Then we obtain that

$$(\nabla_X Y)(t_0) = \frac{1}{3}q(X(t_0),Y(t_0))+\frac{1}{3}q(Y(t_0),X(t_0)) = \frac{1}{3}T(X(t_0),Y(t_0)),$$

or

$$(\mathring{\nabla}_X Y)(t_0) = (\nabla_X Y)(t_0)-\frac{1}{3}T(X(t_0),Y(t_0)) = 0.$$

Thus $\mathcal{T}_e\mathring{\lambda}_e=\mathcal{T}_e\lambda_e\left(\exp_e\frac{2}{3}tX\right)\circ\mathcal{T}_e\varrho_e\left(\exp_e\frac{1}{3}tX\right)$ gives the parallel translation along $\exp_e tX$ with respect to the connection $\mathring{\nabla}$, which was to be proved.

Theorem 4.3. *Let $\mathcal{L}$ be a connected Moufang loop with the multiplication $x\underset{e}{\cdot}y$ and $\mathcal{G}$ is the Lie transformation group generated by the maps $\mathring{\lambda}_e(x):=\lambda_e(x^{2/3})\varrho_e(x^{1/3})$ $(x\in\mathcal{L})$. Let $\mathcal{H}$ be the isotropy subgroup of $\mathcal{G}$ at $e\in\mathcal{L}$. The loop can be represented as a reductive homogeneous space $\mathcal{G}/\mathcal{H}$ with the reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$, respectively. The complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ consists of the tangent vectors of the one parameter subgroups $\{\mathring{\lambda}_e(x(t))\}$ at the identity of $\mathcal{G}$, where the curves $\{x(t)\}$ are the one parameter subgroups in the loop $\mathcal{L}$. Let $\mathring{T}$ be the torsion tensor field of the canonical connection $\mathring{\nabla}$ of $\mathcal{G}/\mathcal{H}$ and let ∇ be the invariant connection of the homogeneous space $\mathcal{G}/\mathcal{H}$ defined by*

$$\nabla_X Y=\mathring{\nabla}_X Y-\mathring{T}(X,Y).$$

Then the multiplication $x\underset{e}{\cdot}y$ locally coincides with the geodesic loop multiplication of ∇.

Proof. Let $\mathcal{L}$ be a connected Moufang loop. The translated multiplications $x\underset{a}{\cdot}y:=x\cdot a\backslash y$ locally coincide with the geodesic loop multiplications of the canonical connection ∇. Let $p\colon\bar{\mathcal{L}}\to\mathcal{L}$ be the universal covering of the loop $\mathcal{L}$. The kernel $p^{-1}(e)$ is a central abelian discrete subgroup of $\bar{\mathcal{L}}$, which is naturally isomorphic to the fundamental group of $\mathcal{L}$ (cf. Proposition IX. 1.24. in [5]). Since $p\colon\bar{\mathcal{L}}\to\mathcal{L}$ is a covering homomorphism it is covering homomorphism for the translated multiplications $x\underset{a}{\cdot}y:=x\cdot a\backslash y$ too. Let $\bar{\nabla}$ denote the covering connection of ∇ defined on the manifold $\bar{\mathcal{L}}$. It is clear from the construction of the covering loop multiplication on $\bar{\mathcal{L}}$ and of the covering connection $\bar{\nabla}$ that $\bar{\mathcal{L}}$ is a Moufang loop and the translated multiplications $x\underset{a}{\cdot}y$ on $\bar{\mathcal{L}}$ locally coincide with the geodesic loop multiplications of ∇. By Proposition 3.3. $\bar{\mathcal{L}}$ is a locally reductive homogeneous space with the connection $\bar{\nabla}$. Since it is simply connected, it can be represented as a global homogeneous space $\bar{\mathcal{L}}=\bar{\mathcal{G}}/\bar{\mathcal{H}}$, where $\bar{\mathcal{G}}$ is the transvection group (cf. Theorem I. 25. in [7]) of $\bar{\mathcal{L}}$ generated by the affine transformations having the local representation

$$\mathring{\lambda}_x(y)=\exp_y\circ\mathcal{T}_x\mathring{\lambda}_x(y)\circ\exp_x^{-1},$$

where

$$\mathring{\lambda}_x(y):=\bar{\lambda}_x(y^{2/3})\bar{\varrho}_x(y^{1/3})$$

and $\bar{\lambda}_x(y)$ is the left multiplication map of the translated covering loop $x\underset{a}{\cdot}y$ on $\bar{\mathcal{L}}$. Since the mappings $\mathring{\lambda}_e(z)$ are isomorphisms between the multiplications $\mathring{\lambda}_e(x)y$ and

$\mathring{\lambda}_z(x)y$ we have $\mathring{\lambda}_e(z)\mathring{\lambda}_e(x)y=\mathring{\lambda}_z(\mathring{\lambda}_e(z)x)\mathring{\lambda}_e(z)y$. With the notation $u=\mathring{\lambda}_e(z)x$, we obtain

$$\mathring{\lambda}_z(u) = \mathring{\lambda}_e(z)\,\mathring{\lambda}_e\big(\mathring{\lambda}_e(z)^{-1}u\big)\,\mathring{\lambda}_e(z)^{-1}.$$

Thus the transvection group $\bar{\mathcal{G}}$ is generated by the maps $\mathring{\lambda}_e(z)$, $z\in\bar{\mathcal{L}}$. Consequently, the subgroup generated by the maps $\mathring{\lambda}_e(t)$, $t\in p^{-1}(e)$ in $\bar{\mathcal{G}}$ is central and the homomorphism $p\colon\bar{\mathcal{L}}\to\mathcal{L}$ can be extended to a homomorphism $\lambda(p)\colon\bar{\mathcal{G}}\to\mathcal{G}$ so that $\lambda(p)(\mathring{\lambda}_e(z))=\mathring{\lambda}_e(z)$ and the group $\mathcal{G}$ is generated by the maps $\mathring{\lambda}_e(z):=\lambda_e(x^{2/3})\,\varrho_e(x^{1/3})$ $(x\in\mathcal{L})$ and acts transitively on $\mathcal{L}$.

Since the complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ correspond to the subspace spanned by the tangent vectors of the parallel translated frames in the linear frame bundle over $\mathcal{L}$, we obtain from Proposition 4.2. that $\mathfrak{m}$ consists of the tangent vectors of the one parameter subgroups $\{\mathring{\lambda}_e(x(t))\}$ at the identity of $\mathcal{G}$. Thus the assertion follows from Theorem 3.4.

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