Acta Sci. Math., 56 (1992), 269—285

Basic cohomology classes of compact Sasakian manifolds

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1. Introduction and preliminaries. It was proved in [G1] that for any compact $(2m+1)$-dimensional Sasakian manifold M the following inequality is satisfied:

$$(1.1)\quad \int_M \left(|S|^2-\frac{1}{2}\varrho^2+2\varrho\right)dV+\frac{m-1}{2m\operatorname{Vol}(M)}\Big(\int_M \varrho\, dV\Big)^2 \geqq 2m(2m+1)\operatorname{Vol}(M),$$

where $|S|$, ϱ, $\operatorname{Vol}(M)$, and dV are the length of the Ricci tensor, the scalar curvature, the volume of M, and the Riemannian measure on M, respectively. Inequality (1.1) was applied in [G1] to a study of cohomologically Einstein—Sasakian manifolds. The purpose of this paper is to prove a set of inequalities for basic cohomology classes of compact Sasakian manifolds. The simplest of these inequalities is equivalent to inequality (1.1).

Let M be a $(2m+1)$-dimensional differentiable manifold (in what follows we assume the all manifolds, maps, differential forms, etc. are of class C^∞). Assume that M carries a global differential 1-form η such that $\eta\wedge(d\eta)^m\neq 0$ everywhere on M. Then we say that η defines a *contact structure* on M. A manifold M furnished with a contact structure η is called a *contact manifold.* It is known, [B], that a contact manifold (M,η) admits a unique global vector field X_0 satisfying $\eta(X_0)=1$ and $d\eta(X_0,X)=0$ for any tangent vector field X on M. X_0 is called the *characteristic vector field* of a contact manifold (M,η). Since vector field X_0 nowhere vanishes, M can be considered as a foliated manifold with 1-dimensional leaves. Let ω be a $\mathbf{F}$-valued differential k-form on a contact manifold (M,η), where $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$. We say that ω is *horizontal* if $i(X_0)\omega=0$, *invariant* if $L_{X_0}\omega=0$, and *basic* if it is horizontal and invariant. Here $i(X_0)$ and L_{X_0} are the inner product by X_0 and the Lie derivative, respectively. Denote by $A_B(M,\eta,\mathbf{F})$ (resp. $A_B^k(M,\eta,\mathbf{F})$) the set of all $\mathbf{F}$-valued basic forms (resp. basic k-forms), and by $C_B(M,\eta,\mathbf{F})$ (resp. $C_B^k(M,\eta,\mathbf{F})$) the set of all $\mathbf{F}$-valued closed basic forms (resp. closed basic k-forms) on M. It is easy to see that $dA_B^{k-1}(M,\eta,\mathbf{F})\subset C_B^k(M,\eta,\mathbf{F})$. Set $H_B^k(M,\eta,\mathbf{F})=C_B^k(M,\eta,\mathbf{F})/$

Received October 11, 1990.

$/dA_B^{k-1}(M,\eta,\mathbf{F})$. $H_B^k(M,\eta,\mathbf{F})$ is called the k^{th} *basic cohomology group* of (M,η) over $\mathbf{F}$. In what follows we shall usually write $H_B^k(M)$ or $H_B^k(M,\mathbf{F})$ instead of $H_B^k(M,\eta,\mathbf{F})$, and similarly for $A_B^k(M,\eta,\mathbf{F})$ and $C_B^k(M,\eta,\mathbf{F})$. It is easy to see that if $\lambda\in C_B^k(M)$, $\mu\in C_B^l(M)$, then $\lambda\wedge\mu\in C_B^{k+l}(M)$, and if $\lambda\in C_B^k(M)$, $\mu\in dA_B^{l-1}(M)$, then $\lambda\wedge\mu\in dA_B^{k+l-1}(M)$. Therefore, for any $\alpha\in H_B^k(M)$, $\beta\in H_B^l(M)$, we have a well-defined product $\alpha\cdot\beta\in H_B^{k+l}$. Clearly,

$$H_B^0(M,\mathbf{F})=\mathbf{F},\quad H_B^k(M)=\{0\}\quad \text{for}\quad k\geqq 2m+1.$$

Generally, $\dim_{\mathbf{F}} H_B^k(M,\mathbf{F})$, $k=1,\dots,2m$, may be infinite. However, for "good" contact structures (such as K-structures or Sasakian structures) $\dim_{\mathbf{R}} H_B^k(M,\mathbf{R})=$ $=\dim_{\mathbf{C}} H_B^k(M,\mathbf{C})<\infty$.

A contact manifold (M,η) is called regular, [B], if X_0 is a regular vector field on M, that is every point $x\in M$ has a cubical coordinate neighborhood $\mathcal{U}$ such that the integral curves of X_0 passing through $\mathcal{U}$ pass through the neighborhood only once. It is known, [B], that any compact regular $(2m+1)$-dimensional contact manifold M is the bundle space of a principle circle bundle $\pi\colon M\to B$ over a $2m$-dimensional simplectic manifold B. It is easy to show that in the case of a compact regular contact manifold $H_B^k(M)$ is the pullback of $H^k(B)$, where $H^k(B)$ is the DeRham cohomology group of B.

Let (M,η) be a contact manifold. In what follows we will always use the following notation:

$$\Phi = d\eta. \tag{1.2}$$

Φ is a closed basic form. Therefore Φ represents a basic cohomology class. In what follows we will denote this cohomology class by Ω. $\Omega\in H_B^2(M)$ is called the *fundamental basic cohomology class.*

For a compact contact $(2m+1)$-dimensional manifold (M,η) we now define a linear function $I\colon A_B(M,\mathbf{F})\to\mathbf{F}$ from the set of all basic $\mathbf{F}$-valued forms on M into $\mathbf{F}$ as follows: If $\omega\in A_B^{2k}(M,\mathbf{F})$, $k=0,1,\dots,m$, then

$$I(\omega)=\frac{1}{2^m m!\,\mathrm{Vol}(M)}\int_M \eta\wedge\Phi^{m-k}\wedge\omega. \tag{1.3}$$

If $\omega\in A_B^{2k+1}(M,\mathbf{F})$, $k=1,\dots,m$, then $I(\omega)=0$. We shall denote by the same symbol I a function $I\colon H_B(M,\mathbf{F})\to\mathbf{F}$ defined as follows: Let $\alpha\in H_B(M,\mathbf{F})$ and let ω be a closed basic form representing α. Then, by definition,

$$I(\alpha)=I(\omega). \tag{1.4}$$

We will show in Sec. 2 that $I(\alpha)$ is well-defined by formula (1.4), that is $I(\alpha)$ does not depend on a particular choice of a basic form ω representing α. It is clear from the definition of I that

$$
(1.5) \qquad \begin{aligned} I(\Phi^k \wedge \omega) &= I(\omega), \quad \text{if} \quad \omega \in A_B^{2l} \quad \text{and} \quad l+k \leqq m; \\ I(\Omega^k \cdot \alpha) &= I(\alpha), \quad \text{if} \quad \alpha \in H_B^{2l} \quad \text{and} \quad l+k \leqq m. \end{aligned}
$$

By [S], page 3—4, $\int \eta \wedge \Phi^m = 2^m m! \operatorname{Vol}(M)$. Therefore

$$
(1.6) \qquad I(\Phi^k) = I(\Omega^k) = 1, \quad 0 \leqq k \leqq m.
$$

Let (M, η) be a contact manifold. An associated contact metric structure, [B], for a contact structure η is a collection (η, X_0, φ, g), where X_0 is the characteristic vector field, φ is a field of automorphisms of the tangent spaces of M, and g is a Riemannian metric on M such that

$$
\begin{aligned} \varphi^2(X) &= -X + \eta(X) X_0, \\ \eta(X) &= g(X, X_0), \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \Phi(X, Y) &= g(X, \varphi Y), \end{aligned}
$$

for any tangent vector fields X and Y on M. An associated contact metric structure for a contact structure η always exists, but is not unique, [B]. We say that a contact metric structure (η, X_0, φ, g) on M is *normal*, [B], if the almost complex structure T on $M \times \mathbf{R}$ defined by $T\left(X, f\frac{d}{dt}\right) = \left(\varphi X - fX_0, \eta(X)\frac{d}{dt}\right)$ is integrable. A differentiable manifold M furnished with a normal contact metric structure (η, X_0, φ, g) is called a *Sasakian manifold.*

Let $(M, \eta, X_0, \varphi, g)$ be a $(2m+1)$-dimensional Sasakian manifold. For $x \in M$, set

$$
(1.7) \qquad D_x = \{X \in TM_x \colon \eta(X) = 0\}.
$$

D_x is called the *horizontal subspace* at the point x. By (1.7), φ induces an almost complex structure (once more denoted dy φ) on D_x. Denote by $D_x^{\mathbf{C}}$ the complexification of D_x. Then $D_x^{\mathbf{C}} = D_x^{1,0} \oplus D_x^{0,1}$, where

$$
(1.8) \qquad \begin{aligned} D_x^{1,0} &= \{X \in D_x^{\mathbf{C}} \colon \varphi X = \sqrt{-1}\, X\}, \\ D_x^{0,1} &= \{X \in D_x^{\mathbf{C}} \colon \varphi X = -\sqrt{-1}\, X\}. \end{aligned}
$$

It follows that the set $\operatorname{Hor}^p(M)$ of all $\mathbf{C}$-valued horizontal p-forms on M may be bigraded as follows;

$$
\operatorname{Hor}^p(M) = \sum_{k+l=p} \operatorname{Hor}^{k,l}(M),
$$

where $\operatorname{Hor}^{k,l}(M)$ is the set of all horizontal $(k+l)$-forms on M which can obtain non-zero values only for sets of vectors $X_1, \dots, X_{k+l} \in TM_x^{\mathbf{C}}$ among which k vectors belong to $D_x^{1,0}$ and l vectors belong to $D_x^{0,1}$.

Let $\alpha \in H_B^{k+l}(M, \mathbf{C})$. We say that α is *of the type* (k, l), if there is a basic form ω representing α, such that $\omega \in \mathrm{Hor}^{k,l}(M)$. We will see in Sec. 2 that for a $(2m+1)$-dimensional compact Sasakian manifold the notion for $\alpha \in H_B^p(M), (0 \leqq p \leqq m)$, to be of the type (k, l) is well defined. That means that if $\omega \in \mathrm{Hor}^{k,l}(M)$ and $\tau \in \mathrm{Hor}^{r,s}(M)$ represent the same basic cohomology class $\alpha \in H_B^p(M, \mathbf{C})$, then $k=r$ and $l=s$. For $0 \leqq k+l \leqq m$, set

$$(1.9) \qquad H_B^{k,l}(M) = \{\alpha \in H_B^{k+l}(M, \mathbf{C}) \colon \alpha \text{ is of the type } (k, l)\}.$$

Then $H_B^{k,l}$ is a subgroup (as an additive group) of $H_B^{k+l}(M, \mathbf{C})$. We will show in Sec. 2 that for a compact Sasakian manifold there is a direct sum decomposition

$$H_B^p(M, \mathbf{C}) = \sum_{k+l=p} H_B^{k,l}(M), \quad 0 \leqq p \leqq m.$$

For $0 \leqq k+l \leqq m$, set

$$h_B^{k,l} = \dim_{\mathbf{C}} H_B^{k,l}(M).$$

$h_B^{k,l}$ will be called the *basic Hodge number of the type* (k, l). By (1.3), $\Phi \in \mathrm{Hor}^{1,1}(M)$. Hence $\Omega^k \in H_B^{k,k}(M)$. By (1.6), $\Omega^k \neq 0$. Therefore

$$h_B^{0,0} = 1, \quad h_B^{k,k} \geqq 1, \quad k = 1, \dots, \left[\frac{m}{2}\right].$$

Moreover, we will show in Sec. 2 that

$$1 = h_B^{0,0} \leqq h_B^{1,1} \leqq \dots \leqq h_B^{[m/2],[m/2]}.$$

In Sec. 3 we prove the main result of this paper:

Theorem 1.1. *Let $(M, \eta, X_0, \varphi, g)$ be a compact $(2m+1)$-dimensional Sasakian manifold and let k be an integer such that $1 \leqq k \leqq \dfrac{m}{2}$. Assume that $h_B^{k-1,k-1} = 1$. Let $\alpha \in H_B^{k,k}(M)$. Then*

$$(1.10) \qquad (-1)^k [I(\alpha \cdot \bar{\alpha}) - I(\alpha) I(\bar{\alpha})] \geqq 0,$$

and the equality holds if and only if $\alpha = t\Omega^k$, $t \in \mathbf{C}$. Here $\bar{\alpha}$ means the complex conjugate of α.

Taking $k=1$ in Theorem 1.1, we obtain

Corollary 1.2. *Let M be a compact Sasakian $(2m+1)$-dimensional $(m \geqq 2)$ manifold and let $\alpha \in H_B^{1,1}(M)$. Then*

$$(1.11) \qquad I(\alpha \cdot \bar{\alpha}) - I(\alpha) I(\bar{\alpha}) \leqq 0$$

and the equality holds if and only if $\alpha = t\Omega$, where $t \in \mathbf{C}$.

It follows easily from the results of Sec. 2 that if $b_2(M)=0$, where $b_2(M)$ is the second Betti number of M, then $h_B^{1,1}=1$. Hence, taking $k=2$ in Theorem 1.1, we obtain

Corollary 1.3. *Let M be a compact Sasakian $(2m+1)$-dimensional $(m\geqq 4)$ manifold and let $\alpha\in H_B^{2,2}(M)$. If $b_2(M)=0$, then*

$$I(\alpha\cdot\bar{\alpha})-I(\alpha)I(\bar{\alpha})\geqq 0, \tag{1.12}$$

and the equality holds if and only if $\alpha=t\Omega^2$, where $t\in\mathbf{C}$.

In Sec. 4 for any $(2m+1)$-dimensional Sasakian manifold and for any $k=$ $=1,\dots,m$ we introduce a canonical real closed basic form $C_k^{(B)}$ of bidegree (k,k). We will call this form the basic Chern form of a Sasakian manifold. Substituting $C_k^{(B)}$ instead of α in (1.10), we obtain an integral inequality similar to inequality (1.1). In the simplest case, when $k=1$, we obtain inequality (1.1).

If M is a regular Sasakian manifold, then M is a unit circle bundle over a Kaehler manifold B. It is easy to see that in this case the basic Chern form $C_k^{(B)}$ belongs to a basic cohomology class which is the pull-back of the Chern class $C_k(B)$. It was shown in [G2] that for $B=P^2(\mathbf{C})\times P^3(\mathbf{C})$,

$$I(C_2(B)\cdot C_2(B))-I(C_3(B))\cdot I(C_2(B))<0.$$

Hence, if a Sasakian manifold M is a unit circle bundle over $B=P^2(\mathbf{C})\times P^3(\mathbf{C})$, then

$$I(C_2^{(B)}(M)\cdot C_2^{(B)}(M))-I(C_2^{(B)}(M))\cdot I(C_2^{(B)}(M))<0.$$

Comparing this inequality with inequality (1.12), we see that the condition $b_2(M)=0$ in Corollary 1.3 cannot be omitted. More generally, this example shows that the condition $h_B^{k-1,k-1}=1$ in Theorem 1.1 is essential.

We conclude Sec. 4 by Remark showing how one can define basic Pontrjagin classes $P_k^{(B)}\in H_B^{4k}(M,\mathbf{R})$, $k=1,\dots,[m/2]$, on K-contact manifolds.

Finally we note that for Kaehler manifolds a theorem similar to Theorem 1.1 has been proved in [G2].

2. Decomposition theorems. For a compact metric manifold (M,η,X_0,φ,g) we will denote by $\langle\,,\rangle$ the local scalar product with respect to the Riemannian metric g, and by $(\lambda,\mu)=\int_M\langle\lambda,\mu\rangle\,dV$ the global scalar product, where λ and μ are differential forms of the same degree. As usual, $*$ will be the Hodge "star" operator and δ will be the adjoint of the operator of exterior differentiation, i.e. $(d\lambda,\mu)=(\lambda,\delta\mu)$, where λ and μ are forms of degrees p and $p+1$, respectively. We also will denote by $e(\eta)\lambda$ the exterior product by η, i.e. $e(\eta)\lambda=\eta\wedge\lambda$. Clearly, $(i(X_0)\lambda,\mu)=(\lambda,e(\eta)\mu)$ for any two differential forms λ and μ of degrees $p+1$ and p respectively.

Lemma 2.1. *Let (M, η) be a compact $(2m+1)$-dimensional contact manifold. Then the function I: $H_B(M, \mathbf{F}) \to \mathbf{F}$ given by formulas* (1.3) *and* (1.4) *is well-defined.*

Proof. Let λ and λ_1 be basic closed $2k$-forms representing the same basic cohomology class $\alpha \in H_B^{2k}(M)$. Then $\lambda - \lambda_1 = d\mu$ where μ is a basic $(2k-1)$-form.

We must prove that $\int_M \eta \wedge \Phi^{m-k} \wedge \lambda = \int_M \eta \wedge \Phi^{m-k} \lambda_1$. Therefore we must prove that $\int_M \eta \wedge d\omega = 0$, where $\omega = \Phi^{m-k} \wedge \mu$. Clearly, ω is a basic form. Let (η, X_0, φ, g) be a contract metric structure on M associated with contact structure η. By [S], page 3—4,

$$*1 = \frac{1}{2^m m!} \eta \wedge \Phi^m. \tag{2.1}$$

Hence,

$$\int_M \eta \wedge d\omega = (\eta \wedge d\omega, *1) = \frac{1}{2^m m!} (e(\eta)\, d\omega, e(\eta)\Phi^m) =$$

$$= \frac{1}{2^m m!} (d\omega, i(X_0)e(\eta)\Phi^m) = \frac{1}{2^m m!} (d\omega, \Phi^m) = \frac{1}{2^m m!} (\omega, \delta\Phi^m).$$

By [SH],

$$\delta\Phi^r = 4r(m-r+1)\eta \wedge \Phi^{r-1}. \tag{2.2}$$

Therefore,

$$\int_M \eta \wedge d\omega = \frac{4m}{2^m m!} (\omega, e(\eta)\Phi^{m-1}) = \frac{4m}{2^m m!} (i(X_0)\omega, \Phi^{m-1}) = 0,$$

since $i(X_0)\omega = 0$.

Corollary. *For any basic form λ, $I(d\lambda) = 0$.*

From now and to the end of this section let $(M, \eta, X_0, \varphi, g)$ be a compact $(2m+1)$-dimensional Sasakian manifold. Let us denote by d_B and $(\,,\,)_B$ the restriction of the exterior differential and of the global scalar product on the space $A_B(M)$ of basic forms on M. Let δ_B: $A_B(M) \to A_B(M)$ be the adjoint operator for d_B with respect to $(\,,\,)_B$. Then $\Delta_B = \delta_B d_B + d_B \delta_B$ is called the *basic Laplacian*. The set $\mathfrak{H}_B^k$ of *basic harmonic k-forms* is the kernel of Δ_B on $A_B^k(M)$. Any Sasakian manifold M can be considered as a foliated manifold with 1-dimensional leaves. By the Main Theorem of [KT] (whose conditions are obviously satisfied for Sasakian manifolds), we have

$$A_B^k(M) \cong \Delta_B(A_B^k) \oplus \mathfrak{H}_B^k(M) \tag{2.3}$$

and $\dim_{\mathbf{C}} \mathfrak{H}_B^k < \infty$. It follows from (2.3) that

$$A_B^K(M, \mathbf{C}) = \operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \mathfrak{H}_B^k(M). \tag{2.4}$$

As usual we obtain from (2.4) that $H_B^k(M, \mathbf{C}) \cong \mathfrak{H}_B^k(M)$.

Let $TM_x^{\mathbf{C}}$ be the complexified tangent space at the point $x \in M$. Then

$$TM_x^{\mathbf{C}} = D_x^{1,0} \oplus D_x^{0,1} \oplus \mathbf{C}X_0, \tag{2.5}$$

where $D_x^{1,0}$ and $D_x^{0,1}$ are defined by (1.8). It is known, [I], that the pair of complex distributions $(D_x^{1,0}, D_x^{0,1})$ defines a C—R structure on M. Hence each of the distributions $D_x^{1,0}$ and $D_x^{0,1}$ is integrable. Let $\{e_i, e_{\bar{i}}, X_0\}$, $i=1, \ldots, m$; $\bar{i}=m+1, \ldots, 2m$, be a local field of frames adapted to the decomposition (2.6). That means that at the point x each $e_i \in D_x^{1,0}$ and each $e_{\bar{i}} \in D_x^{0,1}$. Let $\{\theta^i, \theta^{\bar{i}}, \eta\}$ be the dual basis of $\mathbf{C}$-valued 1-forms on M. Then, by Frobenius' theorem

$$d\theta^i \equiv 0 \ (\operatorname{mod} \theta^j,\ j = 1, \ldots, m) \quad \text{and} \quad d\theta^{\bar{i}} \equiv 0 \ (\operatorname{mod} \theta^{\bar{j}},\ \bar{j} = m+1, \ldots, 2m).$$

Therefore

$$\begin{aligned} d\theta^i &= \sum a^i_{jk}\theta^j \wedge \theta^k + \sum a^i_{j\bar{k}}\theta^j \wedge \theta^{\bar{k}} + \sum b^i_j \eta \wedge \theta^j, \\ d\theta^{\bar{i}} &= \sum a^{\bar{i}}_{\bar{j}k}\theta^{\bar{j}} \wedge \theta^k + \sum a^{\bar{i}}_{\bar{j}\bar{k}}\theta^{\bar{j}} \wedge \theta^{\bar{k}} + \sum b^{\bar{i}}_{\bar{j}} \eta \wedge \theta^{\bar{j}}, \end{aligned}$$

where $a^i_{jk}, a^i_{j\bar{k}}, a^{\bar{i}}_{\bar{j}k}, a^{\bar{i}}_{\bar{j}\bar{k}}, b^i_j, b^{\bar{i}}_{\bar{j}}$ are functions. It follows that for any horizontal form $\omega \in \operatorname{Hor}^{k,l}(M)$ of bidegree (k, l)

$$d\omega = \omega' + \omega'' + \eta \wedge \omega''', \tag{2.6}$$

where $\omega' \in \operatorname{Hor}^{k+1,l}(M)$, $\omega'' \in \operatorname{Hor}^{k,l+1}(M)$, $\omega''' \in \operatorname{Hor}^{k,l}(M)$. Assume now that ω is basic. Then $0 = i(X_0)d\omega = \omega'''$. Therefore $d\omega = \omega' + \omega''$. Set $d\omega' = \lambda' + \eta \wedge \mu'$, $d\omega'' = \lambda'' + \eta \wedge \mu''$, where λ', λ'', μ', and μ'' are horizontal. If follows that $0 = d\omega' + d\omega'' = (\lambda' + \lambda'') + \eta(\mu' + \mu'')$. Hence $\mu' + \mu'' = 0$. Since $\mu' \in \operatorname{Hor}^{k+1,l}(M)$ and $\mu'' \in \operatorname{Hor}^{k,l+1}(M)$, we obtain that $\mu' = \mu'' = 0$. Hence $d\omega'$ and $d\omega''$ are horizontal and therefore ω' and ω'' are basic. It follows that if $\omega \in A_B^{k,l}(M)$, where $A_B^{k,l}(M)$ is the set of basic forms on M of bidegree (k, l), then $d\omega = \omega' + \omega''$, where $\omega' \in A_B^{k+1,l}(M)$ and $\omega'' \in A_B^{k+1,l}(M)$. Set $d'_B\omega = \omega'$, $d''_B\omega = \omega''$. Then we obtain that $d_B = d'_B + d''_B$, where d'_B and d''_B are differential operators on $A_B(M, \mathbf{C})$ of bidegrees (1, 0) and (0, 1), respectively. Let $\delta'_B \colon A_B(M, \mathbf{C}) \to A_B(M, \mathbf{C})$ and $\delta''_B \colon A_B(M, \mathbf{C}) \to A_B(M, \mathbf{C})$ be the adjoint operators for d'_B and d''_B, respectively, with respect to the global scalar product $(,)_B$. Then δ'_B and δ''_B are of bidegree $(-1, 0)$ and $(0, -1)$, respectively, and $\delta_B = \delta'_B + \delta''_B$. Set $\Delta'_B = \delta'_B d'_B + d'_B \delta'_B$, $\Delta''_B = \delta''_B d''_B + d''_B \delta''_B$.

Lemma 2.2. *Let ω be a basic p-form,* $0 \leqq p \leqq m$. *Then*

$$\Delta_B \omega = 2\Delta'_B \omega = 2\Delta''_B \omega.$$

Proof. This lemma is analogous to Theorem 3.7 of [W], Chapter V. A proof Lemma 2.2 can be obtained by repeating the arguments of the proof of the above mentioned theorem from [W], and we omit it.

Denote dy $\Pi_{k,l}$ the natural projection from $A_B(M, \mathrm{C})$ to $A_B^{k,l}(M)$. By Lemma 2.2,

$$\Delta_B \Pi_{k,l} = \Pi_{k,l}\Delta_B, \quad 0 \leqq k+l \leqq m. \tag{2.7}$$

For any differential form ω on M, set $L\omega = \Phi\wedge\omega$, where $\Phi = d\eta$. If ω is basic, then $L\omega$ is also basic. Therefore L induces the map L_B: $A_B(M)\to A_B(M)$. Denote by Λ the adjoint operator of L with respect to (,), and by Λ_B the adjoint operator of L_B with respect to $(,)_B$. Clearly L_B and Λ_B are operators of bidegrees (1, 1) and $(-1, -1)$, respectively.

Lemma 2.3.

(i) *If ω is basic, then $\Pi_{k,l}\omega$ is also basic.*

(ii) *If ω is a basic harmonic p-form and $0\leqq p\leqq m$, then $\Pi_{k,l}\omega$ is also basic harmonic.*

(iii) *If ω is a harmonic p-form and $0\leqq p\leqq m$, then $\Pi_{k,l}\omega$ is also harmonic.*

Remark. By [T1] and [Y], any harmonic p-form, $0\leqq p\leqq m$, is basic harmonic. Therefore the operator $\Pi_{k,l}$ is well-defined on the set of harmonic p-forms, $0\leqq p\leqq m$.

Proof. (i) Let $\omega\in A_B^p(M, \mathrm{C})$. Then $\omega = \omega_{0,p} + \omega_{1,p-1} + \ldots + \omega_{p,0}$, where $\omega_{k,l} = \Pi_{k,l}\omega$. By (2.6), $d\omega_{k,l} = \lambda + \eta\wedge\mu_{k,l}$, where λ is horizontal and $\mu_{k,l}$ is horizontal of bidegree (k, l). Since ω is basic,

$$0 = i(X_0)\,d\omega = i(X_0)(d\omega_{0,p} + \ldots + d\omega_{p,0}) = \mu_{0,p} + \mu_{1,p-1} + \ldots + \mu_{p,0},$$

Hence each $\mu_{k,l} = 0$. Therefore $i(X_0)d\omega_{k,l} = i(X_0)\lambda = 0$. Thus, $\omega_{k,l} = \Pi_{k,l}\omega$ is basic.

(ii) Let ω be a basic harmonic p-form, $0\leqq p\leqq m$. By (2.7), $\Delta_B(\Pi_{k,l}\omega) = \Pi_{k,l}(\Delta_B\omega) = 0$. This proves (ii).

(iii) Let λ and μ be two basic forms on M. For Sasakian manifolds, formula (3.8) from [KT] gives

$$(\Delta\lambda, \mu) = \big((\Delta_B + L\Lambda)\lambda, \mu\big). \tag{2.8}$$

Let ω be a harmonic p-form, $0\leqq p\leqq m$. Then ω and therefore $\Pi_{k,l}\omega$ are basic. Hence, by (2.7) and (2.8),

$$\big(\Delta(\Pi_{k,l}\omega), \Pi_{k,l}\omega\big) = (\Delta_B\Pi_{k,l}\omega + L\Lambda\Pi_{k,l}\omega, \Pi_{k,l}\omega) =$$
$$= (\Pi_{k,l}\Delta_B\omega + L\Pi_{k-1,l-1}\Lambda\omega, \Pi_{k,l}\omega).$$

Since any harmonic p-form, $0\leqq p\leqq m$, is basic harmonic, we have $\Delta_B\omega = 0$. By [T1], any harmonic p-form, $0\leqq p\leqq m$, is effective, i.e. $\Lambda\omega = 0$. Therefore $\big(\Delta(\Pi_{k,l}\omega), \Pi_{k,l}\omega\big) = 0$. It follows that $\big(d(\Pi_{k,l}\omega), d(\Pi_{k,l}\omega)\big) + \big(\delta(\Pi_{k,l}\omega), \delta(\Pi_{k,l})\big) = 0$. Thus, $d(\Pi_{k,l}\omega) = \delta(\Pi_{k,l}\omega) = 0$. Therefore, $\Pi_{k,l}\omega$ is harmonic. This proves (iii).

Let ω be a closed basic form of bidegree (k, l), $0\leqq k+l\leqq m$. Then, by (2.3), $\omega = \psi + \Delta_B\lambda$, where ψ is basic harmonic and λ is basic. By (2.7), $\omega = \Pi_{k,l}\omega =$

$=\Pi_{k,l}\psi+\Delta_B(\Pi_{k,l}\lambda)$. Since ψ is uniquely defined by ω and since, by Lemma 2.3, $\Pi_{k,l}\psi$ is basic harmonic, $\psi=\Pi_{k,l}\psi$. Therefore ψ is of bidegree (k, l). Thus, we obtain that if a basic cohomology class $\alpha\in H_B^{k+l}(M, \mathbf{C})$, $0\leqq k+l\leqq m$, contains a closed basic form of bidegree (k, l), its basic harmonic form is also of bidegree (k, l). Therefore, the cohomology group $H_B^{k,l}(M)$, defined by (1.9), is well-defined. By Lemma 2.3, we have a direct sum decomposition

$$(2.9)\qquad H_B^p(M, \mathbf{C}) = H_B^{0,p}(M)\oplus H_B^{1,p-1}(M)\oplus \dots \oplus H_B^{p,0}(M), \quad 0 \leqq p \leqq m.$$

Similarly, let $H^p(M, \mathbf{C})$, $0\leqq p\leqq m$, be the p^{th} DeRham cohomology group, and let $H^{k,l}(M)$ be the set of all elements of H^p which are represented by a harmonic p-form of bidegree (k, l). Then

$$(2.10)\qquad H^p(M, \mathbf{C}) = H^{0,p}(M)\oplus H^{1,p-1}(M)\oplus \dots \oplus H^{p,0}(M), \quad 0 \leqq p \leqq m.$$

Let $0\leqq p\leqq m$, $0\leqq k\leqq l\leqq m$. Set

$$b_p = \dim_{\mathbf{C}} H^p(M, \mathbf{C}), \quad b_p^{(B)} = \dim_{\mathbf{C}} H_B^p(M, \mathbf{C}),$$
$$h^{k,l} = \dim_{\mathbf{C}} H^{k,l}(M, \mathbf{C}), \quad h_B^{k,l} = \dim_{\mathbf{C}} H_B^{k,l}(M, \mathbf{C}).$$

Here b_p are usual Betti numbers. We will call $b_p^{(B)}$, $h^{k,l}$ and $h_B^{k,l}$ the *basic Betti numbers*, the *Hodge numbers*, and the *basic Hodge numbers*, respectively. By (2.9) and (2.10),

$$(2.11)\qquad \begin{aligned} b_p &= h^{0,p}+h^{1,p-1}+\dots+h^{p,0}, \quad 0 \leqq p \leqq m; \\ b_p^{(B)} &= h_B^{0,p}+h_B^{1,p-1}+\dots+h_B^{p,0}, \quad 0 \leqq p \leqq m. \end{aligned}$$

Denote by C a linear operator C: Hor $(M)\to$Hor (M) such that $C\omega=(\sqrt{-1})^{k-l}\omega$ if ω is of bidegree (k, l), where Hor (M) is the set of all horizontal forms on M. Let $*$ denote the Hodge "star" operator. Remind that ω is called effective, if $\Lambda\omega=0$.

Lemma 2.4. *Let ω be a horizontal and effective p-form, $0\leqq p\leqq m$, and let $0\leqq r\leqq m-p$. Then*

$$*(L^r\omega) = (-1)^{p(p-1)/2}\frac{r!}{2^{m-p-2r}(m-p-r)!}e(\eta)L^{m-p-r}C\omega.$$

Proof. This lemma is similar to Theorem 1.6 from [W], Chapter 5. The proof of Lemma 2.4 is just a repetition of the proof of the above mentioned theorem from [W], and we omit it.

We now prove a decomposition theorem for closed basic forms.

Theorem 2.5. *Let M be a compact $(2m+1)$-dimensional Sasakian manifold and let ω be a closed basic p-form on M. Then*

(i) *ω can be decomposed as*

$$(2.12)\qquad \omega = \sum_{i=(p-m)^+}^{[p/2]} L^i\psi_i+d\lambda,$$

where $(p-m)^+ = \max\{0, p-m\}$, ψ_i *is a harmonic* $(p-2i)$*-form, any* λ *is basic. In addition, harmonic forms* ψ_i, $i=(p-m)^+, \ldots, [p/2]$, *are uniquely defined by* ω.

(ii) *If* ω *is of bidegree* (k, l), *then for each* i, ψ_i *is of bidegree* $(k-i, l-i)$.

Proof. (i) We first consider the case $0 \leqq p \leqq m$. By (2.4), $\omega = \psi + d\lambda$, where ψ is a basic harmonic p-form uniquely defined by ω, and λ is basic. Since ψ is basic harmonic, $d_B\psi = 0$ and $\delta_B\psi = 0$. For Sasakian manifolds formula 3.3 from [KT] takes the form $\delta\psi = \delta_B\psi + e(\eta)\Lambda\psi$. Therefore

$$d\psi = 0, \quad \delta\psi = e(\eta)\Lambda\psi. \tag{2.13}$$

Differential forms on Sasakian manifolds satisfying (2.13) were introduced in [0] and were called there *C-harmonic forms*. By the decomposition theorem for C-harmonic forms of degree p, $0 \leqq p \leqq m$, [T2],

$$\psi = \sum_{i=0}^{[p/2]} L^i \psi_i,$$

where ψ_i are harmonic $(p-2i)$-forms uniquely defined by ψ. This proves (i) in the case $0 \leqq p \leqq m$.

Let now $m+1 \leqq p \leqq 2m$. Once more, $\omega = \psi + d\lambda$, where ψ is a basic harmonic form uniquely defined by ω, and λ is basic. Following [KT], for any basic q-form μ we set

$$\bar{*}\mu = (-1)^q i(X_0) * \mu.$$

Then $\bar{*}\,\bar{*}\mu = (-1)^q \mu$. By Lemma 2.4, for any horizontal and effective q-form μ, $0 \leqq q \leqq m$, and for any r, $0 \leqq r \leqq m-q$,

$$\bar{*}(L^r\mu) = (-1)^{q(q+1)/2} \frac{r!}{2^{m-q-2r}(m-q-r)!} L^{m-q-r} C\mu. \tag{2.14}$$

Set $\tilde{\psi} = \bar{*}\psi$. By [KT], $\bar{*}\Delta_B = \Delta_B \bar{*}$. Therefore $\tilde{\psi}$ is basic harmohic. Since $\tilde{\psi}$ is of degree $2m-p < m$, we have a decomposition

$$\tilde{\psi} = \sum_{j=0}^{[(2m-p)/2]} L^j \tilde{\psi}_j,$$

where $\tilde{\psi}_j$ are harmonic of degree $(2m-p-2j)$. By (2.24),

$$\psi = (-1)^{2m-p} \bar{*}\tilde{\psi} = \sum_{j=0}^{[(2m-p)/2]} \bar{*}(L^j \tilde{\psi}_j) =$$

$$= \sum_{j=0}^{[(2m-p)/2]} (-1)^{p(p+1)/2-m+j} \frac{j!}{2^{p-m}(p-m+j)!} L^{p-m+j} C\tilde{\psi}_j.$$

Set $i=p-m+j$,

$$\psi_i = (-1)^{p(p-1)/2+i}\frac{(i-p+m)!}{2^{p-m}i!}C\tilde{\psi}_{i-p+m}.$$

Then

$$\psi = \sum_{i=p-m}^{[p/2]} L^i\psi_i, \tag{2.15}$$

where we used the identity $p-m+\left[\frac{2m-p}{2}\right]=\left[\frac{p}{2}\right]$. The degree of $\tilde{\psi}_{i-p+m}$ is $(p-2i)$. Therefore deg $\tilde{\psi}_{i-p+m}\leqq m$ for $i=p-m, \dots, [p/2]$. It follows by Lemma 2.3, that $\Pi_{k,l}\tilde{\psi}_{i-p+m}$ is harmonic. Since

$$C\tilde{\psi}_{i-p+m} = \sum_{k+l=p-2i}(\sqrt{-1})^{k-l}\Pi_{k,l}\tilde{\psi}_{i-p+m},$$

we obtain that ψ_i is harmonic of degree $(p-2i)$. By (2.15), $\bar{*}\psi=\sum_{i=p-n}^{[p/2]}\bar{*}(L^i\psi_i)$. Using Lemma 2.4, we easily obtain from the last equality that ψ_i, $i=p-m, \dots, [p/2]$, are uniquely defined by ψ. This completes the proof of (i).

(ii) To prove (ii), assume that ω is of bidegree (k, l). Then, by (2.3), $\omega=\psi+$ $+\Delta_B\mu$, where ψ is basic harmonic and μ is basic. Then

$$\omega = \sum_{i=(p-m)^+}^{[p/2]} L^i\psi_i+\Delta_B\mu,$$

where ψ_i is harmonic of degree $(p-2i)$. It follows that

$$\omega = \Pi_{k,l}\omega = \sum_{i=(p-m)^+}^{[p/2]}\Pi_{k,l}L^i\psi_i+\Pi_{k,l}\Delta_B\mu = \sum_{i=(p-m)^+}^{[p/2]} L^i(\Pi_{k-i,l-i}\psi_i)+\Pi_{k,l}\Delta_B\mu.$$

If $p=k+l\leqq m$, then, by 2.7, $\Pi_{k,l}\Delta_B\mu=\Delta_B\Pi_{k,l}\mu$. Let $p=k+l\geqq m+1$. By [KT] $\Delta_B\xi=(-1)^p\bar{*}\Delta_B\bar{*}\mu$. Therefore, since deg $(\bar{*}\mu)\leqq m$,

$$\begin{aligned}\Pi_{k,l}\Delta_B\mu &= (-1)^p\Pi_{k,l}\bar{*}\Delta_B\bar{*}\mu = (-1)^p\bar{*}\Pi_{m-l,m-k}\Delta_B\bar{*}\mu = \\ &= (-1)^p\bar{*}\Delta_B\Pi_{m-i,m-k}\bar{*}\mu = (-1)^p\bar{*}\Delta_B\bar{*}\Pi_{k,l}\mu = \Delta_B\Pi_{k,l}\mu.\end{aligned}$$

Thus, for any p, $0\leqq p\leqq 2m$,

$$\omega = \sum_{i=(p-m)^+}^{[p/2]} L^i(\Pi_{k-i,l-i}\psi_i)+\Delta_B(\Pi_{k,l}\mu).$$

By Lemma 2.3, $\Pi_{k-i,l-i}\psi_i$ is harmonic and $\Pi_{k,l}\mu$ is basic. By uniqueness of decomposition (2.12), $\psi_i=\Pi_{k-i,l-i}\psi_i$. Hence ψ_i is of bidegree $(k-i, l-i)$. This proves (ii).

In course of the proof of Theorem 2.5 we saw that the notion of a basic harmonic form is the same as the notion of a C-harmonic form. Therefore we can use results of [0] and [T2] on C-harmonic forms. If ω is C-harmonic, then $L\omega$ is also

C-harmonic. Thus we obtain homomorphisms

$$L\colon H_B^{p-2}(M) \to H_B^p(M), \quad p \leqq m,$$

and

$$L\colon H_B^{k-1,l-1}(M) \to H_B^{k,l}(M), \quad k+l \leqq m.$$

These homomorphisms are one-to-one. In addition,

$$\begin{aligned} H_B^p(M) &= LH_B^{p-2}(M) \oplus H^p(M), \quad p \leqq m, \\ H_B^{k,l}(M) &= LH_B^{k-1,l-1}(M) \oplus H^{k,l}(M), \quad k+l \leqq m. \end{aligned} \tag{2.16}$$

It follows that

$$\begin{aligned} b_p &= b_p^{(B)} - b_{p-2}^{(B)}, \quad p \leqq m, \\ h^{k,l} &= h_B^{k,l} - h_B^{k-1,l-1}, \quad k+l \leqq m, \end{aligned} \tag{2.17}$$

where we set $b_{-2}^{(B)} = b_{-1}^{(B)} = h_B^{k,-1} = h_B^{-1,l} = 0$. It follows from (2.17) that

$$\begin{aligned} b_{p-2}^{(B)} &\leqq b_p^{(B)}, \quad p \leqq m, \\ h_B^{k-1,l-1} &\leqq h_B^{k,l}, \quad k+l \leqq m. \end{aligned} \tag{2.18}$$

In particular, we have

$$1 = h_B^{0,0} \leqq h_B^{1,1} \leqq \ldots \leqq h_B^{[m/2],[m/2]}. \tag{2.19}$$

Note that the mapping $\bar{*}$ induces the isomorphisms

$$\begin{aligned} H_B^p(M) &\cong H_B^{2m-p}(M), \quad 0 \leqq p \leqq 2m, \\ H_B^{k,l}(M) &\cong H_B^{m-l,m-k}(M), \quad 0 \leqq k, l \leqq m. \end{aligned} \tag{2.20}$$

In addition, the complex conjugation induces the isomorphism

$$H_B^{k,l}(M) \cong H_B^{l,k}(M), \quad 0 \leqq k, l \leqq m. \tag{2.21}$$

Therefore we have

$$\begin{aligned} b_p^{(B)} &= b_{2m-p}^{(B)}, \quad 0 \leqq p \leqq 2m, \\ h_B^{k,l} &= h_B^{l,k} = h_B^{m-k,m-l} = h_B^{m-l,m-k}, \quad 0 \leqq k, l \leqq m. \end{aligned} \tag{2.22}$$

3. Inequalities for basic cohomology classes. In this section we continue to assume that $(M, \eta, X_0, \varphi, g)$ is a $(2m+1)$-dimensional compact Sasakian manifold.

Lemma 3.1. *Let ω any τ be harmonic forms of degrees $2i$ and $2j$, respectlively. Assume that $0 \leqq i \leqq m/2$, $0 \leqq j \leqq m/2$, and $i \neq j$. Then $I(\omega \wedge \tau) = 0$, where I is defined by formula* (1.3). *In particular, if u is a harmonic form of degree $2i$, where $0 < i \leqq m/2$, then $I(u) = 0$.*

Proof. Let u be a harmonic p-form, $0 \leqq p \leqq m$. Then, by [T1], u is effective (i.e. $\Lambda u = 0$), and therefore, by [T2],

$$\Lambda^r L^{r+s} u = 2^{2r}(s+1) \ldots (s-r)(m-p-s-r+1) \ldots (m-p-s) L^s u.$$

Take $r=m+1$ and $s=m-p+1$ in this formula. Since $r+s>m$, we obtain that $L^{r+s}u=0$. Therefore,

$$L^{m-p+1}u = 0, \tag{3.1}$$

where u is a harmonic p-form, $0\leqq p\leqq m$. By (3.1) we obtain that if $i>j$, then

$$\Phi^{m-i-j}\wedge\omega\wedge\xi = (L^{m-2i+(i-j)}\omega)\wedge\tau = 0.$$

Similarly, if $i<j$, then

$$\Phi^{m-i-j}\wedge\omega\wedge\xi = \omega\wedge(L^{m-2j+(j-i)}\tau) = 0.$$

It follows that

$$I(\omega\wedge\tau) = \frac{1}{2^m m!\,\mathrm{Vol}\,(M)}\int_M \eta\wedge\Phi^{m-i-j}\wedge\omega\wedge\tau = 0.$$

Let ω and τ be closed basic $\mathbf{C}$-valued forms of bidegree (k, k), where $0\leqq k\leqq$ $\leqq m/2$. By Theorem 2.5, we have

$$\omega = \sum_{i=0}^{k} L^i\omega_i+d\lambda, \quad \tau = \sum_{i=0}^{k} L^i\tau_i+d\mu,$$

where ω_i and τ_i are harmonic forms of bidegree $(k-i, k-i)$, and λ and μ are basic forms.

Lemma 3.2.

$$I(\omega\wedge\tau)-I(\omega)I(\tau) = \frac{1}{2^{2k}m!\,\mathrm{Vol}\,(M)}\sum_{i=0}^{k-1}(-1)^{k-i}2^{2i}(m-2k+2i)!\,(\omega_i, \bar{\tau}_i).$$

Proof.

$$I(\omega\wedge\tau) = I\Big(\big(\sum_{i=0}^{k} L^i\omega_i+d\lambda\big)\wedge\big(\sum_{j=0}^{k} L^j\tau_j+d\mu\big)\Big) = \sum_{i,j=0}^{k} I(L^{i+j}\omega_i\wedge\tau_j)+I(dv),$$

where

$$v = \lambda\wedge\big(\sum_{j=0}^{k} L^j\tau_j\big)+\big(\sum_{i=0}^{k} L^i\omega_i\big)\wedge\mu+\lambda\wedge d\mu$$

is a basic form of degree $4k-1$. By Corollary to Lemma 2.1, $I(dv)=0$. By (1.5) and by Lemma 3.1, $I(L^{i+j}\omega_i\wedge\omega_j)=I(\omega_i\wedge\omega_j)=0$, if $i\neq j$. Therefore

$$I(\omega\wedge\tau) = I(\omega_k\wedge\tau_k)+\sum_{i=0}^{k-1} I(\omega_i\wedge\tau_i). \tag{3.2}$$

Since $\deg\omega_k=\deg\tau_k=0$, we obtain that $I(\omega_k\wedge\tau_k)=\omega_k\tau_k I(1)=\omega_k\tau_k$. By Lemma 3.1, $I(\omega)=I\big(\sum_{i=0}^{k} L^i\omega_i+d\lambda\big)=\sum_{i=0}^{k} I(\omega_i)=I(\omega_k)=\omega_k$. Similarly, $I(\tau)=\tau_k$. Therefore, by (3.2),

$$I(\omega\wedge\tau)-I(\omega)I(\tau)=\sum_{i=0}^{k-1}I(\omega_i\wedge\tau_i)=\frac{1}{2^m m!\,\mathrm{Vol}(M)}\sum_{i=0}^{k-1}\int_M \eta\wedge\Phi^{m-2k+2i}\wedge\omega_i\wedge\tau_i=$$

$$=\frac{1}{2^m m!\,\mathrm{Vol}(M)}\sum_{i=0}^{k-1}(\eta\wedge\omega_i,\ *(L^{m-2k+2i}\bar{\tau}_i)).$$

By Lemma 2.4,

$$*(L^{m-2k+2i}\bar{\tau}_i)=(-1)^{k-i}2^{m-2k+2i}(m-2k+2i)!\,\eta\wedge\bar{\tau}_i.$$

Hence

$$I(\omega\wedge\tau)-I(\omega)I(\tau)=\frac{1}{2^{2k}m!\,\mathrm{Vol}(M)}\sum_{i=0}^{k-1}(-1)^{k-i}2^{2i}(m-2k+2i)!(\omega_i,\bar{\tau}_i).$$

This proves the lemma.

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Let ω be a closed basic **C**-valued form of bidegree (k,k), representing $\alpha\in H_B^{k,k}(M)$. By Theorem 2.5, $\omega=\sum_{i=0}^{k}L^i\omega_i+d\lambda$, where ω_i is a harmonic form of bidegree $(k-i,k-i)$, and λ is a basic form. Since $h_B^{k-1,k-1}=1$, we have by (2.19), that $h_B^{k-i,k-i}=1$ for $i=1,\dots,k$. Hence, by (2.17), $h^{k-i,k-i}=0$ for $i=1,\dots,k-1$. Therefore there is no harmonic forms of bidegree $(k-i,k-i)$ for $i=1,\dots,k-1$. By Lemma 3.2, we obtain that

$$(-1)^k[I(\omega\wedge\bar{\omega})-I(\omega)I(\bar{\omega})]=\frac{1}{m!\,\mathrm{Vol}(M)}(m-2k)!(\omega_0,\omega_0)\geqq 0.$$

The equality holds if and only if $\omega_0=0$. In this case $\omega=t\Phi^k+d\lambda$, where $t=\omega_k$. Therefore the equality holds if and only if $\alpha=t\Omega^k$. This proves the theorem.

4. Basic Chern forms. Let (M,η,X_0,φ,g) be a compact $(2m+1)$-dimensional Sasakian manifold and let ∇ be the Riemannian connection on (M,g). A linear connection on M given by the formula, [Ta]:

$$\bar{\nabla}_X Y=\nabla_X Y+\eta(X)\varphi Y+\eta(Y)\varphi X+\Phi(X,Y)X_0 \tag{4.1}$$

will be called *the canonical connection* on M. The following properties of the canonical connection are easily verified by direct computation:

$$\bar{\nabla}_X\eta=0,\quad \bar{\nabla}_X X_0=0,\quad \bar{\nabla}_X\varphi=0 \tag{4.2}$$

for any tangent vector X on M;

$$i(X_0)\tilde{\Theta}=0,\quad i(X_0)\tilde{T}=0, \tag{4.3}$$

where $\tilde{\Theta}$ and $\tilde{T}$ are the curvature form and the torsion form of $\bar{\nabla}$, respectively;

$$\tilde{R}(X,Y)Z = R(X,Y)Z+2\Phi(X,Y)\varphi Z+[\Phi(X,Z)-\eta(X)\eta(Z)]\varphi Y- \tag{4.4}$$
$$-[\Phi(Y,Z)-\eta(Y)\eta(Z)]\varphi X+[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]X_0,$$

where R and $\tilde{R}$ are curvature tensors of ∇ and $\tilde{\nabla}$, respectively;

$$\tilde{R}(\varphi X, \varphi Y) = \tilde{R}(X,Y). \tag{4.5}$$

Consider M as a base of a vector bundle F with the fibre $D_x=\{X\in TM_x\colon \eta(X)=0\}$ at the point $x\in M$. The map $\varphi|_x\colon D_x\to D_x$ defines a complex structure on D_x. Hence F may be considered as a complex vector bundle over M. By (4.2), the canonical connection $\tilde{\nabla}$ induces a complex linear connection in the complex bundle F, which we will denote again by $\tilde{\nabla}$. Let $C_k^{(B)}$ be the k^{th} Chern form of $\tilde{\nabla}$, [C]. $C_k^{(B)}$, $k=1, \ldots, m$, are defined by the formula

$$\det\left[tI+\frac{\sqrt{-1}}{2\pi}\tilde{\Theta}\right] = t^m+\sum_{k=1}^{m} C_k^{(B)}t^{m-k}. \tag{4.6}$$

$C_k^{(B)}$ is closed. By (4.3), $\tilde{\Theta}$ is horizontal. Therefore $C_k^{(B)}$ is horizontal. Hence $C_k^{(B)}$ is basic. Because of (4.5), $C_k^{(B)}$ is real and of bidegree (k,k). Thus, for any $k=1, \ldots$ $\ldots, m$, $C_k^{(B)}$ is a canonically defined real closed basic $2k$-form of bidegree (k,k). We will call $C_k^{(B)}$ the k^{th} *basic Chern form* of a Sasakian manifold. Substituting $C_k^{(B)}$ in (1.10) we obtain that in the case $h_B^{k-1,k-1}=1$ the following integral inequality is satisfied

$$(-1)^k[I(C_k^{(B)}\cdot C_k^{(B)})-I(C_k^{(B)})I(C_k^{(B)})] \geqq 0. \tag{4.7}$$

Using (4.4), we obtain by direct computation that in the case $k=1$ inequality (4.7) is the same as inequality (1.1).

Remark. Let (M,η) be a contact manifold. An associated contact metric structure (η, X_0, φ, g) is called an *associated K-metric structure,* [B], if X_0 is a Killing vector field with respect to g. If a contact manifold (M,η) admits an associated K-metric structure, (M,η) is called a K-contact manifold. We will show now how one can define basic Pontrjagin cohomology classes on a K-contact manifold.

Let (M,η) be a $(2m+1)$-dimensional contact manifold. A linear connection $\tilde{\nabla}$ on M will be called *basic* if

$$\tilde{\nabla}_X\eta = 0, \quad \tilde{\nabla}_X X_0 = 0, \quad i(X_0)\tilde{\Theta} = 0, \quad i(X_0)\tilde{T} = 0, \tag{4.8}$$

where $\tilde{\Theta}$ and $\tilde{T}$ are the curvature form and the torsion form of $\tilde{\nabla}$, respectively.

Assume that (M,η) admits a basic linear connection $\tilde{\nabla}$. Consider M as the base space of a real vector bundle with the $2m$-dimensional fibre $D_x=\{X\in TM_x\colon \eta(X)=0\}$ at the point $x\in M$. By (4.8), $\tilde{\nabla}$ can be considered as a connection in this vector bundle. Put

$$\det\left[tI-\frac{1}{2\pi}\Theta\right] = t^m+\sum_{k=1}^{m} E_k(\tilde{\Theta})t^{m-k}.$$

Then $E_{2k}(\tilde{\Theta})$ is a closed and horizontal (since $\tilde{\Theta}$ is horizontal) $4k$-form, [C], p. 118. Hence $E_{2k}(\tilde{\Theta})$ is basic and therefore defines an element $p_k^{(B)}\in H_B^{4k}(M,\mathbf{R})$. We will show that $p_k^{(B)}$ does not depend on a choice of a basic linear connection. Indeed, let $\tilde{\nabla}'$ be another basic linear connection and let $\tilde{\Theta}'$ and T' be its curvature and torsion forms, respectively. Set $\alpha=\tilde{\nabla}'-\nabla$, $\tilde{\nabla}^t=\tilde{\nabla}+t\alpha$. Let $\tilde{\Theta}^t$ be the curvature form of $\tilde{\nabla}^t$. Then α is a linear form on M of the type ad GL $(2m,\mathbf{R})$, and by (4.8) and (4.9),

$$\alpha(X_0)X = \tilde{\nabla}'_{X_0}X-\tilde{\nabla}_{X_0}X = \tilde{\nabla}_X X_0+[X_0,X]+\tilde{T}'(X_0,X)-$$
$$-\tilde{\nabla}_X X_0-[X_0,X]-\tilde{T}(X_0,X) = 0.$$

Hence α is horizontal. By [C], p. 42, $\tilde{\Theta}^t=\tilde{\Theta}+tD\alpha-t^2\alpha\wedge\alpha$. Taking $t=1$, we obtain $D\alpha=\tilde{\Theta}'-\tilde{\Theta}+\alpha\wedge\alpha$. Therefore $\tilde{\Theta}=(1-t)\tilde{\Theta}+t\tilde{\Theta}'+t(1-t)\alpha\wedge\alpha$. It follows that $\tilde{\Theta}^t$ is horizontal for all t. By [C], p. 115, $E_{2k}(\tilde{\Theta}')-E_{2k}(\tilde{\Theta})=d\varrho$, where $\varrho=\int_0^1 \Psi(t)dt$ and where $\psi(t)$ is a polynomial function of α and $\tilde{\Theta}^t$. It follows that ϱ is horizontal. In addition,

$$L_{X_0}\varrho = [i(X_0)d+di(X_0)]\varrho = i(X_0)\,d\varrho = i(X_0)[E_{2k}(\tilde{\Theta}')-E_{2k}(\tilde{\Theta})] = 0.$$

Hence ϱ is basic. Thus, $E_{2k}(\tilde{\Theta}')$ and $E_{2k}(\tilde{\Theta})$ are homologous within basic forms. Therefore $E_{2k}(\tilde{\Theta}')$ and $E_{2k}(\tilde{\Theta})$ define the same element $p_k^{(B)}\in H_B^{4k}(M,\mathbf{R})$. If (M,η) is a contact manifold which admits a basic linear connection, then $p_k^{(B)}$, $k=1,\dots$ $\dots,[m/2]$, will be called *basic Pontrjagin classes* of (M,η).

Let now (M,η) be a K-contact manifold. Let (η,X_0,φ,g) be an associated K-metric structure and ∇ be the Riemannian connection on M with respect to g. Direct calculation shows that the connection

$$\tilde{\nabla}_X Y = \nabla_X Y+\eta(X)\varphi Y+\eta(Y)\varphi X+\Phi(X,Y)X_0$$

is a basic connection on (M,g). Hence the basic Pontrjagin classes $p_k^{(B)}\in H_B^{4k}(M,\mathbf{R})$, $k=1,\dots,[m/2]$, are well-defined on each K-contact manifold (M,η).

References

[B] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer-Verlag (Berlin, Heidelberg, New York, 1976).

[C] S. Chern, *Complex Manifolds without Potential Theory*, Second Edition, Springer-Verlag (New York, Heidelberg, Berlin, 1979).

[G1] H. GAUCHMAN, An integral inequality for normal contact Riemannian manifolds and its applications, *Geometriae Dedicata* **23** (1987), 53—58.

[G2] H. GAUCHMAN, Inequalities for cohomology classes of Kaehler manifolds, *Geometriae Dedicata,* **31** (1989), 261—266.

[I] S. IANUS, Sulle varietá di Cauchy—Riemann, *Rend dell Accademia di Scienze Fisiche e Matematiche, Napoli,* XXXIX (1972), 191—195.

[KT] F. W. KAMBER and PH. TONDEUR, DeRham—Hodge theory for Riemannian foliations, *Math. Ann.,* **277** (1987), 415—431.

[O] Y. OGAWA, On C-harmonic forms in a compact Sasakian space, *Tôhoku Math. J.,* **19** (1967), 267—296.

[S] S. SASAKI, *Almost Contact Manifolds, Part 1,* Lectures Notes, Mathematical Institute, Tôhoku Univ., 1968.

[SH] S. SASAKI and Y. HATAKEYAMA, On differentiable manifolds with contact metric structure, *J. Math. Soc. Japan,* **14** (1962), 249—271.

[T1] S. TACHIBANA, On harmonic tensors in compact Sasakian spaces, *Tôhoku Math. J.,* **17** (1965), 271—284.

[T2] S. TACHIBANA, On a decomposition of C-harmonic forms in a compact Sasakian space, *Tôhoku Math. J.,* **19** (1967), 198—212.

[Ta] N. TANAKA, On non-degenerate real hypersurfaces, graded Lie algebras, and Cartan connections, *Japanese J. Math.,* **2** (1976), 131—190.

[W] R. O. WELLS, *Differential Analysis on Complex Manifolds,* Prentice Hall, (Englewood Cliffs, N. J., 1973).

[Y] K. YANO, On harmonic and Killing vector fields, *Ann. of Math.,* **55** (1952), 38—45.

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