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On Jessen's inequality

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1. Introduction

Let X be a compact Hausdorff space and let $C(X)$ be the space of all continuous real-valued functions on X, endowed with supremum norm and usual ordering. Let $M_+^1(X)$ be the set of all probability Radon measures on X. The following fact is well-know [7, Sect. 6]:

(1) If μ is a bounded linear functional on $C(X)$ such that $\|\mu\|=\mu(1)=1$ then μ is positive, i.e. $\mu\in M_+^1(X)$.

Let K be a compact convex subset of a locally convex Hausdorff real space B and let B' be the topological dual of B. Let $\mu\in M_+^1(K)$. Then (see [7, Sect. 1 and Sect. 4]):

(2) *There exists a unique* $b\in B$ *such that* $\mu(l)=l(b)$ *for all* $l\in B'$. (In fact, $b\in K$; it is called the barycenter of μ).

(3) $f(b)\leqq\mu(f)$ *for every convex function* $f\in C(K)$.

The inequality (3) is related to the Jessen's inequality (see [1], [4], [5], [6], [10]).

We shall use these results to prove a Jessen-type theorem similar to Theorem 5 of F. V. HUSSEINOV [3] and we shall extend (1), (2), (3) by considering a class of sublinear functionals studied by V. TOTIK [12] instead of linear functionals $\mu\in M_+^1(K)$.

2. A Jessen-type theorem

Let E be a nonempty set and let L be a linear space of real-valued functions defined on E; suppose that the constant function 1 belongs to L. Let $M\colon L\to R$ be a linear isotonic (i.e., $M(f)\leqq M(g)$ whenever $f, g\in L, f\leqq g$) functional such that $M(1)=1$.

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Let B be a locally convex Hausdorff real space and let $\mathscr{L}$ be a set of B-valued functions defined on E such that $l\circ F\in L$ for all $F\in\mathscr{L}$ and all $l\in B'$. Let $\mathscr{M}\colon \mathscr{L}\to B$ be such that $l(\mathscr{M}F)=M(l\circ F)$ for all $l\in B'$ and $F\in\mathscr{L}$.

Let K be a compact convex subset of B, let $\varphi\in C(K)$ be a convex function and let $F\in\mathscr{L}$. Suppose that $F(E)\subset K$ and $\varphi\circ F\in L$. Denote $H=\{h\in C(K)\colon h\circ F\in L\}$.

The following theorem contains a Jessen-type inequality (see [1], [2], [6], [10] and, especially, [3]).

Theorem 1. a) *$\mathscr{M}F\in K$ and $\varphi(\mathscr{M}F)\leqq M(\varphi\circ F)$.*

b) *If φ is a strictly convex function then $\varphi(\mathscr{M}F)=M(\varphi\circ F)$ if and only if $h(\mathscr{M}F)=M(h\circ F)$ for all $h\in H$.*

Proof. Let us remark that $1\in H$, $\varphi\in H$ and $l\in H$ for all $l\in B'$. Consider $\lambda\colon H\to R$, $\lambda(h)=M(h\circ F)$ for all $h\in H$. Then λ is a positive linear functional with $\|\lambda\|=1$. Using the Hahn—Banach theorem we deduce that there exists a bounded linear functional μ on $C(K)$ such that μ coincides with λ on H and $\|\mu\|=\|\lambda\|$. It follows that $\|\mu\|=\mu(1)=1$. Using (1) we infer that $\mu\in M_+^1(K)$.

Now let $l\in B'$. Then $\mu(l)=\lambda(l)=M(l\circ F)=l(\mathscr{M}F)$ and (2) implies that $\mathscr{M}F$ is the barycenter of μ, hence $\mathscr{M}F\in K$. Using (3) we deduce $\varphi(\mathscr{M}F)\leqq\mu(\varphi)=\lambda(\varphi)=$ $=M(\varphi\circ F)$.

Suppose now that φ is strictly convex (such a function exists in $C(K)$ if and only if K is metrizable) and $\varphi(\mathscr{M}F)=M(\varphi\circ F)$. Then $\mu(\varphi)=\varphi(\mathscr{M}F)$. By virtue of [8, Lemma], μ is the Dirac measure corresponding to $\mathscr{M}F$. It follows that $M(h\circ F)=$ $=\lambda(h)=\mu(h)=h(\mathscr{M}F)$ for all $h\in H$.

3. Inequalities for sublinear functionals

Let X be a compact Hausdorff space. Denote by $\mathscr{T}(X)$ the set of all sublinear functionals $A\colon C(X)\to R$ such that $A(1)=1$, $A(-1)=-1$ and $\|A\|=1$ (i.e., $|A(f)|\leqq 1$ for all $f\in C(X)$ with $\|f\|\leqq 1$).

V. Totik has proved in [12] that if $A\in\mathscr{T}(X)$ then $A(f)\leqq\max\limits_X f$ for all $f\in C(X)$; moreover, if X is metrizable then for every $A\in\mathscr{T}(X)$ there exists a sequence $(x_n)_{n\in\mathbf{N}}$ in X such that $A(f)=\limsup\,(f(x_1)+\ldots+f(x_n))/n$ for all $f\in C(X)$.

We shall extend (1), (2) and (3) replacing $M_+^1(X)$ by $\mathscr{T}(X)$.

Example 1. Let $\mu, \nu\in M_+^1(X)$. Define $A(f)=\int\limits_X \max\,(\nu(f), f(x))\,d\mu(x)$ for all $f\in C(X)$. Then $A\in\mathscr{T}(X)$.

Example 2. Let (ν_i) be a net in $M_+^1(X)$. Define $A(f)=\limsup \nu_i(f)$ for all $f\in C(X)$. Then $A\in\mathscr{T}(X)$.

Although the following extension of (1) is a consequence of Proposition 2 below, we insert here a direct proof.

Proposition 1. *Every* $A\in\mathcal{T}(X)$ *is isotonic.*

Proof. Let $f,g\in C(X)$, $f\leqq g$, $A(f)>A(g)$. Then $0<A(f)-A(g)\leqq A(h)$ where $h=f-g\leqq 0$. Let $m=-\min\limits_X h$. Clearly $m>0$. We have $0<A(h)\leqq A(-m)+$ $+A(h+m)$; this yields $A(h+m)>m$. On the other hand $0\leqq h+m\leqq m$ and $\|A\|=1$ imply $A(h+m)\leqq m$, a contradiction.

Propositon 2. $A\in\mathcal{T}(X)$ *if and only if there exists a nonempty set* $S\subset M_+^1(X)$ *such that* $A(f)=\sup\{\mu(f)\colon \mu\in S\}$ *for all* $f\in C(X)$.

Proof. Let $A\in\mathcal{T}(X)$ and let $S=\{\mu\colon C(X)\to R\colon \mu \text{ linear}, \mu\leqq A\}$. Using the Hahn—Banach theorem we deduce that $A(f)=\sup\{\mu(f)\colon \mu\in S\}$ for all $f\in C(X)$. But if $\mu\in S$, then $\|\mu\|=\mu(1)=1$; therefore $S\subset M_+^1(X)$.

The converse is easy to prove.

The following result extends (2) and (3).

Theorem 2. *Let K be a compact convex subset of a locally convex Hausdorff real space B and let* $A\in\mathcal{T}(K)$. *There exists a unique nonempty compact convex subset* Q *of* K *such that* $A(h)=\max\limits_Q h$ for all $h\in B'$. Moreover:

a) *A is linear on* B' *if and only if* Q *reduces to one point.*

b) *Let* $M\subset K$. *Then* $A(f)\geqq\sup\limits_M f$ *for every convex function* $f\in C(K)$ *if and only if* $M\subset Q$.

Proof. By Proposition 2 there exists $S\subset M_+^1(K)$ such that $A(f)=$ $=\sup\{\mu(f)\colon \mu\in S\}$ for all $f\in C(K)$. Let $C=\operatorname{conv}\{b(\mu)\colon\mu\in S\}$ where $b(\mu)$ is the barycenter of μ. Let Q be the closure of C. If $h\in B'$ we have $A(h)=$ $=\sup\{\mu(h)\colon \mu\in S\}=\sup\{h(b(\mu)\colon \mu\in S\}=\sup\{h(x)\colon x\in C\}=\max\limits_Q h$. The uniqueness of Q is an easy consequence of separation theorems.

a) Clearly A is linear on B' if Q reduces to one point. Conversely, let A be linear on B'. For $h\in B'$ we have $\max\limits_Q h=A(h)=-A(-h)=-\max\limits_Q(-h)=\min\limits_Q h$. It follows that every $h\in B'$ is constant on Q, hence Q reduces to one point.

b) Let $M\subset Q$ and let $f\in C(K)$ be convex. Then $A(f)=\sup\{\mu(f)\colon \mu\in S\}\geqq$ $\geqq\sup\{f(b(\mu))\colon \mu\in S\}=\sup\{f(x)\colon x\in C\}=\max\limits_Q f\geqq\sup\limits_M f$.

Conversely, suppose that $A(f)\geqq\sup\limits_M f$ for every convex function $f\in C(K)$. If $t\in M$ and $t\notin Q$, there exists $h\in B'$ such that $h(t)>\max\limits_Q h$. We have $\sup\limits_M h\geqq$ $\geqq h(t)>\max\limits_Q h=A(h)\geqq\sup\limits_M h$, a contradiction. Hence $M\subset Q$.

Example 3. Let $a\in[0,1]$, $A\colon C[0,1]\to R$, $A(f)=\int\limits_0^1\max(f(a),f(x))\,dx$. Then $A\in\mathcal{T}([0,1])$ and $Q=[(1-(1-a)^2)/2,\,(1+a^2)/2]$.

Example 4. Let $K=[0,1]^2$ and $A\in\mathcal{T}(K)$,

$$A=\sup\left\{\frac{1}{2}(\varepsilon_{(0,0)}+\varepsilon_{(0,1)}),\ \frac{1}{2}(\varepsilon_{(1,0)}+\varepsilon_{(1,1)})\right\}$$

where ε_t is the Dirac measure corresponding to t.

Then

$$Q=\left\{\left(x,\frac{1}{2}\right):\ x\in[0,1]\right\}.$$

Example 5. Let $K=[0,1]^2$ and

$$A=\sup\left\{\frac{1}{2}(\varepsilon_{(0,0)}+\varepsilon_{(1,1)}),\ \frac{1}{2}(\varepsilon_{(0,1)}+\varepsilon_{(1,0)})\right\}.$$

Then Q reduces to the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

Remark 1. Let $\mu\in M_+^1(K)$ and let $b=b(\mu)$. For $g\in C(K)$ define

$$B_n g(b)=\int_{K^n} g\big((t_1+\ldots+t_n)/n\big)\,d(\mu\otimes\ldots\otimes\mu)(t_1,\ldots,t_n).$$

(B_n is a Bernstein—Schnabl type operator; see [11] and the references given there.)

For every convex function $f\in C(K)$ we have the following improvement of (3):

$$\mu(f)=B_1f(b)\geqq B_2f(b)\geqq\ldots\geqq f(b)$$

and $\lim B_nf(b)=f(b)$ (see [11]). In [9] it is shown that $\mu(f)=f(b)$ if and only if f is affine on the closure of conv (supp μ).

Remark 2. Let now K be a Choquet simplex. Let $\mu\in M_+^1(K)$ and $b=b(\mu)$. Let ε_b be the Dirac measure and let $\mu_b\in M_+^1(K)$ be the unique maximal measure which represents b. Then for every convex function $f\in C(K)$ we have (see [7, Sect. 9]):

$$\varepsilon_b(f)\leqq\mu(f)\leqq\mu_b(f). \tag{4}$$

For $K=[a,b]\subset R$ inequalities similar to the second inequality are studied and generalized in [1] and [6], Lemma 1.

Let $A\in\mathcal{T}(K)$, $S\subset M_+^1(K)$, $A=\sup\{\mu:\ \mu\in S\}$. Let $f\in C(K)$ be convex. Then $A(f)=\sup\{\mu(f):\ \mu\in S\}\leqq\sup\{\mu_b(f):\ b\in Q\}$. Hence

$$\sup\{\varepsilon_b(f):\ b\in Q\}\leqq A(f)\leqq\sup\{\mu_b(f):\ b\in Q\}. \tag{5}$$

This is an extension of (4).

References

[1] P. R. Beesack and J. E. Pečarić, On Jessen's inequality for convex functions, *J. Math. Anal. Appl.*, **110** (1985), 536—552.

[2] I. M. Bomze, Jensen-Ungleichung in Banach-Räumen: ein kurzer Beweis, *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl.*, **121** (1984), 67—68.

[3] F. V. Husseinov, On Jensen's inequality, *Mat. Zametki*, **41** (1987), 798—806 (Russian).

[4] B. Jessen, Bemaerkinger om konvekse Funktioner og Uligheder imellem Middelvaerdier. I, *Mat. Tidsskrift* **B** (1931), 17—28.

[5] E. J. McShane, Jensen's inequality, *Bull. Amer. Math. Soc.*, **43** (1937), 521—527.

[6] J. E. Pečarić and P. R. Beesack, On Jessen's inequality for convex functions. II, *J. Math. Anal. Appl.*, **118** (1986), 125—144.

[7] R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand (Princeton, 1966).

[8] I. Raşa, On some results of C. A. Micchelli, *Anal. Numér. Théor. Approx.*, **9** (1980), 125—127.

[9] I. Raşa, Sets on which concave functions are affine and Korovkin closures, *Anal. Numér. Théor. Approx.*, **15** (1986), 163—165.

[10] I. Raşa, A note on Jessen's inequality, *Itin. Sem. Funct. Eq. Approx. Conv. Cluj-Napoca*, Univ. "Babes-Bolyai", Preprint 6 (1988), 275—280.

[11] I. Raşa, On the monotonicity of sequences of Bernstein—Schnabl operators, *Anal. Numér. Théor. Approx.*, **17** (1988), 185—187.

[12] V. Totik, Representation of functionals via summability methods, *Acta Sci. Math.*, **48** (1985), 483—498 and **49** (1985), 235—256.

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