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On the divergence phenomena of the differentiated trigonometric projection operators

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1. Introduction

Let r be a nonnegative integer, $C_{2\pi}^r$ the class of 2π-periodic continuous functions which have r continuous derivatives, $C_{2\pi}=C_{2\pi}^0$, $\mathcal{T}_n$ the set of trigonometric polynomials of degree at most n. Let

$$P_n \in C_{2\pi} \to \mathcal{T}_n$$

be projection operators, that is, linear operators with the following properties:

(i) $P_n(f, x) \in \mathcal{T}_n$ if $f \in C_{2\pi}$,

(ii) $P_n(f, x) \equiv f(x)$ if $f \in \mathcal{T}_n$.

Furthermore,

$$\|P_n^{(r)}\| := \sup_{0 \not\equiv f \in C_{2\pi}} \frac{\|P_n^{(r)}(f, x)\|}{\|f\|}$$

is the norm of the r times differentiated operator, $\|\cdot\|$ denotes supremum norm over the real line, $E_n(f)$ is the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree n, and

$$\omega(f, \delta) = \max_{|h| \leq \delta} \|f(x+h)-f(x)\|.$$

Recently, P. O. RUNCK, J. SZABADOS and P. VÉRTESI [1] studied the convergence of differentiated trigonometric projection operators, they established

THEOREM A. *If $f \in C_{2\pi}^r$ and $P_n \in C_{2\pi} \to \mathcal{T}_n$, then*

$$\|f^{(r)}(x) - P_n^{(r)}(f, x)\| = O\big(E_n(f^{(r)}) + E_n(f)\,\|P_n^{(r)}\|\big).$$

On the divergence phenomena [1] showed

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Theorem B. *Given* $r\geqq 0$, *a modulus of continuity* $\omega(t)$ *such that*

(1)
$$\lim_{t\to 0+}\frac{t}{\omega(t)}=0,$$

and a sequence of projection operators $P_n\in C_{2\pi}\to\mathcal{T}_n$, *then there exists an* $f_r(x)\in C_r(\omega)$ *such that*

$$\limsup_{n\to\infty}\frac{\|f_r^{(r)}(x)-P_n^{(r)}(f_r,x)\|}{\omega(n^{-1})\log n}>0,$$

where

$$C_r(\omega)=\left\{f(x)\colon f\in C_{2\pi}^r,\ \sup_{t>0}\frac{\omega(f^{(r)},t)}{\omega(t)}<\infty\right\}.$$

Theorem C. *Given* $r\geqq 0$, *a sequence of projection operators* $P_n\in C_{2\pi}\to\mathcal{T}_n$ *and a sequence* $\varepsilon_1\geqq\varepsilon_2\geqq\ldots$, $\lim_{n\to\infty}\varepsilon_n=0$, *then there exists an* $f_r(x)\in W^r\,\mathrm{Lip}\,1$ *such that*

$$\limsup_{n\to\infty}\frac{\|f_r^{(r)}(x)-P_n^{(r)}(f_r,x)\|}{\varepsilon_n n^{-1}\log n}>0,$$

where

$$W^r\,\mathrm{Lip}\,1=\{f(x)\in C_{2\pi}^r\colon f^{(r)}\in\mathrm{Lip}\,1\}.$$

From the above discussions, a natural question thus arises: Can we remove condition (1) eventually while the same conclusion of Theorem B still holds? Using some new ideas, together with the basic methods given in [1] and careful calculation, the present paper will prove this fact.

2. Result and Proof

Lemma 1. *Let*

$$S_{N,n}(x)=\sum_{j=N}^{n}\frac{\sin jx}{j},$$

then for $|x|\leqq n^{-3/2}$,

$$\|S_{N,n}(x)\|=O(n^{-1/2}),$$

and for $|x|\geqq N^{-1/2}$,

$$\|S_{N,n}(x)\|=O(N^{-1/2}).$$

Proof. It is easy to see that

(2)
$$|S_{N,n}(x)|\leqq n|x|$$

since

$$|\sin jx|\leqq j|x|.$$

At the same time, by Abel transform,

$$\sum_{j=N}^{n}\frac{\sin jx}{j}=\sum_{j=N}^{n-1}\frac{1}{j(j+1)}\sum_{k=N}^{j}\sin kx+\frac{1}{n}\sum_{k=N}^{n}\sin kx,$$

while

$$\sum_{k=N}^{j}\sin kx=\frac{\cos\left(j+\frac{1}{2}\right)x+\cos\left(N-\frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

so [1])

(3) $$|S_{N,n}(x)|\leqq C|x|^{-1}\left(\sum_{j=N}^{n-1}\frac{1}{j(j+1)}+\frac{1}{n}\right)=O(N^{-1}|x|^{-1}).$$

Now Lemma 1 follows from (2) and (3).

Lemma 2. *Given $r\geqq 0$ and $n\geqq 1$, there exists a function $g_{nr}(x)\in C_{2\pi}^{\infty}$ such that*

(4) $$\|g_{nr}(x)\|\leqq Cn^{r},$$

(5) $$\|g_{nr}^{(j)}(x)\|\leqq Cn^{j},\quad j=r,r+1,$$

(6) $$|g_{nr}^{(r+1)}(x)|=\begin{cases}O(n^{r+2/3}), & |x|\leqq n^{-1}\\ O(n^{r+3/4}), & |x|\geqq n^{-1/4},\end{cases}$$

and

(7) $$\frac{1}{\pi}\int_{-\pi}^{\pi}g_{nr}^{(r)}(t)D_n(t)\,dt\geqq Cn^{r}\log n,$$

where

$$D_n(t)=\frac{\sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}}$$

is the nth Dirichlet kernel.

Proof. Set

$$g_{nr}(x)=n^{r}\left(\sum_{j=[n^{1/2}]}^{[n^{2/3}]}\frac{\cos\left((n-j)x+\frac{r\pi}{2}\right)}{j(n-j)^{r}}-\sum_{j=[n^{1/2}]}^{[n^{2/3}]}\frac{\cos\left((n+j)x+\frac{r\pi}{2}\right)}{j(n+j)^{r}}\right),$$

then

$$g_{nr}^{(r)}(x)=n^{r}\left(\sum_{j=[n^{1/2}]}^{[n^{2/3}]}\frac{\cos(n-j)x}{j}-\sum_{j=[n^{1/2}]}^{[n^{2/3}]}\frac{\cos(n+j)x}{j}\right)=2n^{r}\sin nx\sum_{j=[n^{1/2}]}^{[n^{2/3}]}\frac{\sin jx}{j}.$$

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[1]) In the whole paper, C always indicates some positive constant depending upon r but independent of n which may have different values at different places.

Furthermore,

$$|g_{nr}^{(r+1)}(x)| = \left|2n^{r+1}\cos nx \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\sin jx}{j} + 2n^r \sin nx \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right| \leqq$$

$$\leqq 2n^r \left(n\left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{\sin jx}{j}\right| + |\sin nx| \left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right|\right).$$

Altogether with Lemma 1 and

$$|\sin nx| \leqq n|x|,$$

$$\left|\sum_{j=[n^{1/2}]}^{[n^{2/3}]} \cos jx\right| = O(\min\{n^{2/3}, |x|^{-1}\}),$$

we then get for $|x| \leqq n^{-1}$,

$$g_{nr}^{(r+1)}(x) = O(n^{r+2/3}),$$

and for $|x| \geqq n^{-1/4}$,

$$g_{nr}^{(r+1)}(x) = O(n^{r+3/4}),$$

(6) is completed. By above discussions and the well-known estimate

$$\left|\sum_{j=1}^{n} \frac{\sin jx}{j}\right| = O(1)$$

for all n, (5) is trivial. The estimate (4) is also not difficult. If $r=0$, then (5) implies (4). Let $r \geqq 1$, we have

$$\|g_{nr}(x)\| \leqq 2n^r \sum_{j=[\sqrt{n}]}^{[n^{2/3}]} \frac{1}{j(n-j)^r} = o(n^r), \quad n \to \infty,$$

which implies (4). At last,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g_{nr}^{(r)}(t) D_n(t)\, dt = n^r \sum_{j=[n^{1/2}]}^{[n^{2/3}]} \frac{1}{j} \geqq Cn^r \log n,$$

that is (7).

Theorem. *Given $r \geqq 0$, a modulus of continuity $\omega(t)$ and a sequence of projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, then there exists an $f_r(x) \in C_r(\omega)$ such that*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(x) - P_n^{(r)}(f_r, x)\|}{\omega(n^{-1}) \log n} > 0.$$

Proof. Considering Theorem B, we only need to prove our theorem in Lip 1 case. If for any fixed N,

$$\limsup_{n\to\infty} \frac{\|g_{Nr}^{(r)}(x) - P_n^{(r)}(g_{Nr}, x)\|}{n^{-1} \log n} > 0,$$

then $g_{Nr}(x)$ is the required function. Now we suppose otherwise for any fixed N,

(8) $$\lim_{n\to\infty} \frac{\|g_{Nr}^{(r)}(x)-P_n^{(r)}(g_{Nr}, x)\|}{n^{-1}\log n} = 0.$$

Using the argument from [1], p. 291, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}\big(g_{nr}(\cdot+u), -u\big)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt.$$

By (7) we get

(9) $$\|P_n^{(r)}(g_{nr}, x)\| \geqq Cn^r \log n.$$

We select a subsequence from natural numbers $n_1<n_2<\dots$ by induction. Let $n_1=1$. After n_k, we choose n_{k+1} satisfying the following properties:

(10) $$n_{k+1}^{-1} \leqq \frac{1}{2} n_k^{-4},$$

(11) $$n_{k+1}^{-1} \leqq \min\left\{\frac{1}{2}, \|P_{n_k}^{(r)}\|^{-1}\right\} n_k^{-1},$$

(12) $$\frac{\|g_{n_jr}^{(r)}(x)-P_{n_{k+1}}^{(r)}(g_{n_jr}, x)\|}{n_{k+1}^{-1}\log n_{k+1}} \leqq k^{-1}\log^{-1}k, \quad j = 1, 2, \dots, k.$$

Due to (8), (12) is possible. Define

$$f_r(x) = \sum_{j=1}^{\infty} g_{n_jr}(x)\,n_j^{-r-1}.$$

Clearly, $f_r\in C_{2\pi}^r$. For $\delta>0$, let $n_{k+1}^{-1}\leqq\delta<n_k^{-1}$. Then by mean value theorem there is a $\theta_k\in[0, 1]$ such that

$$|f_r^{(r)}(x+\delta)-f_r^{(r)}(x)| \leqq \delta\sum_{j=1}^{k} |g_{n_jr}^{(r+1)}(x+\theta_k\delta)|\,n_j^{-r-1}+2\sum_{j=k+1}^{\infty}\|g_{n_jr}^{(r)}(x)\|\,n_j^{-r-1} :=$$

$$:= \Sigma_1+\Sigma_2.$$

Due to (5), (6) and (11),

$$\Sigma_2 \leqq C\sum_{j=k+1}^{\infty} n_j^{-1} = O(n_{k+1}^{-1}) = O(\delta).$$

Meanwhile since $(n_i^{-1}, n_i^{-1/4})\cap(n_j^{-1}, n_j^{-1/4})=\emptyset$, $1\leqq i<j$ (by (10)), if $x+\theta_k\delta\in(n_{i_0}^{-1}, n_{i_0}^{-1/4})$ for some i_0, $1\leqq i_0\leqq k$, then

$$x+\theta_k\delta\notin(n_j^{-1}, n_j^{-1/4})$$

for $1\leqq j\leqq k, j\neq i_0$, that is, $x+\theta_k\delta$ satisfies (6). Therefore,

$$\Sigma_1 \leqq \delta\big(|g_{n_{i_0}r}^{(r+1)}(x+\theta_k\delta)|\, n_{i_0}^{-r-1} + \sum_{1\leqq j\leqq k, j\neq i_0} |g_{n_jr}^{(r+1)}(x+\theta_k\delta)|\, n_j^{-r-1}\big) \leqq$$

$$\leqq C\delta\big(1+\sum_{j=1}^{\infty} n_j^{-1/4}\big) = O(\delta),$$

thus we have proved $f_r\in W^r$ Lip 1. On the other hand,

$$\|f_r^{(r)}(x)-P_{n_k}^{(r)}(f_r, x)\| \geqq \|P_{n_k}^{(r)}(g_{n_kr}, x)\|\, n_k^{-r-1} -$$

$$-\sum_{j=1}^{k-1} \|g_{n_jr}^{(r)}(x)-P_{n_k}^{(r)}(g_{n_jr}, x)\|\, n_j^{-r-1} - \sum_{j=k}^{\infty} \|g_{n_jr}^{(r)}(x)\|\, n_j^{-r-1} -$$

$$-\sum_{j=k+1}^{\infty} \|P_{n_k}^{\gamma}(g_{n_jr}, x)\|\, n_j^{-r-1} := \Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6.$$

From (9),

$$\Sigma_3 \geqq Cn_k^{-1}\log n_k, \tag{13}$$

while (5), (11) and (12) imply that

$$\Sigma_5 = O\big(\sum_{j=k}^{\infty} n_j^{-1}\big) = O(n_k^{-1}), \tag{14}$$

$$\Sigma_4 = O(k^{-1}n_k^{-1}\log n_k), \tag{15}$$

finally using (4) and (11) we get

$$\Sigma_6 = O\big(\|P_{n_kr}^{(r)}\| \sum_{j=k+1}^{\infty} n_j^{-1}\big) = O(\|P_{n_kr}^{(r)}\|\, n_{k+1}^{-1}) = O(n_k^{-1}). \tag{16}$$

Combining (13)—(16), we thus have

$$\|f_r^{(r)}(x)-P_{n_k}^{(r)}(f_r, x)\| \geqq Cn_k^{-1}\log n_k$$

for sufficiently large k. Theorem is proved.

Reference

[1] P. O. Runck, J. Szabados and P. Vértesi, On the convergence of the differentiated trigonometric projection operators, *Acta Sci. Math.*, 53 (1989), 287—293.

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