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On composition operators

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1. Preliminaries

Let $(X, \mathcal{S}, \lambda)$ be a σ-finite measure space and let Φ be a measurable non-singular ($\lambda\Phi^{-1}(E)=0$ whenever $\lambda(E)=0$) transformation from X into itself. Then the *composition transformation* C_Φ on $L^2(X, \mathcal{S}, \lambda)$ is defined as

$$C_\Phi f = f \circ \Phi \quad \text{for every} \quad f \in L^2(X, \mathcal{S}, \lambda).$$

In case C_Φ is a bounded operator with range in $L^2(X, \mathcal{S}, \lambda)$ we call it a *composition operator*. In this paper we generalize the Theorem 1 [11] and prove that the result is also true in case $(X, \mathcal{S}, \lambda)$ is a σ-finite standard Borel space. In the subsequent sections we characterize composition operators with ascent 1 and descent 1 and give a criterion for partial isometry and co-isometry composition operator. In the last section hyponormal composition operators on $l^2(\mathrm{N}, \mathcal{S}, \lambda)$, the square summable weighted sequence space, have been characterized and a necessary condition for C_Φ to be hyponormal on $L^2(X, \mathcal{S}, \lambda)$ is given, where $(X, \mathcal{S}, \lambda)$ is a standard Borel space.

Let $B(L^2(\lambda))$, $R(C_\Phi)$, $R(C_\Phi)^\perp$ denote the Banach algebra of all bounded linear operators on $L^2(\lambda)$, the range of C_Φ and the orthogonal complement of the range of C_Φ respectively. We denote by

$$f_0 = \frac{d\lambda\Phi^{-1}}{d\lambda} \quad \text{and} \quad g_0 = \frac{d\lambda(\Phi\circ\Phi)^{-1}}{d\lambda}$$

the Radon—Nikodym derivative of the measure $\lambda\Phi^{-1}$ and $\lambda(\Phi\circ\Phi)^{-1}$ with respect to the measure λ, respectively. The symbols X_0 and X_0' will stand for the set $\{x: f_0(x)=0\}$ and $\{x: g_0(x)=0\}$ respectively. The multiplication operators induced by f_0 and g_0 are denoted by M_{f_0} and M_{g_0}.

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If E and F are two measurable sets, then "$E=F$" will indicate that their symmetric difference is of measure zero. We denote the characteristic function of a measurable set E by χ_E.

Definition. An operator A on a Hilbert space H is called a *Fredholm operator* if the range of A is closed and the dimensions of the kernel and co-kernel are finite.

Definition. A *standard Borel space* X is a Borel subset of a complete separable metric space T. The class $\mathcal{S}$ will consist of all the sets of form $X\cap B$, where B is a Borel subset of T.

2. Fredholm composition operators

Theorem 1. *Let $(X,\mathcal{S},\lambda)$ be a σ-finite non-atomic standard Borel space and $C_\Phi\in B(L^2(\lambda))$. Then C_Φ is a Fredholm operator if and only if it is invertible.*

Proof. If C_Φ is invertible, then C_Φ is clearly a Fredholm operator.

Since $\operatorname{Ker} C_\Phi=\operatorname{Ker} C_\Phi^*C_\Phi=\operatorname{Ker} M_{f_0}$ [17, p. 82], where M_{f_0} is the multiplication operator induced by f_0, and X is a non-atomic, the nullity of C_Φ is either zero or infinite. Suppose C_Φ is a Fredholm operator. Then, since C_Φ is one-to-one with closed range, to prove that C_Φ is invertible it is enough to show that Φ is one-to-one a.e. [18, Theorem 2; 13, Corollary 2.4]. Suppose Φ is not one-to-one. By Corollary 8.2 [22] there exist two Borel sets Y_1 and Z_1 such that Φ is one-to-one on Y_1 onto Z_1 and $\lambda(X\setminus Y_1)\neq 0$. Now $(X\setminus Y_1)$ is a Borel set. Let $\Phi_1=\Phi|(X\setminus Y_1)$. Again by Corollary 8.2 [22] we get two Borel sets $Y_2\subseteq(X\setminus Y_1)$ and Z_2 such that Φ is one-to-one on Y_2 onto Z_2. Since X is a non-atomic, we can write $Y_2=\bigcup_{n=1}^{\infty} E_n$ such that $0<\lambda(E_n)<\infty$, $E_n\cap E_m=\emptyset$ whenever $n\neq m$, and $\lambda(Y_1\cap\Phi^{-1}(\Phi[E_n]))\neq 0$. From the fact that $R(C_\Phi)=L^2(X,\Phi^{-1}(\mathcal{S}),\lambda)$ [13, Lemma 2.4], where $\Phi^{-1}(\mathcal{S})=\{\Phi^{-1}(E)\colon E\in\mathcal{S}\}$, it follows that, for every $n\in\mathbf{N}$, there exists $K_n\in\Phi^{-1}(\mathcal{S})$ such that $\langle\chi_{E_n},\chi_{K_n}\rangle\neq 0$. This shows that χ_{E_n} does not belong to $R(C_\Phi)^\perp$. Since $\lambda(Y_1\cap\Phi^{-1}(\Phi[E_n]))\neq 0$, we have $E_n\neq\Phi^{-1}(E)$ for any $E\in\mathcal{S}$, and hence χ_{E_n} can not belong to $R(C_\Phi)$ also. Let $L^2(\lambda)=R(C_\Phi)^\perp\oplus R(C_\Phi)$. Consider $\{\chi_{E_n}\}=\{f_n+g_n\}$, where $f_n\in R(C_\Phi)^\perp$ and $g_n\in R(C_\Phi)$. In view of the remark [22, page 3] $\Phi[E_n]=\{\Phi(x)\colon x\in E_n\}$ is a Borel set. Let $F_n=\Phi^{-1}(\Phi[E_n])$. Since $\Phi|Y_2$ is one-to-one, $\{F_n\}$ is a disjoint sequence of sets. We claim that $g_n=g_n\cdot\chi_{F_n}$. Suppose $g_n\neq g_n\cdot\chi_{F_n}$. Then $\lambda(G_n\setminus F_n)\neq 0$, wheer $G_n= =\{x\colon g_n(x)\neq 0\}$. Since

$$f_n(x)=\begin{cases}1-g_n(x) & \text{for } x\in E_n,\\ -g_n(x) & \text{for } x\in X\setminus E_n.\end{cases}$$

We can find a set $G'\subseteq(G_n\setminus F_n)$ belonging to the σ-algebra $\Phi^{-1}(\mathcal{S})$ such that

$\langle f_n, \chi_{G'} \rangle \neq 0$ which is a contradiction. Thus $g_n = g_n \cdot \chi_{F_n}$. Since

$$\langle f_n, f_m \rangle = \langle \chi_{E_n} - g_n, \chi_{E_m} - g_m \rangle = 0 \quad \text{whenever} \quad n \neq m,$$

$\{f_n\}$ is an infinite orthogonal sequence in $R(C_\Phi)^\perp$ which contradicts the assumption that dim $R(C_\Phi)^\perp$ is finite. Hence Φ is one-to-one (a.e.). Thus C_Φ has dense range and hence C_Φ is invertible.

3. Ascent and descent of a composition operator

Definition. Let A be an operator on a Hilbert space H. Then the *ascent* $\alpha(A)$ of A is the least non-negative integer such that $\operatorname{Ker} A^k = \operatorname{Ker} A^{k+1}$ for all $k \geqq \alpha(A)$ and the *descent* $\delta(A)$ of A is the least non-negative integer such that $\overline{R(A^k)} = \overline{R(A^{k+1})}$ for all $k \geqq \delta(A)$, where $\overline{R(A)}$ is the closure of the range of A.

We shall prove the following theorem which characterizes composition operators with ascent 1.

Theorem 2. *Let* $C_\Phi \in B(L^2(X, \mathcal{S}, \lambda))$. *Then* C_Φ *has ascent* 1 *if and only if* $\lambda(\Phi \circ \Phi)^{-1}(E) = 0$ *implies* $\lambda \Phi^{-1}(E) = 0$ *for* $E \in \mathcal{S}$.

Proof. Since C_Φ is a composition operator, there exists an $M < \infty$ such that

$$\lambda \Phi^{-1}(\Phi^{-1}(E)) \leqq M \lambda \Phi^{-1}(E) \leqq M^2 \lambda(E)$$

for every $E \in \mathcal{S}$ [20, Theorem 1]. This shows that the measure $\lambda(\Phi \circ \Phi)^{-1}$ defined as $\lambda(\Phi \circ \Phi)^{-1}(E) = \lambda \Phi^{-1}(\Phi^{-1}(E))$ is absolutely continuous with respect to the measure λ, and consequently for every $E \in \mathcal{S}$ we have $\lambda(\Phi \circ \Phi)^{-1}(E) = \int_E g_0 \, d\lambda$. Suppose $\lambda \Phi^{-1}(E) = 0$ whenever $\lambda(\Phi \circ \Phi)^{-1}(E) = 0$. Then, by absolute continuity of $\lambda(\Phi \circ \Phi)^{-1}$ and the equation $\lambda \Phi^{-1}(E) = \int_E f_0 \, d\lambda$, it follows that $X_0 = X_0'$ and hence by [17, page 82] we conclude that

$$\operatorname{Ker} C_\Phi = \operatorname{Ker} M_{f_0} = L^2(X_0) = \operatorname{Ker} M_{g_0} = \operatorname{Ker} C_\Phi^2.$$

This shows that C_Φ is of ascent 1.

Conversely, suppose $\operatorname{Ker} C_\Phi = \operatorname{Ker} C_\Phi^2$. Since $\operatorname{Ker} C_\Phi = L^2(X_0)$ and $\operatorname{Ker} C_\Phi^2 = L^2(X_0')$, it follows that $X_0 = X_0'$. Since $\lambda \Phi^{-1}(E) = \int_E f_0 \, d\lambda$ and $\lambda(\Phi \circ \Phi)^{-1}(E) = \int_E g_0 \, d\lambda$, it follows that $\lambda(\Phi \circ \Phi)^{-1}(E) = 0$ implies $\lambda \Phi^{-1}(E) = 0$.

Theorem 3. *Let X be a σ-finite standard Borel space and let* $C_\Phi \in B(L^2(X, \mathcal{S}, \lambda))$. *Then the operator* C_Φ *has ascent* 1 *if and only if* $\Phi[X_1] \supseteq X_1$, *where* $\Phi[X_1] = \{\Phi(x_1) \colon x_1 \in X_1\}$ *and* $X_1 = X \setminus X_0$.

Proof. Suppose $\Phi[X_1]\supseteqq X_1$. Then, since $\operatorname{Ker} C_\Phi=L^2(X_0)$ and $L^2(X)=$ $=L^2(X_0)\oplus L^2(X_1)$, every $f\in\operatorname{Ker} C_\Phi^2$ can be written as

$$f=f_1+g_1,$$

where $f_1\in\operatorname{Ker} C_\Phi$ and $g_1\in L^2(X_1)$. Since

$$g_1\circ\Phi\circ\Phi = C_\Phi^2 g_1 = C_\Phi^2 f = 0 \quad\text{and}\quad \Phi[X_1]\supseteqq X_1,$$

it follows that $g_1=0$ a.e. on X_1. Hence $f=f_1$. Thus $\operatorname{Ker} C_\Phi^2\subseteqq\operatorname{Ker} C_\Phi$. The inclusion otherway is true in general for every operator. This shows that $\operatorname{Ker} C_\Phi=\operatorname{Ker} C_\Phi^2$.

Conversely, suppose $\Phi[X_1]\not\supseteqq X_1$. Now, if E is a measurable subset of $X_1\setminus\Phi[X_1]$ of non-zero finite measure, then $C_\Phi^2\chi_E=0$. Since E is a subset of X_1, it follows that $C_\Phi\chi_E\neq 0$, which implies $\operatorname{Ker} C_\Phi\neq\operatorname{Ker} C_\Phi^2$. Hence the proof of the theorem is completed.

Corollary 4. *Let* $C_\Phi\in B(l^2(\mathbf{N},\mathscr{S},\lambda))$. *Then* $\operatorname{Ker} C_\Phi=\operatorname{Ker} C_\Phi^2$ *if and only if* $(\Phi\circ\Phi)[\mathbf{N}]=\Phi[\mathbf{N}]$, *where* $\Phi[\mathbf{N}]$ *is the range of* Φ.

Example 5. Let $X=\mathbf{R}$, the set of real numbers, and let $\Phi(x)=|x|$, $x\in\mathbf{R}$. Then C_Φ is a composition operator with ascent 1 on $L^2(\mathbf{R})$.

In the following theorem, we shall characterize composition operator with descent 1.

Theorem 6. *Let* $C_\Phi\in B(L^2(\lambda))$. *Then* $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$ *if and only if* $\Phi^{-1}(\mathscr{S})=$ $=(\Phi\circ\Phi)^{-1}(\mathscr{S})$, *where* $\mathscr{S}$ *is the σ-algebra of measurable subsets of* X.

Proof. Suppose $\Phi^{-1}(\mathscr{S})=(\Phi\circ\Phi)^{-1}(\mathscr{S})$. Then, since the ranges of C_Φ and C_Φ^2 are dense in $L^2(X,\Phi^{-1}(\mathscr{S}),\lambda)$ and $L^2(X,(\Phi\circ\Phi)^{-1}(\mathscr{S}),\lambda)$ respectively [13, Lemma 2.4], it follows that $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

Conversely, suppose $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$. Then

$$L^2(X,\Phi^{-1}(\mathscr{S}),\lambda)=L^2(X,(\Phi\circ\Phi)^{-1}(\mathscr{S}),\lambda).$$

We claim that $\Phi^{-1}(\mathscr{S})=(\Phi\circ\Phi)^{-1}(\mathscr{S})$. Suppose $\Phi^{-1}(\mathscr{S})\neq(\Phi\circ\Phi)^{-1}(\mathscr{S})$. Then, since $(\Phi\circ\Phi)^{-1}(\mathscr{S})$ is a subfamily of $\Phi^{-1}(\mathscr{S})$, there exists an element $E=\Phi^{-1}(F)$ which does not belong to $(\Phi\circ\Phi^{-1}(\mathscr{S})$. Since X is σ-finite, we can write

$$X=\bigcup_{i=1}^{\infty}X_i=\bigcup_{i=1}^{\infty}\Phi^{-1}(X_i)=\bigcup_{i=1}^{\infty}(\Phi\circ\Phi)^{-1}(X_i),$$

where $\{X_i\}$ is a disjoint sequence of sets of finite measure. Let $F_i=E\cap(\Phi\circ\Phi)^{-1}(X_i)$. Then F_i does not belong to $(\Phi\circ\Phi^{-1}(\mathscr{S})$ for some $i\in\mathbf{N}$ or otheswise E will be in

$(\Phi\circ\Phi)^{-1}(\mathcal{S})$. This shows that $\chi_{F_i}\notin\overline{R(C_\Phi^2)}$ for that $i\in\mathbf{N}$, which is a contradiction. Thus the proof of the theorem is complete.

In the following theorem we characterize composition operators with descent 1 in particular case.

Theorem 7. *Let X be a σ-finite standard Borel space and let $C_\Phi\in B(L^2(\lambda))$. Then $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$ if and only if $\Phi|X_1$ is injective a.e., where $X_1=\{x: f_0(x)\neq 0\}$.*

Proof. Since $\overline{R(A)}\supseteq\overline{R(A^2)}$ for any bounded operator A on a Hilbert space, it follows that $\overline{R(C_\Phi)}\supseteq\overline{R(C_\Phi^2)}$. We will show that if $\Phi|X_1$ is injective a.e., then the equality prevails. Suppose $\chi_E\in\overline{R(C_\Phi)}$. Then there exists a measurable subset $F\subseteq X_1$ such that $E=\Phi^{-1}(F)$. Since $\Phi|X_1$ is injective a.e. and X is a σ-finite standard Borel space, $\Phi[F]$ is a Borel set and $\Phi[F]=\bigcup_{n=1}^{\infty}E_n$ for some disjoint sequence $\{E_n\}$ of measurable subsets of finite measure. Consider the sum

$$\sum_{n=1}^{\infty}\chi_{E_n}\circ\Phi\circ\Phi=\sum_{n=1}^{\infty}C_\Phi^2\chi_{E_n}.$$

It is easy to see that the sum converges to χ_E a.e. By the Lebesgue dominated convergence theorem it converges to χ_E in L^2-norm. Hence χ_E belongs to the closure of $R(C_\Phi^2)$. From this it follows that all simple functions which belong to $\overline{R(C_\Phi)}$ also belongs to $\overline{R(C_\Phi^2)}$. This is enough to establish the equality $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

Conversely, suppose $\Phi|X_1$ is not injective a.e. Then, since X_1 is a Borel set, by Corollary 8.2 [22], there exists two Borel sets A and Z such that $\Phi_1=\Phi|X_1$ is one-to-one on A onto Z, $\lambda\Phi_1^{-1}(X_1\setminus Z)=0$ and $\lambda(X_1\setminus A)\neq 0$. Let $F\subseteq(X_1\setminus A)$ be a measurable set of finite measure such that $\lambda(A\cap\Phi^{-1}(\Phi[F]))\neq 0$. Then $\chi_{\Phi^{-1}(F)}=C_\Phi\chi_F\in\in R(C_\Phi)$. We claim that $\chi_{\Phi^{-1}(F)}$ does not belong to $\overline{R(C_\Phi^2)}$. If $\chi_{\Phi^{-1}(F)}\in\overline{R(C_\Phi^2)}==L^2(X,(\Phi\circ\Phi)^{-1}(\mathcal{S}),\lambda)$, then there exists $E\in\mathcal{S}$ such that $\Phi^{-1}(F)=(\Phi\circ\Phi)^{-1}(E)==\Phi^{-1}(G)=\Phi^{-1}(G\cap A)\cup\Phi^{-1}(G\setminus(G\cap A))$, where $G=\Phi^{-1}(E)$. Since $\lambda(A\cap\cap\Phi^{-1}(\Phi[F])\neq 0$, we can conclude that $\lambda(G\cap A)\neq 0$, and hence $\lambda(\Phi^{-1}(G\cap A))==\int_{G\cap A}f_0\neq 0$ which is a contradiction.

Corollary 8. *Let $\inf\{\lambda(n): n\in\mathbf{N}\}>c>0$ and $\sup\{\lambda(n): n\in\mathbf{N}\}<\infty$ and let $C_\Phi\in B(l^2(\mathbf{N},\mathcal{S},\lambda))$. Then $R(C_\Phi)=R(C_\Phi^2)$ if any only if $\Phi|\Phi[\mathbf{N}]$ is one-to-one.*

Example 9. Let $X=[-1,1]$ and λ be the Lebesgue measure on the Borel subsets of X. Let $\Phi(x)=|x|$. Then $C_\Phi\in B(L^2(\lambda))$ and $R(C_\Phi)=R(C_\Phi^2)$.

We shall give an example of a composition opeartor when $R(C_\Phi)\neq R(C_\Phi^2)$ but $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

Example 10. Let $X=\mathbf{R}$, and let $\mathcal{S}$ be the σ-algebra of Borel subsets of $\mathbf{R}$

wint λ as the Lebesgue measure. Define the measurable function Φ as follows

$$\Phi(x) = \begin{cases} 1/x & \text{if } x\in]0,1[, \\ -x-(n-2) & \text{if } x\in [1,\infty[\text{ and } n \leqq x < n+1, \quad n = 1,2,3,\dots, \\ x-(n-1) & \text{if } x\in]-\infty, 0] \text{ and } n \leqq -x < n+1, \quad n = 0,1,2,\dots. \end{cases}$$

This Φ induces a composition operator C_Φ on $L^2(\mathbf{R})$. Since $\chi_{]-1,0]}\notin R(C_\Phi^2)$ and $\chi_{]-1,0]}\in R(C_\Phi)$, it follows that $R(C_\Phi)\neq R(C_\Phi^2)$. But, since $(\Phi\circ\Phi)^{-1}(\mathcal{S})=\Phi^{-1}(\mathcal{S})$, we have $\overline{R(C_\Phi)}=\overline{R(C_\Phi^2)}$.

4. Partial isometry and co-isometry

Definition. An operator A on a Hilbert space is said to be a *partial isometry* if it is an isometry on the orthogonal complement of its kernel.

Theorem 11. *Let C_Φ be a composition operator on $L^2(X, \mathcal{S}, \lambda)$. Then C_Φ is a partial isometry if and only if f_0 is a characteristic function.*

Proof. Suppose C_Φ is a partial isometry. Then $C_\Phi=C_\Phi C_\Phi^* C_\Phi$, [8, Corollary 3, Problem 98] and it follows that $C_\Phi^* C_\Phi=C_\Phi^* C_\Phi C_\Phi^* C_\Phi$ which is equivalent to $M_{f_0}=M_{f_0}\cdot M_{f_0}=M_{f_0^2}$. From this we conclude that f_0 is a characteristic function.
Conversely, suppose f_0 is a characteristic function. Then, since $\operatorname{Ker} C_\Phi= =L^2(X_0)$ and $(\operatorname{Ker} C_\Phi)^\perp=L^2(X_1, \mathcal{S}_1, \lambda)$, where $X_1=X\setminus X_0$ and $\mathcal{S}_1=\{E\cap X_1\colon E\in\mathcal{S}\}$, it follows that

$$C_\Phi^* C_\Phi f = M_{f_0} f = f \quad \text{for all} \quad f\in(\operatorname{Ker} C_\Phi)^\perp.$$

This shows that C_Φ is an isometry on the orthogonal complement of its kernel.

Corollary 12. *Let $C_\Phi\in B(l^2(N))$, where $l^2(N)=\{\{a_n\}\colon \Sigma|a_n|^2<\infty\}$. Then C_Φ is a partial isometry if and only if Φ is one-to-one.*

Proof. Since

$$f_0(n) = \frac{\lambda\Phi^{-1}(n)}{\lambda(n)} = \lambda\Phi^{-1}(n),$$

the Corollary follows.

Example 13. Let $X=[0,\infty[$ and λ be the Lebesgue measure on the Borel subsets of X. Let $\Phi_c(x)=x+c$, where $c\in X$. Then $C_{\Phi_c}\in B(L^2(\lambda))$; $f_0(x)=1$, for $c\leqq x<\infty$, and $f_0(x)=0$, for $0\leqq x<c$. Hence by the above theorem $\{C_{\Phi_c}\colon c\in X\}$ is a family of partial isometries on $L^2(X)$.

Definiton. An operator A on a Hilbert space is called a *co-isometry if* $AA^*=I$.

Theorem 14. *Let* $C_\Phi \in B(L^2(\lambda))$. *Then* C_Φ *is a co-isometry if and only if* C_Φ *is onto and* $f_0 \circ \Phi = 1$ *a.e.*

Proof. Since, for every $f \in R(C_\Phi)$,

$$f = C_\Phi C_\Phi^* f = C_\Phi C_\Phi^* C_\Phi g = C_\Phi M_{f_0} g = C_\Phi (f_0 \cdot g) = f_0 \circ \Phi \cdot f,$$

where $C_\Phi g = f$, C_Φ is is co-isometry if and only if C_Φ is onto and $f_0 \circ \Phi = 1$ a.e.

Corollary 15. *Let* $C_\Phi \in B(l^2(N))$. *Then the following statements are equivalent:*

(i) C_Φ *is partial isometry,*

(ii) C_Φ *is co-isometry,*

(iii) C_Φ *is onto,*

(iv) Φ *is one-to-one.*

5. Hyponormal composition operators

Definition. An operator A on a Hilbert space H is called *hyponormal* if $A^*A - AA^* \geqq 0$.

In [9] hyponormal composition operators have been studied but it remains an open problem to find measure theoretic condition which is both necessary and sufficient for the hyponormality of C_Φ.

Lemma 16. *Let* C_Φ *be a composition operator on* $L^2(\lambda)$. *Then* C_Φ *is hyponormal only if* C_Φ *is one-to-one.*

Proof. Suppose C_Φ is hyponormal. Then

$$\operatorname{Ker} C_\Phi C_\Phi^* \supseteqq \operatorname{Ker} C_\Phi^* C_\Phi = \operatorname{Ker} C_\Phi = L^2(X_0).$$

Since $\operatorname{Ker} C_\Phi C_\Phi^* = \operatorname{Ker} C_\Phi^* = R(C_\Phi)^\perp = L^2(X, \Phi^{-1}(\mathcal{S}), \lambda)^\perp$, then, for every measurable subset E of X_0 of non-zero finite measure, there exists an element F in $\mathcal{S}$ such that

$$(\chi_E, C_\Phi \chi_F) = \langle \chi_E, \chi_{\Phi^{-1}(F)} \rangle \neq 0,$$

which is contradiction. Thus it follows that the measure of X_0 is zero. This shows that C_Φ is one-to-one.

Corollary 17. *Let* $C_\Phi \in l^2(\mathrm{N}, \mathcal{S}, \lambda)$. *Then* C_Φ *is hyponormal only if* Φ *is onto.*

Lemma 18. *Let* $C_\Phi \in B(l^2(\mathrm{N}, \mathcal{S}, \lambda))$. *Then*

$$e_n = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \chi_{\Phi^{-1}(\Phi(n))} + e_n',$$

where e_n is the characteristic function of $\{n\}$, and

$$e_n' \in \left(l^2(\mathrm{N}, \Phi^{-1}(\mathcal{S}), \lambda)\right)^{\perp}, \quad \Phi^{-1}(\mathcal{S}) = \{\Phi^{-1}(E) \colon E \in \mathcal{S}\}.$$

Proof. Since

$$l^2(\mathrm{N}, \mathcal{S}, \lambda) = \overline{R(C_\Phi)} \oplus R(C_\Phi)^{\perp} = l^2(\mathrm{N}, \Phi^{-2}(\mathcal{S}), \lambda) \oplus \left(l^2(\mathrm{N}, \Phi^{-1}(\mathcal{S}), \lambda)\right)^{\perp},$$

e_n admits the form

$$e_n = c\chi_{\Phi^{-1}(\Phi(n))} + e_n', \quad e_n' \in R(C_\Phi)^{\perp},$$

and it follows that

$$e_n' = e_n - c\chi_{\Phi^{-1}(\Phi(n))}.$$

Since $c\chi_{\Phi^{-1}(\Phi(n))} \perp e_n'$,

$$\langle c\chi_{\Phi^{-1}(\Phi(n))}, e_n - c\chi_{\Phi^{-1}(\Phi(n))} \rangle = 0,$$

which implies that

$$c = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))}.$$

This completes the proof of the Lemma.

Using the notation

$$f_0(n) = \frac{\lambda\Phi^{-1}(n)}{\lambda(n)},$$

C_Φ^*, the adjoint of C_Φ, can be expressed as follows:

$$C_\Phi^* e_n = C_\Phi^* \left(\frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \chi_{\Phi^{-1}(\Phi(n))} + e_n' \right) = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} C_\Phi^* C_\Phi \chi_{\Phi(n)} =$$

$$= \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} f_0 \cdot \chi_{\Phi(n)} = \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \cdot f_0 \circ \Phi(n) \cdot \chi_{\Phi(n)} =$$

$$= \frac{\lambda(n)}{\lambda\Phi^{-1}(\Phi(n))} \cdot \frac{\lambda\Phi^{-1}(\Phi(n))}{\lambda(\Phi(n))} \cdot \chi_{\Phi(n)} = \frac{\lambda(n)}{\lambda\Phi(n)} \cdot \chi_{\Phi(n)}.$$

The proof of the following theorem is analogous to the proof of the Proposition 11.5 [2].

Theorem 19. *Let $C_\Phi \in B(l^2(\mathrm{N}, \mathcal{S}, \lambda))$. Then C_Φ is hyponormal if and only if Φ is onto and*

$$\sum_{m \in \Phi^{-1}(n)} \frac{(\lambda(m))^2}{\lambda\Phi^{-1}(m)} \leqq \lambda(n).$$

Proof. Suppose C_Φ is hyponormal. Then, by Corollary 17., Φ is onto. Let ζ_n be the subspace spanned by $\{e_m\}_{m \in \Phi^{-1}(n)}$, $f \in \zeta_n$ and $f = \sum c_m e_m$. Then

$$\int |f \circ \Phi|^2 \, d\lambda = \langle C_\Phi f, C_\Phi f \rangle \geqq \langle C_\Phi^* f, C_\Phi^* f \rangle = \left\langle C_\Phi^* \sum c_m e_m, C_\Phi^* \sum c_m e_m \right\rangle$$

and thus, by the above computation,

$$\int |f|^2 \, d\lambda\Phi^{-1} \geqq \left\langle \sum c_m \frac{\lambda(m)}{\lambda\Phi(m)} e_{\Phi(m)}, \sum c_m \frac{\lambda(m)}{\lambda\Phi(m)} e_{\Phi(m)} \right\rangle.$$

This implies

$$\|f\|^2_{\lambda\Phi^{-1}} \geqq \left|\sum c_m \frac{\lambda(m)}{\lambda\Phi(m)}\right|^2 \cdot \lambda(n) = \frac{1}{\lambda(n)} \left|\sum c_m \lambda(m)\right|^2 = \frac{1}{\lambda(n)} \left|\langle f, h|\Phi^{-1}(n)\rangle_{\lambda\Phi^{-1}}\right|^2,$$

where

$$h = \frac{d\lambda}{d\lambda\Phi^{-1}}.$$

Since the inner product with $h|\Phi^{-1}(n)$ in $L^2(\mathrm{N}, \mathcal{S}, \lambda\Phi^{-1})$ induces a linear functional,

$$\lambda(n) \geqq \|h|\Phi^{-1}(n)\|^2_{\lambda\Phi^{-1}} = \sum \left(\frac{\lambda(m)}{\lambda\Phi^{-1}(m)}\right)^2 \lambda\Phi^{-1}(m) = \sum_{m\in\Phi^{-1}(n)} \frac{(\lambda(m))^2}{\lambda\Phi^{-1}(m)}.$$

Conversely, suppose the hypothesis of the theorem holds. Then

$$\langle C_\Phi C_\Phi^* f, f\rangle = \langle C_\Phi^* f, C_\Phi^* f\rangle = \left\langle C_\Phi^* \sum c_n e_n, C_\Phi^* \sum c_n e_n \right\rangle =$$

$$= \left\langle \sum c_n \frac{\lambda(n)}{\lambda\Phi(n)} \chi_{\Phi(n)}, \sum c_n \frac{\lambda(n)}{\lambda\Phi(n)} \chi_{\Phi(n)} \right\rangle =$$

$$= \left\langle \sum_n \sum_{i\in\Phi^{-1}(n)} c_i \frac{\lambda(i)}{\lambda\Phi(i)} e_n, \sum_n \sum_{i\in\Phi^{-1}(n)} c_i \frac{\lambda(i)}{\lambda\Phi(i)} e_n \right\rangle =$$

$$= \sum_n \frac{1}{\lambda(n)} \Big|\sum_{i\in\Phi^{-1}(n)} c_i \lambda(i)\Big|^2 = \sum_n \left|\left\langle f|\Phi^{-1}(n), \frac{1}{\sqrt{\lambda(n)}} h \right\rangle_{\lambda\Phi^{-1}}\right|^2 \leqq$$

$$\leqq \sum \|f|\Phi^{-1}(n)\|^2_{\lambda\Phi^{-1}} = \|f\|^2_{\lambda\Phi^{-1}} = \langle C_\Phi f, C_\Phi f\rangle = \langle C_\Phi^* C_\Phi f, f\rangle,$$

which shows that C_Φ is hyponormal.

Let X be a σ-finite standard Borel space and X_1 be the maximal subset of X such that $\Phi^{-1}(\Phi(x)\cap(X_1\setminus\{x\}))\neq\emptyset$ for $x\in X_1$. Let $X_2=\Phi[X_1]=\{\Phi(x_1)\colon x_1\in X_1\}$. Then X_2 is a Borel set [22, page 3]. Let $f_0(x)=c_n$ for $x\in X_2^{(n)}$, where $\{X_2^{(n)}\}$ is a disjoint sequence of sets such that $\bigcup_n X_2^{(n)}=X_2$, and let $Y_2^{(n)}=\Phi^{-1}(X_2^{(n)})$.

In the following theorem we consider measurable transformation Φ on a σ-finite standard Borel space such that f_0 satisfies the above property and find necessary condition for C_Φ induced by such Φ to be hyponormal which would explain $f_0\geqq \geqq f_0\circ\Phi$ [9, Theorem 9, Example 16] is not a necessary condition for hyponormality of C_Φ.

Theorem 20. *Let C_Φ be a composition operator induced by above type measurable transformation. Then C_Φ is hyponormal only if*

$$\int_E f_0\, d\lambda \geqq \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \int_E f_0 \circ \Phi\, d\lambda, \quad E \subset Y_2^{(n)}$$

and

$$f_0(x) \geqq f_0 \circ \Phi(x) \quad \text{a.e. on} \quad x \in X \setminus X_1,$$

where X_1 is the maximal subset of X, σ-finite standard Borel space, such that $\Phi^{-1}(\Phi(x)) \cap (X_1 \setminus \{x\}) \neq \emptyset$ for $x \in X_1$.

Proof. It follows similarly as in Lemma 18 that

$$\chi_E = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \chi_{\Phi^{-1}(\Phi(E))} + g, \quad g \in R(C_\Phi)^\perp, \quad E \subset Y_2^{(n)}$$

and

$$C_\Phi^* \chi_E = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} C_\Phi^* C_\Phi \chi_{\Phi(E)} = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} f_0 \cdot \chi_{\Phi(E)}.$$

Since C_Φ is hyponormal, then for $E \subset Y_2^{(n)}$

$$\int_E f_0\, d\lambda = \langle C_\Phi^* C_\Phi \chi_E, \chi_E \rangle \geqq \langle C_\Phi C_\Phi^* \chi_E, \chi_E \rangle = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \langle f_0 \circ \Phi \chi_{\Phi^{-1}(\Phi(E))}, \chi_E \rangle =$$

$$= \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \langle f_0 \circ \Phi \cdot \chi_E, \chi_E \rangle = \frac{\lambda(E)}{\lambda\Phi^{-1}(\Phi(E))} \int_E f_0 \circ \Phi\, d\lambda.$$

If $E = \{x : f_0(x) < f_0 \circ \Phi(x),\ x \in X \setminus X_1\}$ has a positive measure, then for a finite set $F \subset E$.

$$\int_F f_0\, d\lambda = \langle C_\Phi^* C_\Phi \chi_F, \chi_F \rangle < \int_F f_0 \circ \Phi\, d\lambda = \langle C_\Phi C_\Phi^* \chi_F, \chi_F \rangle,$$

which is a contradiction. This proves the theorem.

The above theorem explains why the function in Example 10 [9, p. 131] does not induce hyponormal composition operator.

Since in Example 10 [9, p. 131]

$$\int_{[1,3/2]} f_0\, d\lambda = 1/4 \cdot 1/2 = 1/8 < \frac{\lambda[(1, 3/2])}{\lambda\Phi^{-1}(\Phi[1, 3/2]]} \int_{[1,3/2]} f_0 \circ \Phi\, d\lambda =$$

$$= \frac{1/2}{1+1/2+1} \cdot 5/2 \cdot 1/2 = 1/4,$$

C_Φ is not hyponormal.

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