Acta Sci. Math., 56 (1992), 347—358

On the coadjoint orbits of connected Lie groups

L. PUKANSZKY

Introduction. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$, O an orbit, of positive dimension, of the coadjoint representation and ω_0 the corresponding canonical 2-form (cf. [2], Proposition 5.2.2, p. 182). It is well-known, that pairs like (O, ω_O) play an important role in many questions of the unitary representation theory of G. The objective of the present paper is to analyse (O, ω_0) by aid of suitable ideals of $\mathfrak{g}$. In more details, given an element g of O, we define $B(x, y)\equiv([x, y], g)$ $(x, y\in\mathfrak{g})$. Let $\mathfrak{m}$ be an ideal of $\mathfrak{g}$, different from $\mathfrak{g}$. We say, that it is admissible, if it contains its orthogonal complement, with respect to B, for one and hence for all g of O. Such ideals always exist if $\mathfrak{g}$ is nilpotent, and are of a common occurrence when $\mathfrak{g}$ is solvable (cf. Section 4 below). Let $\mathcal{O}$ be the projection of O on $\mathfrak{m}^*$, the dual of the underlying space of $\mathfrak{m}$. Then $\mathfrak{m}$ determines a subbundle $\mathfrak{M}$ of the tangent bundle $T(O)$ of O. Let O' be the subbundle, orthogonal to $\mathfrak{M}$, of the cotangent bundle $T^*(O)$ of O. O and O' carry canonically the structure of a principal bundle, with the structure group $\mathfrak{m}^\perp$, over $\mathcal{O}$; O is acted upon by $\mathfrak{m}^\perp$ through translations and both bundles are trivial. Let s be a global section of O; it determines an isomorphism φ of principal $\mathfrak{m}^\perp$-bundles over $\mathcal{O}$, from O onto O' (cf. Lemma 9 and Lemma 11 in Section 2). We set $\eta=s^*\omega_O\in Z^2(O)$ and write p for the canonical projection from O' onto O. Let ϑ be the canonical 1-form on $T^*(O)$. Our principal result (cf. Theorem 1 in Section 3) states, that

$$\omega_O = \varphi^*(p^*\eta - d\vartheta).$$

As an application, in Section 4 we give a new proof for the existence of global Darboux coordinates in the case, when G is is solvable and O is simply connected (cf. [5], Theorem 3, p. 208).

The organization of the paper is as follows. Section 1 discusses the bundle structure of O, and Section 2 the relation of O to O'. Section 3 contains the proof of the result quoted above, and Section 4 the discussion of the Darboux coodordinates.

Received December 18, 1990.

The reader is advised to consult the end of the paper, where some key notational conventions, employed throughout the paper, are explained.

1. As stated above, the objective of this section is the investigation of some bundle structure on O. The proof of the principal statement (cf. Proposition) could be abbreviated by the use of standard results (cf. in particular [1], 16.14.1, p. 87) but some elements of the proof below will be needed later.

Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. If $\mathfrak{a}$ is a subspace of g, $\mathfrak{a}_B^{\perp} \subset \mathfrak{g}$ will stand for its orthogonal complement with respect to B, belonging to some $g \in \mathfrak{g}^*$ specified by the context and $\mathfrak{a}^{\perp}$ for the orthogonal complement in $\mathfrak{g}^*$. We fix an orbit O, of positive dimension, of the coadjoint representation and an ideal $\mathfrak{m}$, admissible with respect to O, that is $\mathfrak{m}_B^{\perp} \subset \mathfrak{m}$ for one, and thus for all elements of O. Fixing an element g of O, we set $K = G_g$, and consider O as a C^∞-manifold by transfer from G/K. Let us note, that the identity map from O into $\mathfrak{g}^*$ is smooth. We write h for $g|\mathfrak{m}$, and set $T = G_h$; $\mathcal{O}$ has a differentiable structure as G/T. Let π be the restriction map $\mathfrak{g}^* \to \mathfrak{m}^*$. We recall (cf. [1], 16.14.9, p. 94) that with the above definitions $(O, \mathcal{O}, \pi)$ is a fiber bundle with a fiber diffeomorphic to T/K. In the following we show that this fibration is identical with the orbit space of $\mathfrak{m}^{\perp}$, acting on O by translation.

Lemma 1. *With the above notation we have:* $(G_h)_0\, g = g + \mathfrak{m}^{\perp}$.

Proof. (i) For $n = 2, 3, \ldots$, let $\{l_j : 1 \leqq j \leqq n\}$ be some subset of $\mathfrak{g}_h = \mathfrak{m}_B^{\perp}$. We claim, that $l_1 \ldots l_n g = 0$. In fact, let L be the left-hand side. Given an element $k \in \mathfrak{g}$, we put $l = (-1)^n [l_{n-1} \ldots [l, k] \ldots]$. Since $n \geqq 2$ and $\mathfrak{g}_h \subset \mathfrak{m}$, l belongs to $\mathfrak{m}$, and thus we conclude that $(k, L) = ([l_n, l], g) = 0$ by virtue of $l_n \in \mathfrak{g}_h$. Since $(G_h)_0$ is generated by elements of the form $\exp(l)$ $(l \in \mathfrak{g}_h)$ we conclude that $(G_h)_0 g = g + \mathfrak{g}_h g$. — (ii) This being so it is enough to prove that if $\mathfrak{m}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{g}_g$, we have $\mathfrak{m}^{\perp} = \mathfrak{g}_h g$. Note, that if $\mathfrak{a}$ is a subspace of $\mathfrak{m}$, then $\mathfrak{a}_B^{\perp} = (\mathfrak{a}g)^{\perp}$. We have therefore

$$\mathfrak{m} = \mathfrak{m} + \mathfrak{g}_g = (\mathfrak{g}_h)_B^{\perp} = (\mathfrak{g}_h g)^{\perp},$$

whence $\mathfrak{g}_h g = \mathfrak{m}^{\perp}$. Summing up, we have proved that

$$(G_h)_0 g = g + \mathfrak{m}^{\perp}.$$

From here we can conclude

Lemma 2. *The triple* $(O, \mathcal{O}, \pi)$ *is a principal* $\mathfrak{m}^{\perp}$*-space.*

Lemma 3. *The map* $t \mapsto tg$ $(t \in T)$ *induces a diffeomorphism* $T/K \cong g + \mathfrak{m}^{\perp}$.

Proof. We recall (cf. [1], 16.10.7, p. 62) that if G acts smoothly on the C^∞-manifold X, and $x \in X$ is such that Gx is locally closed, then Gx carries a differentiable structure, well-determined by the condition that $s \mapsto sx$ be a diffeomorphism from G/G_x onto Gx. We apply this by replacing X, G, x through $\mathfrak{g}^*$, G_h and g respectively. To conclude our proof it is enough to note that, by Lemma 1, we have: $Tg=g+m^{\perp}$, which is closed in $\mathfrak{g}^*$.

Lemma 4. *There is a global section* $s\colon \mathcal{O}\mapsto O$.

Proof. We recall (cf. [1], 16.12.2, p. 82) that if (X, B, π) is a fiber bundle, with a fiber diffeomorphic to $\mathbf{R}^N$, then there is a global section $s\colon B\mapsto X$. Thus it is enough to note that in our case, by Lemma 3, we have $T/K\cong g+\mathfrak{m}^{\perp}$.

For a fixed $s\in\Gamma(O)$, we define $f\colon O\mapsto\mathfrak{g}^*$ by $f(g)\equiv g-s(\pi(g))$. We can note at once that f is smooth, takes its values in $\mathfrak{m}^{\perp}$ and satisfies $f(g+v)=f(g)+v$ for any $g\in O$ and $v\in\mathfrak{m}^{\perp}$. We set $X=\mathcal{O}\times\mathfrak{m}^{\perp}$ and define $\Psi\colon O\mapsto X$ by $\Psi(g)==\pi\{(g),f(g)\}$ $(g\in O)$.

Lemma 5. Ψ *is a smooth bijection* $O\to X$.

Proof. Smoothness being evident, it is enough to show that is bijective. In fact, (i) Assume, that $\Psi(g)=\Psi(g')$. Then, in particular, $\pi(g)=\pi(g')$ and thus $g'==g+v$ with some $v\in\mathfrak{m}^{\perp}$. We have, however, also $f(g)=f(g')=f(g)+v$ and hence $v=0$ and $g=g'$. — (ii) We claim that Ψ is surjective. In fact, let $\{h, w\}\in X$ be given. Suppose that $g\in O$ satisfies $\pi(g)=h$. Defining $g'=g+w-f(g)$ we have clearly $\Psi(g')=\{h, w\}$. Summing up, we have shown, that Ψ is a smooth bijection $O\to X$. We recall that $K=G_g$, $h=g|m$ and $T=G_h$.

Lemma 6. *The restriction of the canonical map* $G/K\to O$ *to a fiber of* $G/K\to\to G/T$ *is an isomorphism of this fiber to an* $\mathfrak{m}^{\perp}$*-orbit of* O *(the latter considered as a submanifold of* g^**).*

Proof. Suppose that $g'\in O$ is given, and, say, $g'=ag$ $(a\in G)$. Then $a(T/K)$ is the fiber corresponding to g'. It is enough to show that the map $t\mapsto atg$ $(t\in T)$ induces an isomorphism $T/K\to g'+\mathfrak{m}^{\perp}$. But, by Lemma 3, the map of loc. cit. (h, say) from T/K onto $g+\mathfrak{m}^{\perp}$ induces an isomorphism and thus it suffices to observe that

$$\begin{tikzcd} a(T/K) \arrow[r] & g'+\mathfrak{m}^{\perp} \\ T/K \arrow[u, "a(\ )"] \arrow[r, "h"'] & g+\mathfrak{m}^{\perp}. \arrow[u, "a(\ )"'] \end{tikzcd}$$

Lemma 7. *With the above notation* $\Psi\colon O\to X$ *is an isomorphism of fiber bundles.*

Proof. By Lemma 5, Ψ is a smooth fiber preserving bijection $\Psi\colon O\to X$ and by Lemma 6, the restriction of Ψ to any fiber in O is an isomorphism with its image. Thus it is enough to recall (cf. [1], 16.21.2, p. 75) that (in particular) if (X, B, π) and (X', B, π') are fiber bundles and $f\colon X\to X'$ is a fiber-preserving smooth map, then it is an isomorphism of fiber bundles, if its restriction to any fiber is an isomorphism with its image.

Proposition. *$(O, \mathcal{O}, \pi)$ is a principal bundle with the structure group $\mathfrak{m}^\perp$ acting on O by translations.*

Proof. By what we have seen above, it is enough to observe that Ψ is equivariant with respect to the action of $\mathfrak{m}^\perp$ on O and X respectively.

2. The objective of this section is to present some material needed in the next section for the proof of Theorem 1. We continue to assume that O is a fixed orbit, of positive dimension, of the coadjoint representation and $\mathfrak{m}$ is an admissible ideal (cf. Introduction). We start by introducing some notational conventions. 1) If Y is a left G-space, $m \in Y$, and $x \in \mathfrak{g} = \mathrm{Lie}\,(G)$, we set

$$\sigma_m(x) = (d/dt)\exp(tx)m|_{t=0}.$$

Given $g \in \mathfrak{g}^*$, we denote by τ_g the canonical translation $T_g(\mathfrak{g}^*) \to \mathfrak{g}^*$ (cf. [1], p. 22). Note that we have clearly: $\tau_g\sigma_g(x) = xg$. 3) With the above notation we can write for $x, y \in \mathfrak{g}$:

$$\omega_O(\sigma_g(x) \wedge \sigma_g(y)) = B(x, y).$$

We remark, that if $t = \sigma_g(x)$ and $v \in T_g(O)$, then $\omega_O(t \wedge v) = (x, \tau_g v)$. In fact, assuming $v = \sigma_g(y)$ we have $\omega_O(t \wedge v) = ([x, y], g) = (x, yg) = (x, \tau_g v)$. We denote by $\mathfrak{N}$ the distribution on O such that $\tau_g N_g = \mathfrak{m}g$. Let us observe, that if $v_g \in \mathfrak{N}$ and $t \in T_g(O)$ is such that $T_g(\pi)t = 0$, then $\omega_O(t \wedge v_g) = 0$. In fact, assuming $t = \sigma_g(x)$, we have $0 = T_g(\pi)t = \sigma_h(x)$, whence $x \in \mathfrak{g}_h = \mathfrak{m}_B^\perp \subseteq \mathfrak{m}$. If $v_g = \sigma_g(y)$ for a $y \in \mathfrak{m}$ we have: $\omega_O(t \wedge v_g) = ([x, y], g) = 0$ by $x \in \mathfrak{g}_h$. We conclude from all this that there is a map $P\colon \mathfrak{N}_g \to (T_h(\mathcal{O}))^*$ such that $P(v_g)(T_g(\pi)t) = \omega_O(t \wedge v_g)$ $(t \in T_g(O))$. Writing $p\colon T^*(\mathcal{O}) \to \mathcal{O}$ and $pr\colon T(O) \to O$ for the canonical projections, we note

$$\begin{tikzcd}
\mathfrak{N} \arrow[r, "P"] \arrow[d, "pr"'] & T^*(\mathcal{O}) \arrow[d, "p"] \\
O \arrow[r, "\pi"'] & \mathcal{O}.
\end{tikzcd}$$

Let σ be a section $\mathcal{O} \to O$ (cf. Lemma 4) and form as loc. cit. $f(g) \equiv g - \sigma(\pi(g))$ $(g \in O)$. If f is any smooth map $O \to \mathfrak{m}^\perp$ we can define $F(g) \in T_g(\mathfrak{g}^*)$ by $\tau_g F(g) = f(g)$, and note that F is a vector field on O taking its values in $\mathfrak{N}$. In fact, to see this, it is enough to have $\mathfrak{m}^\perp \subseteq \mathfrak{m}g$; but this is equivalent to $\mathfrak{m}_B^\perp = (\mathfrak{m}g)^\perp \subset \mathfrak{m}$ or $\mathfrak{m}_B^\perp \subset \mathfrak{m}$, which we assume. All this being so, for $g \in O$ we set: $\varphi(g) = P(F(g))$; we have clearly

$$\begin{tikzcd}
O \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & T^*(\mathcal{O}) \arrow[dl, "\varrho"] \\
& \mathcal{O}. &
\end{tikzcd}$$

Lemma 8. *With notation as above, we have for* $g\in O$ *and* $x\in\mathfrak{g}$:

$$\varphi(g)\big(\sigma_h(x)\big) = \big(x, f(g)\big).$$

Proof. Writing $t=\sigma_g(x)$, we obtain

$$\varphi(g)\big(\sigma_h(x)\big) = P\big(F(g)\big)\big(T_g(\pi)\,t\big) = \omega_O\big(t\wedge F(g)\big) = \big(x, f(g)\big)$$

and thus: $\varphi(g)\big(\sigma_h(x)\big)=\big(x, f(g)\big)$ $(g\in O, x\in\mathfrak{g})$.

We denote by $\mathfrak{M}$ the subbundle of $T(\mathcal{O})$ such that $\tau_h M_h=\mathfrak{m}h$. Recalling, that $X=\mathcal{O}\times\mathfrak{m}^\perp$, we note that there is a canonical identification between $\mathfrak{M}^\perp$ and X. In fact, given $\lambda\in\mathfrak{M}^\perp$ let us put $h=p(\lambda)$. We define $\lambda'\in\mathfrak{m}^\perp$ by $\lambda'(x)\equiv\lambda(\sigma_h(x))$ $(x\in\mathfrak{g})$. This being so, we set $\Phi(\lambda)=\{h, \lambda'\}$. We observe that Φ is a bijection $\mathfrak{M}^\perp\to X$. In fact, if $\Phi(\mu)=\Phi(\nu)=\{h, \lambda\}$, say, we have $\mu, \nu\in(T_h(\mathcal{O}))^*$ and $\mu(\sigma_h(x))\equiv\equiv\lambda(x)\equiv\nu(\sigma_h(x))$ $(x\in\mathfrak{g})$, and thus $\mu=\nu$ and Φ is injective. Let now $\{h, \lambda'\}\in X$ be given. If $\sigma_h(x)=0$, we have $x\in\mathfrak{g}_h\subseteq\mathfrak{m}$ and hence we can define $\lambda\in(T_h(\mathcal{O}))^*$ by $\lambda(\sigma_h(x))\equiv\lambda'(x)$ $(x\in\mathfrak{g})$. In this fashion $\Phi(\lambda)=\{h, \lambda'\}$, and Φ is surjective. Below we shall write O' for $\mathfrak{M}^\perp$. We can define on O' the structure of a principal $\mathfrak{m}^\perp$-bundle as follows. Given $v\in\mathfrak{m}^\perp$, let $A_h(v)\in(T_h(\mathcal{O}))^*$ such that $A_h(v)(\sigma_h(x))\equiv\equiv(x, v)$ $(x\in\mathfrak{g})$. Then if $\lambda\in O'$ and $p(\lambda)=h$, we can set $\lambda v=\lambda+A_h(v)$. We note, that $\Phi(\lambda v)=\Phi(\lambda)v$. — We remark that if $g\in O$, then we have: $\varphi(g)\in O'$. In fact, Lemma 8 implies, that $\varphi(g)(\sigma_h(x))\equiv(x, f(g))$ $(x\in\mathfrak{g})$; but by $f(g)\in\mathfrak{m}^\perp$, the right-hand-side vanishes for $x\in\mathfrak{m}$.

Lemma 9. φ: $O\to O'$ *is an isomorphism of principal* $\mathfrak{m}^\perp$*-bundles over* $\mathcal{O}$.

Proof. Let Ψ: $O\to X$ be as in Lemma 5, corresponding to the section $\mathcal{O}\to O$ employed in the definition of φ. To obtain the desired conclusion, it is enough to note that clearly $\Phi\circ\varphi=\Psi$.

Lemma 10. *Let* ϑ *be the canonical* 1*-form on* $T^*(\mathcal{O})$. *Then, with notation as above, we have:* $\varphi^*\vartheta=-\iota(F)\omega_O$.

Proof. Assume that $t\in T_g(O)$. We have

$$\begin{aligned}(\varphi^*\vartheta)(t) &= \vartheta\big(T_g(\varphi)\,t\big) = \big(T_{\varphi(g)}(p)T_g(\varphi)\,t, \varphi(g)\big) = \\ &= \varphi(g)\big(T_g(p\circ\varphi)\,t\big) = \varphi(g)\big(T_g(\pi)\,t\big) = P\big(F(g)\big)\big(T_g(\pi)\,t\big) = \\ &= \omega_O\big(t\wedge F(g)\big) = -\big(\iota(F)\,\omega_O\big)(t)\end{aligned}$$

whence $\varphi^*\vartheta=-\iota(F)\omega_O$.

Lemma 11. *Let* φ: $O\to O'$ *be an isomorphism of principal* $\mathfrak{m}^\perp$*-bundles over* $\mathcal{O}$. *Then there is a section* $s\in\Gamma(O)$ *giving rise to* φ *as described before Lemma 8.*

Proof. (i) Given $t\in T_g(O)$, by virtue of the computation of the proof of Lemma 10 we have: $(\varphi^*\vartheta)(t)=\varphi(g)(T_g(\pi)t)$. — (ii) We define the vector field F on O by

$\varphi^*\vartheta=-\iota(F)\omega_O$ and set $f(g)\equiv\tau_g(F(g))$. We claim, that for all $x\in g$: $\varphi(g)(\sigma_h(x))=$ $=(x,f(g))$. In fact, writing $t=\sigma_g(x)$ we have

$$\varphi(g)(\sigma_h(x)) = \varphi(g)(T_g(\pi)t) = (\varphi^*\vartheta)(t) = \omega_O(t\wedge F(g)) = (x,f(g))$$

whence $\varphi(g)(\sigma_h(x))\equiv(x,f(g))$, as stated above. — (iii) a) We observe that f takes its values in $\mathfrak{m}^\perp$. In fact, we have $\varphi(g)\in O'$ and thus, by (ii) above: $(x,f(g))=$ $=\varphi(g)(\sigma_h(x))=0$ for all $x\in\mathfrak{m}$. b) We note that for any $g\in O$ and $v\in\mathfrak{m}^\perp$: $f(g+v)=$ $=f(g)+v$. In fact, we have for all $x\in\mathfrak{g}$:

$$(x,f(g+v)) = \varphi(g+v)(\sigma_h(x)) =$$

$$= \varphi(g)(\sigma_h(x))+A_h(v)(\sigma_h(x)) = (x,f(g))+(x,v) = (x,f(g)+v)$$

and thus $f(g+v)=f(g)+v$. In this manner we can define $s\in\Gamma(O)$ by $s(\pi(g))\equiv$ $\equiv g-f(g)$ $(g\in O)$. — (iv) We observe that $F(g)\in N_g$. We have, in fact $\tau_g F(g)=$ $=f(g)\in\mathfrak{m}^\perp\subset\mathfrak{m}g$, since $\mathfrak{m}$ is admissible with respect to O. In this fashion we can form $\psi(g)\equiv P(F(g))$. — (v) We show finally, that $\varphi=\psi$. In fact, we have by Lemma 8 and (ii) above: $\psi(g)(\sigma_h(x))=(x,f(g))=\varphi(\sigma_h(x))$ $(x\in\mathfrak{g})$, providing the desired conclusion.

3. The principal objective of this section is Theorem 1. We start with the following definition. Let us write $\mathfrak{b}$ for the quotient algebra $\mathfrak{g}/\mathfrak{m}$ and α for the canonical morphism $\mathfrak{g}\to\mathfrak{b}$. Given $x\in\mathfrak{g}$, we write X for the vector field on O satisfying $X_g=$ $=\sigma_g(x)$. This being so, we define the $\mathfrak{b}$-valued 1-form δ by $\delta(t_g)=\alpha(x)$, if $t_g=$ $=\sigma_g(x)$.

Lemma 12. *With the above notation we have:* $d\delta=[\delta,\delta]$

Proof. Let t, t' be in $T_g(O)$, $t=\sigma_g(x)$, $t'=\sigma_g(y)$, say $(x,y\in\mathfrak{g})$. We have

$$d\delta(t\wedge t') = d\delta(X_g\wedge Y_g) = X_g\delta(Y)-Y_g\delta(X)-\delta([X,Y]_g).$$

But $\delta(Y_g)\equiv\delta(\sigma_g(y))\equiv\alpha(y)$ and thus $X_g(\delta(Y))\equiv0$ and similarly, $Y_g\delta(X)\equiv0$. Writing $z=[x,y]$, we have $Z=-[X,Y]$. From this we conclude that

$$d\delta(t\wedge t') = \delta(Z_g) = \alpha(z) = \alpha([x,y]) = [\alpha(x),\alpha(y)] = [\delta(t),\delta(t')],$$

and therefore: $d\delta(t\wedge t')=[\delta(t),\delta(t')]$ $(t,t'\in T_g(O))$.

We note that there is a canonical identification between the dual $\mathfrak{b}^*$ and $\mathfrak{m}^\perp$. Given a $\mathfrak{b}$-valued k-form γ on O, and a smooth map $f\colon O\to\mathfrak{m}^\perp$, we shall write γ_f for the numerical-valued k-form defined at $g\in O$ by $\gamma_f(\ )=(\gamma(\ ),f(g))$. In particular, if $f(g)\equiv v\in\mathfrak{m}^\perp$ is fixed, we write γ_v for γ_f. — Below, given $v\in\mathfrak{m}^\perp$, we denote by L_v the map $L_v g=g+v$ $(g\in O)$.

Lemma 13. *With the above notation we have*

$$L_v^* \omega_O = \omega_O - (d\delta)_v.$$

Proof. Let t and t' be in $T_g(O)$ such that $t = \sigma_g(x)$, $t' = \sigma_g(y)$, say. There is an $\bar{x} \in \mathfrak{g}$ such that

$$xg = \tau_g t = \tau_{g+v}(T_g(L_v)t) = \bar{x}(g-v)'$$

and analogously for y. From this we conclude, that

$$(L_v^* \omega_O)(t \wedge t') = \omega_O(T_g(L_v)t \wedge T_g(L_v)t') =$$
$$= ([\bar{x}, \bar{y}], g+v) = (\bar{x}, \bar{y}(g+v)) = (\bar{x}, yg) = ([\bar{x}, y], g)$$

and therefore: $(L_v^* \omega_O - \omega_O)(t \wedge t') = ([\bar{x} - x, y], g)$. In this manner it will be enough to show that $([\bar{x} - x, y], g) = -(d\delta)_v(t \wedge t')$. To this end we note that a) $\bar{x} - x \in g_h$. In fact, we have by definition: $(\bar{x} - x)g = -\bar{x}v$, and thus for all $l \in \mathfrak{m}$: $([x-x, l], g) = = (l, x\bar{v}) = 0$. Next we note, that $xv = \bar{x}v$. In fact, to see this, by a) it suffices to observe that $av = 0$ for all $a \in \mathfrak{g}_n$. In this fashion we can conclude, that

$$([\bar{x} - x, y], g) = -(y, (\bar{x} - x)g) = (y, \bar{x}v) = (y, xv) =$$
$$= -([x, y], v) = -([\delta(t), \delta(t')], v) = -(d\delta)_v(t \wedge t')$$

where we have made use of Lemma 12. Summing up, we have thus obtained $L_v^* \omega_O = = \omega_O - (d\delta)_v$, as claimed at the beginning.

Since, as we have seen in Section 1, O is a principal bundle with the structure group $\mathfrak{m}^\perp$, below, whenever convenient, we shall write gv in place of $g + v = L_v g$ ($v \in \mathfrak{m}^\perp$). Note that gv can stand also for

$$(d/d\tau)(g + \tau v)|_{v=0} \in T_g(O).$$

Let $f: O \to \mathfrak{m}^\perp$ be a smooth map satisfying $f(g+v) \equiv f(g) + v$ ($g \in O$, $v \in \mathfrak{m}^\perp$). We define the $\mathfrak{m}^\perp$-valued 1-form ζ by $\zeta(t) = \tau_{f(g)}(T_g(f)t)$. We have for $g \in O$, $t \in T_g(O)$ and $v \in \mathfrak{m}^\perp$: 1) $\zeta_{gv}(T_g(L_v)t) = \zeta_g(t)$, 2) $\zeta_g(gv) = v$. In this manner ζ defines a connection form on the principal $\mathfrak{m}^\perp$-bundle $(O, \mathcal{O}, \pi)$. We shall write $V_g(O)$ for the collection of all vertical vectors at g that is $V_g(O) = \{t;\, t \in T_g(O)$ such that $T_g(\pi) = 0\}$. We recall that the dual $\mathfrak{b}^*$ of $\mathfrak{b}$ is canonically identifiable with $\mathfrak{m}^\perp$.

Lemma 14. *We have for* $t \in T_g(O)$, $w \in V_g(O)$:

$$\omega_O(t \wedge w) = (\delta(t), \zeta(v)).$$

Proof. To this end it is enough to note that, if $t = \sigma_g(x)$ and $w = \sigma_g(y)$ ($y \in \mathfrak{g}_h$), then we have

$$\omega_O(t \wedge w) = (x, yg) = (\delta(t), \zeta(w)).$$

Let us observe that, in particular, $V_g(O)$ is orthogonal to itself with respect to $(\omega_O)_g$.

Below we assume to be given a fixed choice of $s\in\Gamma(O)$; $\varphi: O\to O'$ will correspond to it as preceeding Lemma 8. — We recall that ϑ is the canonical 1-form on $T^*(O)$.

Lemma 15. *With the previous notation we have:* $\varphi^*\vartheta=\delta_f$.

Proof. Let $t\in T_g(O)$ be such that $t=\sigma_g(x)$ $(x\in \mathfrak{g})$. We have, as in the proof of Lemma 10, using Lemma 8,

$$(\varphi^*\vartheta)(t)=\varphi(g)\big(\sigma_h(x)\big)=\big(x, f(g)\big)=\delta_f(t)$$

and hence $(\varphi^*\vartheta)(t)=\delta_f(t)$.

Theorem 1. *With the previous noration let us put* $\eta=s^*\omega_O\in Z^2(\mathcal{O})$. *Then we have* $\omega_O=\varphi^*(p^*\eta-d\vartheta)$.

Proof. (i) Writing $L=\varphi^*p^*\eta\in Z^2(O)$, we have:

$$L=(p\circ\varphi)^*\eta=\tau^*s^*\omega_O=(s\circ\pi)^*\omega_O.$$

(ii) We have, by virtue of the flat connection, corresponding to ζ on the principal $\mathfrak{m}^{\perp}$-bundle $(O, \mathcal{O}, \pi)$, the following representation of $t_g\in T_g(O)$ as the sum of horizontal and vertical components

$$t_g=\big(T_g(s\circ\pi)(t_g)\big)f(g)+g\zeta(t_g).$$

Denoting by P the horizontal projection, we thus obtain:

$$T_g(s\circ\pi)\,t_g=T_g(L_{-f(g)})(Pt_g).$$

Given $t, t'\in T_g(O)$, we have by Lemma 13:

$$L(t\wedge t')=\omega_O\big(T_g(L_{-f(g)})\,Pt\wedge T_g(L_{-f(g)})\,Pt'\big)=(\omega_O)_P(t\wedge t')+(d\delta)_f(Pt\wedge Pt').$$

(iii) We claim that $d\delta(Pt\wedge Pt')=d\delta(t\wedge t')$. In fact, we have by Lemma 12: $d\delta(Pt\wedge Pt')=[\delta(Pt), \delta(Pt')]$ and thus it suffices to show, that $\delta(t)=\delta(Pt)$, or that $\delta(t)=0$ if $t\in V_g(O)$. To see this we can assume that $t=\sigma_g(x)$ $(x\in\mathfrak{g}_h)$. But then $\delta(t)=\alpha(x)=0$, by $\mathfrak{g}_h\subseteq\mathfrak{m}=\ker(\alpha)$. In this manner, by the end of (ii) above we obtain:

$$L(t\wedge t')=(\omega_O)_P(t\wedge t')+(d\delta)_f(t\wedge t').$$

(iv) For the definition to be used below, of the wedge product between two vector-valued 1-forms we refer to [1], 16.20.15.5, p. 141. — We maintain, that

$$(\omega_O)_P=\omega_O+\zeta\wedge\delta.$$

In fact, let us write P_v for the vertical projection. We have by Lemma 14:

$$(\omega_O)_P(t\wedge t') = \omega_O(Pt\wedge Pt') = \omega_O\big((t-P_v(t))\wedge(t'-P_v(t'))\big) =$$
$$= \omega_O(t\wedge t') - \omega_O\big(t\wedge P_v(t)\big) - \omega_O\big(P_v(t)\wedge t'\big) =$$
$$= \omega_O(t\wedge t') - \big(\delta(t), \zeta(t')\big) + \big(\delta(t'), \zeta(t)\big) = \omega_O(t\wedge t') + (\zeta\wedge\delta)(t\wedge t')$$

or $(\omega_O)_P(t\wedge t') = \omega_O(t\wedge t') + (\zeta\wedge\delta)(t\wedge t')$, proving our assertion. In this fashion we can conclude that $L=\omega_O+\zeta\wedge\delta+(d\delta)_f$. — (v) We assert next that $d(\delta_f)= =\zeta\wedge\delta+(d\delta)_f$. In fact, this is implied by the following simple proposition. Let V be a real vector space of dimension m, M a C^∞-manifold, γ a V-valued 1-form and $f\colon M\to V^*$ a smooth map. Then we have $d(\gamma_f)=df\wedge\gamma+(d\gamma)_f$. In fact, let (v_j) be a basis in V and (v_j^*) the dual basis. Then we can write

$$\gamma = \sum_{j=1}^m \gamma_j v_j, \quad f = \sum_{j=1}^m f_j v_j^* \quad \text{where} \quad (\gamma_j)\subset\mathscr{E}_1(M).$$

We have

$$\gamma_f = \sum_{j=1}^m f_j\gamma_j \quad \text{and thus} \quad d(\gamma_f) = \sum_{j=1}^m (df_j\wedge v_j) + \sum_{j=1}^m f_j\cdot d\gamma_j.$$

Hence it is enough to note that for any pair h, k of tangent vectors we have: $(df\wedge\gamma)(h\wedge k)=\big(df(h), \gamma(k)\big)-\big(df(k), \gamma(h)\big)$, which concludes our proof. — (vi) Summing up, we have by (iv)—(v) above: $L=\omega_O+d(\delta_f)$. Lemma 15 asserts that $\delta_f= =\varphi^*\vartheta$ and thus $d(\delta_f)=\varphi^*(d\vartheta)$. Since $L=\varphi^*p^*\eta$ we get finally

$$\omega_O = \varphi^*(p^*\eta - d\vartheta)$$

as claimed in Theorem 1.

4. The objective of this concluding section is an alternative approach to the following result, first proved in [5], Theorem 3, p. 208.

Theorem 2. *Let O be a simply connected coadjoint orbit of the connected and simply connected solvable Lie group G. Then there is a diffeomorphism $\beta\colon R^d\to O$ such that $\beta^*\omega_O$ is constant.*

Proof. We denote by $\mathfrak{g}$ the Lie algebra of G and proceed by induction according to $\dim(\mathfrak{g})$. We distinguish the following two major possibilities:

A. There is an ideal $\mathfrak{m}$ of codimension one such that $\mathfrak{g}=\mathfrak{g}_g+\mathfrak{m}$ for some g in O. Let π be the restriction map $g^*\to m^*$, and M the connected subgroup of G determined by $\mathfrak{m}$. Then $\pi(O)$ is a coadjoint orbit of M and $\pi|O$ is a diffeomorphism $O\to\pi(O)$. We have, in addition, that $\omega_O=(\pi|O)^*\omega_{\pi(O)}$. By virtue of the assumption of our induction there is a diffeomorphism $\gamma\colon R^\delta\to\pi(O)$ such that $\gamma^*(\omega_{\pi(O)})$ is constant. But then it is enough to take $\beta=(\pi^{-1}|O)\circ\gamma$.

B. Here we assume that the hypothesis of A cannot be realized. Let $\mathfrak{m}$ be a fixed ideal of codimension one. Then, for any $g\in\mathfrak{g}^*$, we have $\mathfrak{g}_g\subseteq\mathfrak{m}$. Putting $h=g|\mathfrak{m}$

we claim that $\mathfrak{g}_h \subseteq \mathfrak{m}$. In fact, if k is in $\mathfrak{g}_h - \mathfrak{m}$, k is orthogonal, with respect to B belonging to g, to $\mathfrak{m}$. But then k is orthogonal to $\mathfrak{g}$ and thus $k \in \mathfrak{g}_g$ and $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_g$, contrary to our assumption. We note, in particular, that in this case $\mathfrak{m}$ is admissible with respect to O.

(i) We fix an element y in O, and write $x = \pi(y)$ and $O_0 = Mx \subseteq \mathfrak{m}^*$. We claim that O_0 is simply connected. In fact, let us put $\mathcal{O} = \pi(O)$. We have, by Lemma 1, $O = \pi^{-1}(\mathcal{O})$, and thus $\mathcal{O}$ is simply connected and G_x is connected. But, by what we have seen above, $\mathfrak{g}_x \subseteq \mathfrak{m}$ and thus $G_x \subset M$, and $G_x = M_x$, and $O_0 = M/M_x$ is simply connected. — (ii) We omit the straightforward verification of the following result. Let G be an arbitrary connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let α be an automorphism of $\mathfrak{g}$; we set $\beta = (\alpha^{-1})^* \in \operatorname{End}(\mathfrak{g}^*)$. Then, if O is any coadjoint orbit, then so is $\beta(O)$ and $\beta^*(\omega_{\pi(O)}) = \omega_O$. — (iii) By virtue of the assumption of our inductive procedure, there is a diffeomorphism g_0 from R^δ onto O_0 such that $g_0^*(\omega_{O_0})$ is constant. We fix an element $k \in \mathfrak{g} - \mathfrak{m}$, write $\gamma(t) \equiv \exp(tk)$ and define a map $h\colon R^{\delta+1} \to \mathcal{O}$ by $h(t, T) \equiv \gamma(t) g_0(T)$ $(t \in \mathfrak{R}, T \in R^\delta)$. Then h is a diffeomorphism from $R^{\delta+1}$ onto $\mathcal{O}$. Let $\mathfrak{a}$ be the subspace spanned by k; we have $g = \mathfrak{m} + \mathfrak{a}$. Let j be the projection onto the second summand. We define $\iota\colon \mathfrak{m}^* \to \mathfrak{g}^*$ such for $h \in \mathfrak{m}^*$ we have $\iota(h)|\mathfrak{m} = h$ and $\iota(h)|\mathfrak{a} = 0$. We write ι also for $\iota|\mathcal{O} \in \Gamma(O)$ and set $\eta = \iota^* \omega_O$. In the following we shall proove that $h^*(\eta)$ is constant. a) Let $h \in O$ be fix and $g = \iota(h)$. Assuming that $t, t' \in T_h(O)$ are given and $t = \sigma_h(u)$, $t' = \sigma_h(v)$ $(u, v \in \mathfrak{g})$ we claim that $\eta(t \wedge t') = B(u, v) - B(ju, v) - B(u, jv)$. In fact, 1) we have for any real τ: $\exp(\tau u) g - i(\exp(\tau u) h) \in \mathfrak{m}^\perp$. Hence there is an $n \in \mathfrak{m}^\perp$ such that $ug = i(uh) + n$. 2) By virtue of (ii) in the proof of Lemma 1, we have $\mathfrak{g}_h g = \mathfrak{m}^\perp$, and thus there is $\bar{u} \in \mathfrak{g}_h$ with $n = \bar{u} g$. From this we can conclude that $\tau_g \iota_{*h}(t) = \iota(uh) = (u - \bar{u}) g$. Similarly, there is $\bar{v} \in \mathfrak{g}_h$ such that $\tau_g \iota_{*h}(t) = (v - \bar{v}) g$. 3) We conclude from this that

$$\eta(t \wedge t') = \omega_O(\iota_{*h}(t) \wedge \iota_{*h}(t')) = ([u - \bar{u}, v - \bar{v}], g) = ([u, v], g) - ([u, \bar{v}], g) - ([\bar{u}, v], g).$$

4) We note that $([u, \bar{v}], g) = (u, \bar{v} g) = (ju, \bar{v} g)$. But, by 2), $\bar{v} g = vg - i(uh)$ and the last term is orthogonal to $\mathfrak{a}$. Hence $([u, \bar{v}], g) = (ju, vg) = B(ju, v)$, and similarly $([\bar{u}, v], g) = B(u, jv)$. In this manner we obtain for $t = \sigma_h(u)$, $t' = \sigma_h(v)$: $\eta(t \wedge t') = B(u, v) - B(ju, v) - B(u, jv)$ as claimed above. — b) Let U be an M-orbit in $\mathcal{O}$. We claim that $(id_U)^* \eta = \omega_U$. In fact, suppose that $h \in U$ and $t, t' \in T_h(U)$. Then there are $u, v \in \mathfrak{m}$ such that $t = \sigma_h(u)$, $t' = \sigma_h(v)$. Since $ju = 0 = jv$, we have by a):

$$\big((id_U)^* \eta\big)(t \wedge t') = ([u, v], h) = \omega_U(t \wedge t')$$

and thus $(id_U)^* \eta = \omega_U$ as claimed above. For $T = (t_1, \ldots, t_\delta)$ we form vector fields on $\mathcal{O}$ by

$$D_0 = \partial/\partial t, \quad D_j = \partial/\partial t_j \quad (1 \leq j \leq \delta).$$

To prove that $h^*\eta$ is constant, it will be enough to show that $(h^*\eta)(D_i\wedge D_j)$ is constant for $0\leqq i,j\leqq\delta$. — c) We start by proving the last claim for i,j such that $1\leqq i,j\leqq\delta$. In fact, let h be an element of $\mathcal{O}$, $h=h(t,T)$, say. We write $h_0=h(0,T)\in O_0$, and thus $h=\gamma(t)h_0$. Putting $O_t=\gamma(t)O_0$ we recall (cf. (ii)), that this is an M-orbit in $\mathcal{O}$. We have also $D_j|_h=T_{h_0}(\gamma(t))(D_j|_{h_0})\in T_h(O_t)$. Using b) above we conclude from this that

$$\eta(D_i|_h\wedge D_j|_h)=\omega_{O_t}(D_i|_h\wedge D_j|_h)=$$
$$=\omega_{O_t}\big(T_{h_0}(\gamma(t))(D_i|_{h_0})\big)\wedge T_{h_0}(\gamma(t))(D_j|_{h_0})=\big(\gamma(t)^*\omega_{O_t}\big)(D_i|_{h_0}\wedge D_j|_{h_0}).$$

But the last expression, by virtue of (ii), is equal to

$$\omega_{O_0}(D_i|_{h_0}\wedge D_j|_{h_0})$$

which, by the choice of $g_0\colon R^\delta\to O_0$ is constant, as h_0 varies over O_0. — d) We claim now that $\eta(D_0|_h\wedge D_j|_h)\equiv 0$ $(1\leqq j\leqq\delta)$. To this end it is enough to show that $\eta(\sigma_h(k)\wedge\sigma_h(u))\equiv 0$ if $u\in\mathfrak{m}$. But, by $jk=k$ and $ju=0$ this is implied by a). In this manner we have completed proving that $h^*\eta$ is constant, as we claimed at the start of (iii). — (iv) Let $\zeta\in\mathscr{E}_1(\mathcal{O})$ be such that $h^*(\zeta)=dt$. We define $f\colon R^{\delta+2}\to O'$ by $f(u,t,T)\equiv(h(t,T),u\zeta)$. Then f is a diffeomorphism from $R^{\delta+2}$ onto O'. Also, $\vartheta'=u\zeta$ is the pullback of the canonical 1-form on $T^*(\mathcal{O})$, to O'. By virtue of what we have seen in (iii), $f^*(p^*\eta-d\vartheta')$ is constant. — (v) We recall that by Theorem 1, there is a diffeomorphism $\varphi\colon O\to O'$ such that $\omega_O=\varphi^*(p^*\eta-d\vartheta')$. Hence $\beta=$ $=\varphi^{-1}\circ f$ is a diffeomorphism $R^d\to O$ such that $\beta^*\omega_O$ is constant, completing the proof of Theorem 2.

Some notational conventions. 1) Given a Lie group G with the Lie algebra $\mathfrak{g}$, $\mathfrak{g}$ is considered as a G-module with respect to the adjoint representation. Similarly, $\mathfrak{g}$ is a $\mathfrak{g}$-module with respect to the adjoint representation of $\mathfrak{g}$. Also $\mathfrak{g}^*$, the dual of the underlying space of $\mathfrak{g}$, is a G or $\mathfrak{g}$-module with respect to the coadjoint representation and its differential respectively. — 2) If a Lie group G acts smoothly on a C^∞-manifold X, for $x\in X$, G_x stands for the stabilizer of x in G, and $\mathfrak{g}_x$ for the subalgebra corresponding to G_x. — 3) A distribution on X will be denoted by a capital German letter. If $\mathfrak{M}$ is such, $M_x\subset T_x(X)$ will denote its value at x X. — 4) Given a principal bundle $\mathscr{B}$ with the structure group G, given $x\in B$ and $g\in G$, we shall write sometimes xg even if the action of g derives from an abelian group structure.

References

[1] J. Dieudonné, *Éléments d'Analyse, Vol. 3,* 2^{nd} ed., Bordas (Paris, 1974).

[2] B. Kostant, Quantization and unitary representations, in: *Lecture Notes in Math.*, vol. 170, Springer (Berlin, 1970); pp. 87—208.

[3] N. PEDERSEN, On the symplectic structure of coadjoint orbits of solvable Lie groups and applications. I, *Math. Ann.*, **281** (1988), 633—669.

[4] L. PUKANSZKY, Unitary representations of solvable Lie groups, *Ann. Sci. Ec. Norm Sup.*, **4** (1971), 457—608.

[5] L. PUKANSZKY, On a property of the quantization map for the coadjoint orbits of connected Lie groups, in: *The orbit method in representation theory*, Birkhäuser, 1990, pp. 187—212.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA., U. S. A.