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Compatibility and incompatibility of Calkin equivalence with the Nagy—Foias calculus

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Introduction. We have shown in [2] that there exist two absolutely continuous (a.c.) contractions T_1 and T_2 which commute and are such that T_2-T_1 is compact, and such that there exists a function h in H^∞ with $h(T_2)-h(T_1)$ not compact.

In this article, we give sufficient conditions on T_1 and T_2 which guarantee that $h(T_2)-h(T_1)$ is compact for any h in H^∞. In the particular case where T_1 is a diagonal operator whose eigenvalues are simple we characterize the a.c. T_2 which commute with T_1 and which verify $h(T_2)-h(T_1)$ is compact for any h in H^∞.

Notations. Let H be a separable infinite-dimensional complex Hilbert space, $\mathscr{L}(H)$ the Banach algebra of bounded linear operators on H and $\mathscr{K}(H)$ the space of compact operators on H. For $T\in\mathscr{L}(H)$, we denote by $r(T)$ the spectral radius of T; if T is an absolutely continuous contraction, we denote by $h(T)$, $h\in H^\infty$, the image of h under the Sz.-Nagy—Foias functional calculus.

Proposition 1. *Let T_1 and T_2 be two a.c. contractions in $\mathscr{L}(H)$ such that $T_1T_2=T_2T_1$ and T_2-T_1 is of finite rank. Then for every $h\in H^\infty$, $h(T_2)-h(T_1)$ is of finite rank.*

Proof. Set $A=T_2-T_1$ and let k be the rank of A. Then, for $n\in\mathbf{N}$, we have $T_2^n=T_1^n+AV_n$, where V_n is an element of $\mathscr{L}(H)$. Now, let h be in H and (p_j) a polynomial sequence which converges to h in the weak*-topology. Then $p_j(T_2)=$ $=p_j(T_1)+AW_j$, $W_j\in\mathscr{L}(H)$. Since the rank of AW_j is less than k, by taking the limit in the weak*-topology, we obtain $h(T_2)=h(T_1)+W$, where W is an operator whose rank is less than k. (It is well-known and easy to see that the set of operators T whose rank is less than k is weak*-closed in $\mathscr{L}(H)$). This completes the proof of the proposition.

We have the following observation for T_1 and T_2 with compact difference whose spectral radii are less than 1.

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Observation 2. *Let T_1 and T_2 be two contractions satisfying* $r(T_1)<1$, $r(T_2)<1$ *and* $T_2-T_1\in K(H)$. *Then:*

$$h(T_2)-h(T_1)\in K(H), \quad h\in H^{\infty}.$$

Indeed, let $h(z)=\sum_{k=0}^{\infty} a_k z^k$ be a function in H^{∞}. Then:

$$h(T_2)-h(T_1) = \sum_{k=0}^{\infty} a_k(T_2^k-T_1^k)$$

and $T_2^k-T_1^k$ can be written in the form:

$$T_2^k-T_1^k = \sum_{j=0}^{k-1} T_2^j(T_2-T_1)T_1^{k-j-1}.$$

Hence $T_2^k-T_1^k$ is compact for every $k\geqq 1$ and so $h(T_2)-h(T_1)$ is a norm-limit of compact operators, hence, it is compact.

The following theorem gives another example of a.c. contractions T_1 and T_2 such that $h(T_2)-h(T_1)\in\mathcal{K}(H)$, $h\in H^{\infty}$.

Theorem 3. *Let T_1 and T_2 be two a.c. contractions such that $T_1=S\oplus 0$ and $T_2=S\oplus K$, $K\in\mathcal{K}(H)$. Then, S and K are a.c. contractions, $r(K)<1$ and $h(T_2)-h(T_1)\in\mathcal{K}(H)$ for every $h\in H^{\infty}$.*

Proof. It is clear that K is absolutely continuous. If $r(K)=1$, then K will have a eigenvalue of modulus 1 which contradicts the absolute continuity of T_2. Hence $r(K)<1$ and if $h(z)=\sum_{k=0}^{\infty} a_k z^k$ is in H^{∞}, we have:

$$h(T_2)-h(T_1) = \big(h(S)\oplus h(K)\big)-\big(h(S)\oplus h(0)\big) = \sum_{k=1}^{\infty} a_k K^k$$

which is compact.

We examine now the particular case where T_1 and T_2 are diagonal operators.

Let (e_n) be an orthonormal basis for H, let (α_n) and (β_n) be two sequences in the unit disc $\mathbf{D}$ and let T_α and T_β be the diagonal operators associated to (α_n) and (β_n) respectively. Then:

Theorem 4. *The following assertions are eguivalent:*

a) $\lim_{n\to\infty} \frac{\beta_n-\alpha_n}{1-|\beta_n|} = 0,$

b) $h(T_\beta)-h(T_\alpha)$ *is compact for every* $h\in H^{\infty}$.

The proof uses the following

Lemma 5. *Let (u_n) and (v_n) be two complex sequences.*

a) *If (u_n) and (v_n) are in* $\mathbf{D}$, *then:*

$$\lim_{n\to\infty} \frac{v_n-u_n}{1-\overline{v_n}u_n}=0 \Leftrightarrow \lim_{n\to\infty} \frac{v_n-u_n}{1-|v_n|}=0$$

b) *If* $\lim_{n\to\infty} u_n=\lim_{n\to\infty} v_n=0$, *then there exists an increasing sequence* $(n_k)\subset \mathbf{N}$ *such that:*

$$\frac{|u_{n_i}|}{|v_{n_j}|}\leqq 2^{j-i-1} \quad \textit{if} \quad j<i \quad \textit{and} \quad \frac{|v_{n_j}|}{|u_{n_i}|}\leqq 2^{i-j-1} \quad \textit{if} \quad i<j.$$

Proof. Assertion a) results from:

$$\frac{|v_n-u_n|}{|1-\overline{v_n}u_n|}\leqq \frac{|v_n-u_n|}{1-|v_n|}$$

and

$$\frac{|v_n-u_n|}{1-|v_n|}=\frac{|v_n-u_n|(1+|v_n|)}{(1-\overline{v_n}u_n)\left(1+\overline{v_n}\,\frac{(u_n-v_n)}{1-\overline{v_n}u_n}\right)}.$$

Assertion b) can be obtained by using a simple induction.

Proof of Theorem 4. To prove a)$\Rightarrow$b), it is sufficient to show that if a) holds then:

$$\lim_{n\to\infty} |h(\beta_n)-h(\alpha_n)|=0.$$

For $h\in H^\infty$ and $a\in \mathbf{D}$, we can write the function $g(z)=h(z)-h(a)$ under the form $g(z)=(z-a)g_a(z)$, $g_a\in H^\infty$ and $\|g_a\|\leqq 2\|h\|_\infty/(1-|a|)$. This implies that

$$|h(\beta_n)-h(\alpha_n)|\leqq 2\|h\|_\infty \frac{|\beta_n-\alpha_n|}{1-|\beta_n|}, \quad h\in H^\infty$$

and so a)$\Rightarrow$b).

Now, suppose that $h(T_\beta)-h(T_\alpha)$ is compact for every $h\in H^\infty$ and the sequence (v_n), $v_n=|(\beta_n-\alpha_n)/(1-\bar\beta_n\alpha_n)|$ does not converge to zero. Since the sequence (v_n) is bounded, it contains a subsequence (v_{n_k}) which converges to a positive limit l. As $T_\alpha-T_\beta$ is compact, we have $0\neq\beta_{n_k}-\alpha_{n_k}\to 0$ and so $|\alpha_{n_k}|\to 1$ and $|\beta_{n_k}|\to 1$. Therefore, for example, the sequence (β_{n_k}) contains a Blaschke subsequence $(\beta_{n_{k_l}})$ that is $\sum_{l=0}^{\infty}(1-|\beta_{n_{k_l}}|)<\infty$. From Lemma 5, by extracting another subsequence, we can suppose that the subsequence (β_{n_k}) is a Blaschke sequence and:

$$\frac{1-|\beta_{n_j}|}{1-|\alpha_{n_i}|}\leqq 2^{i-j-1} \quad \text{if} \quad i<j, \quad \text{and} \quad \frac{1-|\alpha_{n_j}|}{1-|\beta_{n_i}|}\leqq 2^{i-j-1} \quad \text{if} \quad i<j.$$

For $0\neq a\in\mathbf{D}$, denote by e_a the function:

$$e_a(z)=\frac{|a|}{a}\frac{a-z}{1-\bar{a}z},\quad z\in\mathbf{D}.$$

We have:

$$\big|1-|e_a(z)|\big|\leqq 2\frac{1-|a|}{1-|z|}$$

and as $|e_a(z)|=|e_z(a)|$ we have also:

$$\big|1-|e_a(z)|\big|\leqq 2\frac{1-|z|}{1-|a|}.$$

It results that:

$$\big|1-|e_{\beta_{n_i}}(\alpha_{n_j})|\big|=2^{-|i-j|},\ i\neq j\quad\text{so}\quad |e_{\beta_{n_j}}(\alpha_{n_i})|\geqq 1-2^{-|i-j|},\ i\neq j$$

and for any fixed j

$$\prod_{i\neq j}|e_{\beta_{n_i}}(\alpha_{n_j})|\geqq\prod_{i<j}(1-2^{-|i-j|})\prod_{i>j}(1-2^{-|i-j|})\geqq\Big(\prod_{k=1}^{\infty}(1-2^{-k})\Big)^2=c>0.$$

Let

$$B(z)=\prod_{k=1}^{\infty}\frac{|\beta_{n_k}|}{\beta_{n_k}}\frac{\beta_{n_k}-z}{1-\overline{\beta_{n_k}}z}$$

be the Blaschke product associated to the sequence (β_{n_k}). Then:

$$|B(\alpha_{n_j})|=\left|\prod_{k\neq j}\frac{|\beta_{n_k}|}{\beta_{n_k}}\frac{\beta_{n_k}-\alpha_{n_j}}{1-\overline{\beta_{n_k}}\alpha_{n_j}}\right||e_{\beta_{n_j}}(\alpha_{n_j})|\geqq c\,|e_{\beta_{n_j}}(\alpha_{n_j})|\to cl.$$

hence $B(\beta_{n_j})-B(\alpha_{n_j})=-B(\alpha_{n_j})$ does not converge to 0. This contradicts the compactness of $B(T_\beta)-B(T_\alpha)$, and the theorem is proved.

Remark 6. If $T=T_\alpha$, where $\alpha=(\alpha_n)$ is a sequence of distinct elements of $\mathbf{D}$, then every element S of the commutant of T can be written $S=T_\beta$, where $\beta=(\beta_n)$ is a sequence of complex numbers. If S is an a.c. contraction, then $\beta_n\in\mathbf{D}$, $n\in\mathbf{N}$. Therefore we see that $h(S)-h(T)$ is compact for every $h\in H^\infty$ if and and only if

$$\lim_{n\to\infty}\frac{\beta_n-\alpha_n}{1-|\alpha_n|}=0.$$

If $\sup|\alpha_n|=1$, T is a completely nonunitary contraction with $r(T)=1$. Hence we see that there exist a.c. contractions $S\neq T$ such that $r(T)=1$, $ST=TS$ and $h(S)-h(T)\in\mathcal{K}(H)$ for every $h\in H^\infty$ and a.c. contractions S' such that $r(S')=1$, $S'T=TS'$, $S'-T\in\mathcal{K}(H)$ and $h(S')-h(T)\notin\mathcal{K}(H)$ for some $h\in H^\infty$.

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