Acta Sci. Math., 56 (1992), 369—375

On the compositions of (α, β)-derivations of rings, and applications to von Neumann algebras

MATEJ BREŠAR

Introduction

There are two motivations for this research. The first one is an old and well-known result of E. POSNER [12]:

Theorem A. *Let R be a prime ring of characteristic not 2. If the composition of derivations d, g of R is a derivation, then either* $d=0$ *or* $g=0$.

A number of authors have proved extensions of this theorem; we refer the reader to some ring-theoretic results [3, 5, 9] and to some results from analysis [4, 10, 11].

The other motivation comes from the theory of von Neumann algebras. In a series of papers A. B. Thaheem and some other authors have studied the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ where α and β are automorphisms of a von Neumann algebra. This identity plays an important role in the Tomita—Takesaki theory (see, e.g., [6, 7, 8]). In [13 and 14] and in a joint paper with AWAMI [18], THAHEEM has given various proofs of the following theorem.

Theorem B. *Let R be a von Neumann algebra and* α, β *be* **-automorphisms of R satisfying* $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. *If* α *and* β *commute then there exists a central projection p in R such that* $\alpha(p)=\beta(p)=p$, $\alpha=\beta$ *on pR, and* $\alpha=\beta^{-1}$ *on* $(1-p)R$.

For other results concerning the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ we refer to some recent papers [1, 15, 16, 17] where further references can be found.

It is our aim in this paper to extend Theorem A to more general mappings on more general rings, so that the special case of this extension gives a generalization of Theorem B. In particular, our research can be viewed as a new, more elementary

Received March 5, 1991.

approach to the study of the identity $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. In a subsequent paper we hope to consider this identity without assuming the commutativity of α and β.

Let R be a ring and α, β be automorphisms of R. An additive mapping d of R into itself is called an (α, β)-derivation if

$$d(xy) = \alpha(x)\,d(y)+d(x)\,\beta(y) \quad \text{for all} \quad x, y\in R.$$

An (α, β)-derivation d is said to be inner if there exists $a\in R$ such that $d(x)=\alpha(x)a-a\beta(x)$ for all $x\in R$. Of course, derivations are (1,1)-derivations where 1 is the identity on R. We will study the case where the composition of an (α, β)-derivation d and a (γ, δ)-derivation g is an $(\alpha\gamma, \beta\delta)$-derivation. We will first generalize Theorem A by proving that if R is prime of characteristic not 2 and g commutes with both γ and δ, then either $d=0$ or $g=0$ (Corollary 1). An abbreviated version of our main theorem reads as follows.

Theorem 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R, and g be a (γ, δ)-derivation of R. Suppose that d commutes with both α and β, and that g commutes with both γ and δ. If dg is an $(\alpha\gamma, \beta\delta)$-deriavtion then there exist ideals U and V of R such that $U\oplus V$ is an essential ideal of R, $d=0$ on V and $g=0$ on U. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U\oplus V=R$.*

As an immediate consequence of Theorem 1 we obtain that the decomposition of Theorem B holds in arbitrary semiprime ring in which the annihilator of any ideal is a direct summand (Corollary 2). Moreover, the assumption that α and δ preserve adjoints is removed (in fact, we do not work in rings with involution).

Preliminaries

Throughout, R will represent an associative ring. Recall that R is prime if $aRb=0$ implies that $a=0$ or $b=0$. R is said to be semiprime if $aRa=0$ implies that $a=0$. Equivalently, R is semiprime if it has no nonzero nilpotent ideals. Every C^*-algebra is semiprime (for $0\neq aa^*a\in aRa$ if $a\neq 0$). A von Neumann algebra is prime if and only if it is a factor (i.e., its center consists of scalar multiples of the identity).

Let R be semiprime. Suppose that $aRb=0$ for some $a, b\in R$. Then we also have $(bRa)R(bRa)=0$, $abRab=0$, $baRba=0$, and therefore $bRa=0$, $ab=0$, $ba=0$ since R is semiprime. Note that the left and right and two-sided annihilators of an ideal U in R coincide. It will be denoted by Ann (U). Note also that $U\cap \text{Ann}\,(U)=0$ and $U\oplus \text{Ann}\,(U)$ is an essential ideal (i.e., $(U\oplus \text{Ann}\,(U))\cap I\neq 0$ for every nonzero ideal I of R). We will be especially concerned with semiprime rings R in which the annihilator of any ideal is a direct summand; that is, $\mathrm{Ann}(U)\oplus \oplus\mathrm{Ann}(\mathrm{Ann}(U))=R$ for any ideal U of R. Every von Neumann algebra has this property; namely, the annihilator of any ideal in a von Neumann algebra R is σ-weakly closed, therefore it is of the form pR for some central projection p in R. More generally, the same is true for Baer *-rings, and, therefore, for AW^*-algebras (see, e.g., [2]).

The results

Lemma 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R and g be a (γ, δ)-derviation of R. Suppose that the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation, and suppose that g commutes with both γ and δ. Then $g(x)R(\alpha^{-1}d)(y)=0$ for all $x, y\in R$.*

Proof. We have $h=dg$ is a $(\alpha\gamma, \beta\delta)$-derivation. Consequently $(\beta^{-1}d)(g\delta^{-1})= =\beta^{-1}h\delta^{-1}$; that is, the composition of a $(\beta^{-1}\alpha, 1)$-derivation $\beta^{-1}d$ and a $(\gamma\delta^{-1}, 1)$-derivation $g\delta^{-1}$ is a $((\beta^{-1}\alpha)(\gamma\delta^{-1}), 1)$-derivation $\beta^{-1}h\delta^{-1}$. We will show that $g\delta^{-1}$ commutes with $\gamma\delta^{-1}$. Note that this implies that there is no loss of generality in assuming $\beta=1$ and $\delta=1$.

Thus, let us prove that $g\delta^{-1}$ and $\gamma\delta^{-1}$ commute. Since g commutes with γ and δ, it suffices to show that $g\gamma\delta^{-1}=g\delta^{-1}\gamma$. By the definition of (γ, δ)-derivations we have

$$(\gamma g)(xy) = \gamma^2(x)(\gamma g)(y)+(\gamma g)(x)(\gamma\delta)(y),$$

$$(g\gamma)(xy) = \gamma^2(x)(g\gamma)(y)+(g\gamma)(x)(\delta\gamma)(y).$$

Since we have assumed that $\gamma g=g\gamma$ the relations imply that $(g\gamma)(x)(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$; but γ is onto, so we also have $g(x)(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$. Substituting xz for x it follows easily that $g(x)R(\gamma\delta-\delta\gamma)(y)=0$ for all $x, y\in R$. In particular, $g\big((\gamma\delta-\delta\gamma)(x)\big)R(\gamma\delta-\delta\gamma)\big(g(x)\big)=0$ for every x in R. Since g commutes with $\gamma\delta-\delta\gamma$, and since R is semiprime, it follows that $g\gamma\delta=g\delta\gamma$. Multiplying this relation from the right and from the left by δ^{-1} we arrive at $g\delta^{-1}\gamma=g\gamma\delta^{-1}$.

Now, we may assume that $\beta=\delta=1$. A direct computation shows that

$$(dg)(xy) = (\alpha\gamma)(x)(dg)(y)+(d\gamma)(x)g(y)+(\alpha g)(x)d(y)+(dg)(x)y.$$

On the other hand, since dg is an $(\alpha\gamma, 1)$-derivation, we have

$$(dg)(xy) = (\alpha\gamma)(x)(dg)(y)+(dg)(x)y.$$

Comparing the two expressions so obtained for $(dg)(xy)$, we see that

$$(1)\qquad (d\gamma)(x)g(y)+(\alpha g)(x)d(y) = 0 \quad \text{for all} \quad x, y\in R.$$

Replacing y by yz in (1) we obtain

$$(d\gamma)(x)\gamma(y)g(z)+(d\gamma)(x)g(y)z+(\alpha g)(x)\alpha(y)d(z)+(\alpha g)(x)d(y)z = 0.$$

By (1) this relation reduces to

$$(2) \qquad (d\gamma)(x)\gamma(y)g(z)+(\alpha g)(x)\alpha(y)d(z) = 0 \quad \text{for all} \quad x, y, z\in R.$$

Replacing y by $g(y)$ in (2) and using the assumption that g commutes with γ, we then get

$$(d\gamma)(x)g(\gamma(y))g(z)+(\alpha g)(x)(\alpha g)(y)d(z) = 0.$$

On the other hand, using (1) twice we obtain

$$\{(d\gamma)(x)g(\gamma(y))\}g(z) = -(\alpha g)(x)\{(d\gamma)(y)g(z)\} = (\alpha g)(x)(\alpha g)(y)d(z).$$

Comparing the last two relations we get $2(ag)(x)(\alpha g)(y)d(z)=0$ for all $x, y, z\in R$. Since R is 2-torsion free we then have

$$0 = \alpha^{-1}((\alpha g)(x)(\alpha g)(y)(d)(z)) = g(x)g(y)(\alpha^{-1}d)(z).$$

Thus $g(x)g(y)(\alpha^{-1}d)(z)=0$ for all $x, y, z\in R$. Replacing x by xu it follows at once that $g(x)Rg(y)(\alpha^{-1}d)(z)=0$; similarly we see that $g(x)Rg(y)R(\alpha^{-1}d)(z)=0$. The semiprimeness of R then yields $g(y)R(\alpha^{-1}d)(z)=0$ and so the lemma is proved.

As an immediate consequence of Lemma 1 we obtain the following generalization of Posner's theorem.

Corollary 1. *Let R be a prime ring of characteristic not 2, d be an (α, β)-derivation of R, and g be an (γ, δ)-derivation of R. Suppose that g commutes with both γ and δ. If the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation then either $d=0$ or $g=0$.*

Example. The assumption that g commutes with both γ and δ is not superfluous. Moreover, the following simple example shows that it cannot be replaced by the assumption that d commutes with both α and β. Suppose that a prime ring with unit element 1 contains elements a and b such that $a^2=0$, $b^2=1$, $ab+ba=0$, and a, b do not lie in the center of R (for example, in the ring of 2×2 matrices the elements

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfy these conditions). Define the inner automorphism γ by $\gamma(x)=bxb$ and the $(\gamma, 1)$-derivation g by $g(x)=\gamma(x)ba-bax$; $g\neq0$ since $g(x)a=-baxa\neq0$ for some $x\in R$ by the primeness of R. If d is the inner derivation, $d(x)=ax-xa$, then $dg=0$.

We need two easy lemmas.

Lemma 2. *Let R be any ring and Θ be an automorphism of R. If Θ maps an ideal W onto itself then Θ maps* Ann (W) *onto itself.*

Proof. Given $w\in W$, $u\in \mathrm{Ann}(W)$ we have $0=\Theta(uw)=\Theta(u)\Theta(w)$ and similarly, $\Theta(w)\Theta(u)=0$. By assumption, $\Theta(w)$ is an arbitrary element in W, so it follows that $\Theta(u)\in \mathrm{Ann}(W)$. Thus Θ maps $\mathrm{Ann}(W)$ into itself. Analogously, Θ^{-1} maps $\mathrm{Ann}(W)$ into itself, which means that Θ is onto on $\mathrm{Ann}(W)$.

Lemma 3. *Let R be a semiprime ring, and let d be an (α, β)-derivation of R which commutes with both α and β. If d maps R into an ideal W of R, then d is zero on* $\mathrm{Ann}(W)$.

Proof. Given $w\in W$, $u\in \mathrm{Ann}(W)$ we have $u(\alpha^{-1}d)(w)=ud(\alpha^{-1}(w))\in \in \mathrm{Ann}(W)W=0$. Thus $\alpha(u)d(w)=\alpha(u(\alpha^{-1}d)(w))=0$. Hence $d(u)\beta(w)=\alpha(u)d(w)+ +d(u)\beta(w)=d(uw)=0$. But then also $0=\beta^{-1}(d(u)\beta(w))=d(\beta^{-1}(u))w$. That is, $d(\beta^{-1}(u))\in \mathrm{Ann}(W)$ for any $u\in \mathrm{Ann}(W)$. However, by assumption $d(\beta^{-1}(u))$ lies in W, so we are forced to conclude that $d(\beta^{-1}(u))=0$. Since d and β^{-1} commute, $d(u)=0$ as well.

We now have enough information to prove the main theorem of this paper.

Theorem 1. *Let R be a 2-torsion free semiprime ring, d be an (α, β)-derivation of R, and g be an (γ, δ)-derivation of R. Suppose that d commutes with both α and β, and that g commutes with both γ and δ. If the composition dg is an $(\alpha\gamma, \beta\delta)$-derivation, then there exist ideals U and V of R such that:*

(i) $U\cap V=0$ and $U\oplus V$ is an essential ideal of R. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra), then $U\oplus V=R$,

(ii) If Θ is any automorphism of R which commutes with d then Θ maps U onto U and V onto V,

(iii) d maps R into U and d is zero on V,

(iv) g maps R into V and d is zero on U.
In particular, $dg=gd=0$.

Proof. Let U_0 be the ideal of R generated by all $d(x)$, $x\in R$. Let $V=\mathrm{Ann}(U_0)$ and $U=\mathrm{Ann}(V)$. Thus (i) holds. If an automorphism Θ of R commutes with d, then $\Theta(xd(y)z)=\Theta(x)d(\Theta(y))\Theta(z)\in U_0$ for all $x, y, z\in R$. Similarly, $\Theta(xd(y))\in U_0$, $\Theta(d(y)z)\in U_0$ and $\Theta(d(y))\in U_0$. Thus U_0 is invariant under Θ. Likewise U_0 is invariant under Θ^{-1}. Hence Θ maps U_0 onto itself. From Lemma 2 it follows that Θ maps V onto V, and therefore also U onto U. Thus (ii) is proved. Since d maps R into $U_0\subseteq U$, (iii) follows immediately from Lemma 3. It remains to prove (iv). In view of Lemma 3 it suffices to show that $g(x)$ lies in v for every $x\in R$. By Lemma 1, since d and α^{-1} commute, we have $g(x)Rd(y)=0$ for all $x, y\in R$. Thus $g(x)\in \in \mathrm{Ann}(U_0)=V$. Combining (iii) and (iv) we see that $dg=gd=0$. The proof of the theorem is complete.

Let R be any ring. Suppose that automorphisms α and β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ and $\alpha\beta=\beta\alpha$. Multiply the first relation by α, and observe that the relation which we obtain can be written in the form $(\alpha-\beta)(\alpha-\beta^{-1})=0$. That is, the composition of the (α, β)-derivation $\alpha-\beta$ and the (α, β^{-1})-derivation $\alpha-\beta^{-1}$ is equal to zero. Note that all the requirements of Theorem 1 are fulfilled. Thus we have

Corollary 2. *Let R be a 2-torsion free semiprime ring. Suppose that automorphisms α and β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then there exist ideals U and V of R such that:*

(i) $U\cap V=0$ and $U\oplus V$ is an essential ideal. Moreover, if the annihilator of any ideal in R is a direct summand (in particular, if R is a von Neumann algebra) then $U\oplus V=R$,

(ii) α and β map U onto U and V onto V,

(iii) $\alpha=\beta$ on V,

(iv) $\alpha=\beta^{-1}$ on U.

We conclude this paper with the following direct consequence of Corollary 2.

Corollary 3. *Let R be a prime ring of characteristic not 2. Suppose that automorphisms α, β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then either $\alpha=\beta$ or $\alpha=\beta^{-1}$.*

We leave as an open question whether or not the assumption that α and β commute can be removed in Corollary 3 (it certainly cannot be removed in the case R is semiprime, as Thaheem [17] has shown).

References

[1] C. J. K. Batty, On certain pairs of automorphisms of C^*-algebras, *J. Austral. Math. Soc. (Series A)*, **46** (1989), 197—211.

[2] S. K. Berberian, *Baer *-rings*, Springer-Verlag (Berlin/New York, 1972).

[3] J. Bergen, I. N. Herstein and J. Kerr, Lie ideals and derivations of prime rings, *J. Algebra*, **71** (1981), 259—267.

[4] M. Brešar, On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math. J.*, **33** (1991), 89—93.

[5] M. Brešar and J. Vukman, Orthogonal derivations and an extension of a theorem of Posner, *Rad. Mat.*, **5** (1989), 237—246.

[6] A. van Daele, A new approach to the Tomita—Takesaki theory of generalized Hilbert algebras, *J. Funct. Anal.*, **15** (1974), 387—393.

[7] U. Haagerup and H. Hanche-Olsen, Tomita—Takesaki theory for Jordan algebras, *J. Operator Theory*, **11** (1984), 343—364.

[8] U. Haagerup and C. F. Skau, Geometric aspects of the Tomita—Takesaki theory. II, *Math. Scand.*, **48** (1981), 241—252.

[9] C. Lanski, Differential identities, Lie ideals, and Posner's theorems, *Pacific J. Math.*, **134** (1988), 275—297.

[10] M. MATHIEU, Properties of the product of two derivations of a C^*-algebra, *Canad. Math. Bull.*, **32** (1989), 490—497.

[11] M. MATHIEU, More properties of the product of two derivations of a C^*-algebra, *Bull. Austral. Math. Soc.*, **42** (1990), 115—120.

[12] E. POSNER, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093—1100.

[13] A. B. THAHEEM, On a decomposition of a von Neumann algebra, *Rend. Sem. Mat. Univ. Padova*, **65** (1981), 1—7.

[14] A. B. THAHEEM, On the operator equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$, *Internat. J. Math. & Math. Sci.*, **9** (1986), 767—770.

[15] A. B. THAHEEM, On certain decompositional properties of von Neumann algebras, *Glasgow Math. J.*, **29** (1987), 177—179.

[16] A. B. THAHEEM, On the operator equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. II, *Rad. Mat.*, **4** (1988), 343—349.

[17] A. B. THAHEEM, On pairs of automorphisms of von Neumann algebras, *Internat. J. Math. & Math. Sci.*, **12** (1989), 285—290.

[18] A. B. THAHEEM and M. AWAMI, A short proof of a decomposition theorem of a von Neumann algebra, *Proc. Amer. Math. Soc.*, **92** (1984), 81—82.

UNIVERSITY OF MARIBOR
DEPARTMENT OF MATHEMATICS
PF, KOROŠKA 160
62 000 MARIBOR
SLOVENIA