A CONCISE FORMULATION OF FRESNEL'S FORMULAE

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It is shown that the well known formulae of Fresnel can be obtained in a rather straightforward way using consequent vector formalism. The aim of the article is to clarify some general questions connected with Maxwell's theory and, in particular, to improve from the didactical point of view the method of derivation of the formulae.

Fresnel's formulae give the relations between polarization and intensities of the beams which appear if a primary beam is reflected on the plane boundary of two homogeneous media. The considerations of Fresnel are well-known and can be regarded as a well established result of classical electrodynamics. From the didactical point of view it is, however, disturbing that the formulae are obtained as a result of tiresome calculations; we shall give presently a rather symmetric treatment of the problem—at the same time we make a few remarks about the physical contents of the theory. We hope that our consideration may facilitate the teaching of this phenomenon.

Maxwell's equations of the electromagnetic field can be written in the form of the wave equations

$$\begin{aligned} \nabla^2 \mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} &= -4\pi\mathbf{i}, \\ \nabla^2 \Phi - \frac{1}{c^2}\ddot{\Phi} &= -4\pi\varrho, \\ \operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\Phi} &= 0. \end{aligned} \tag{1}$$

The charge and current densities arise partly or wholly as charges and currents inside the atoms of the material in which the field spreads. If we consider $\mathbf{P}$ the electric and $\mathbf{I}$ the magnetic polarization of the material, then we can write

$$\begin{aligned} \varrho &= -\operatorname{div}\mathbf{P}, \\ \mathbf{i} &= \frac{1}{c}\dot{\mathbf{P}} + \operatorname{rot}\mathbf{I}, \end{aligned} \tag{2}$$

provided no outer current and charges appear.

One usually supposes

$$\mathbf{P} = \varkappa\mathbf{E}, \quad \mathbf{I} = \chi'\mathbf{B}, \tag{3}$$

where

$$\left.\begin{aligned} 1+4\pi\varkappa = \varepsilon, \quad 1+4\pi\chi &= \mu, \\ \chi' &= \chi/\mu, \end{aligned}\right\} \tag{4}$$

ε and μ are the dielectric constant and the magnetic permeability; usually such cases are considered when (3) is valid and ε, χ' and therefore ε, μ are given functions of the coordinate vector $\mathbf{r}$.

We note that for rapidly changing fields (3) cannot be taken to be valid — since the polarizations come about by the displacements and changes of velocities of the atomic electrons. Because of the inertia of the electrons the polarizations follow the field only with some delay. One might *e.g.* suppose in place of the first equation (3) the following dynamical equation

$$\frac{1}{\omega_0^2}\ddot{\mathbf{P}}+\mathbf{P} = \varkappa\mathbf{E}, \tag{3a}$$

where ω_0 is one of the characteristic frequencies of the atom. Introducing the inertia of the electrons by (3*a*) or some more elaborate expressions, one is led to a theory which accounts for the dispersion phenomena of light. We cannot deal with this problem—we note here only that, considering waves of a given frequency ν, we can suppose (3) to be valid in a good approximation only; $\varkappa$ and χ' have to be taken as the dynamical polarization depending on the frequency ν.

We make this remark because it is often incorrectly stated that Maxwell's theory is incomplete as it does not give an account of the phenomena of dispersion. The fact seems, however, that the theory does give the correct description of dispersion phenomena—provided the correct relation between polarization and field strength are made use of. The question of the correct relation between polarization and field is rather a question of describing the structure of matter than one of the theory of the electromagnetic field.* The electromagnetic field strength can be derived from the potentials as

$$\begin{aligned} \mathbf{E} &= -\operatorname{grad}\Phi-\frac{1}{c}\dot{\mathbf{A}}, \\ \mathbf{B} &= \operatorname{rot}\mathbf{A}. \end{aligned} \tag{5}$$

Thus making use of (2) and (3) we find

$$\begin{aligned} \varrho &= -\operatorname{div}\varkappa\mathbf{E} = -\varkappa\operatorname{div}\mathbf{E}-\mathbf{E}\operatorname{grad}\varkappa, \\ \mathbf{i} &= \frac{1}{c}\varkappa\dot{\mathbf{E}}+\chi'\operatorname{rot}\mathbf{B}+\operatorname{grad}\chi'\times\mathbf{B}. \end{aligned} \tag{6}$$

* See *e.g. L. Jánossy:* Theory of relativity based on physical reality, Publishing House of the Hungarian Academy of Sciences, Budapest, 1971. p. 197.

With the help of (5) and (6) the current and charge densities can be expressed in terms of the potentials. If we insert the latter expression into the wave equations (1), we find

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = \frac{4\pi\varkappa}{c}\operatorname{grad}\dot{\Phi} + \frac{4\pi\varkappa}{c^2}\ddot{\mathbf{A}} - 4\pi\chi' \operatorname{rot}\operatorname{rot}\mathbf{A} - 4\pi \operatorname{grad}\chi' \times \mathbf{B},$$

$$\nabla^2 \Phi - \frac{1}{c^2}\ddot{\Phi} = -4\pi\varkappa\left(\nabla^2\Phi + \frac{1}{c}\operatorname{div}\dot{\mathbf{A}}\right) + 4\pi\mathbf{E}\operatorname{grad}\varkappa,$$

since

$$\frac{1}{c}\operatorname{div}\dot{\mathbf{A}} = -\frac{1}{c^2}\ddot{\Phi}.$$

The equation can also be written

$$\begin{aligned} &\nabla^2 \mathbf{A} - \frac{1}{V^2}\ddot{\mathbf{A}} + (\varepsilon\mu - 1)\operatorname{grad}\operatorname{div}\mathbf{A} = -4\pi\mu(\operatorname{grad}\chi' \times \mathbf{B}), \\ &\nabla^2 \Phi - \frac{1}{c^2}\ddot{\Phi} = \frac{4\pi}{\varepsilon}\mathbf{E}\operatorname{grad}\varkappa, \end{aligned} \tag{7a}$$

$$V = \frac{c}{\sqrt{\varepsilon\mu}}.$$

Thus in homogeneous regions where

$$\operatorname{grad}\chi' = \operatorname{grad}\varkappa = 0,$$

we find

$$\nabla^2 \mathbf{A} - \frac{1}{V^2}\ddot{\mathbf{A}} = 0, \tag{7}$$

$$\operatorname{div}\mathbf{A} = 0, \quad V = \frac{c}{\sqrt{\varepsilon\mu}},$$

when supposing $\Phi = 0$. The latter supposition is permissible, as it can be shown by a more detailed analysis. Particular solutions of (7) can be written as

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 \cos(\mathbf{kr} - \omega t + \varphi), \\ \mathbf{kA}_0 &= 0, \quad k = \omega/V. \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\dot{\mathbf{A}} = -\frac{\omega}{c}\mathbf{A}_0 \sin(\mathbf{kr} - \omega t + \varphi), \\ \mathbf{B} &= \operatorname{rot}\mathbf{A} = -\mathbf{k} \times \mathbf{A}_0 \sin(\mathbf{kr} - \omega t + \varphi), \end{aligned} \tag{8a}$$

thus

$$\mathbf{B} = n\frac{\mathbf{k}}{k} \times \mathbf{E}.$$

The general solutions of (7) can be obtained as a superposition of solutions of the form (8). Because of the inertia of the electrons, $V = c/n$ is a function of ω and n is thus the refractive index for the circular frequency ω.

Boundary conditions

Consider two homogeneous regions I and II with velocities $\mathbf{V}^{\mathrm{I}}$ and $\mathbf{V}^{\mathrm{II}}$ and a boundary surface S separating them. The waves propagating in I or in II can be obtained as superpositions of plane waves of the form (8). In the region in the immediate vicinity of S, instead of the homogeneous wave equation (7) the inhomogeneous equations (7*a*) are valid. We can take that in this region χ' and $\varkappa$ change rapidly from the values χ_1' and $\varkappa_1$ which they take up in the one region into values χ_2' and $\varkappa_2$ which they take up in the second region. Owing to those rapid change some of the components of $\mathbf{A}$ and Φ suffer also rapid changes, which can be taken in the limit as discrete jumps of those quantities across S.

The detailed analysis shows that on the boundary S a surface charge density

$$\sigma = 4\pi(P_N^{\mathrm{II}} - P_N^{\mathrm{I}}); \tag{9a}$$

$\mathbf{P}^{\mathrm{I}}$, $\mathbf{P}^{\mathrm{II}}$ and P_N^{I}, P_N^{II} their components perpendicular to S, are the polarization vectors on both sides of S.

Similarly a surface current density $\mathbf{i}$ appears

$$\mathbf{i} = 4\pi(\mathbf{I}_p^{\mathrm{II}} - \mathbf{I}_p^{\mathrm{I}}) \tag{9b}$$

where $\mathbf{I}_p^{\mathrm{II}}$ and $\mathbf{I}_p^{\mathrm{I}}$ are the components of the magnetic polarizations $\mathbf{I}^{\mathrm{II}}$ and $\mathbf{I}^{\mathrm{I}}$ parallel to the surface.

The discontinuity and thus the boundary conditions can be expressed more conveniently in terms of the field strength than of the potentials.*

Denoting by $\mathbf{K}$ the unit vector perpendicular to S, the boundary conditions expressing the presence of the surface charges and currents (9*a*) and (9*b*) can be written as

$$\mathbf{K}(\mathbf{E}^{\mathrm{II}} - \mathbf{E}^{\mathrm{I}}) = -4\pi\mathbf{K}(\mathbf{P}^{\mathrm{II}} - \mathbf{P}^{\mathrm{I}}), \tag{10a}$$

$$\mathbf{K}\times(\mathbf{B}^{\mathrm{II}} - \mathbf{B}^{\mathrm{I}}) = 4\pi\mathbf{K}\times(\mathbf{I}^{(\mathrm{II})} - \mathbf{I}^{(\mathrm{I})}). \tag{10b}$$

The tangential component of $\mathbf{E}$ and the normal components of $\mathbf{B}$ must be continuous* — this can be expressed as

$$\mathbf{K}\times(\mathbf{E}^{(\mathrm{II})} - \mathbf{E}^{(\mathrm{I})}) = 0, \tag{11a}$$

$$\mathbf{K}(\mathbf{B}^{(\mathrm{II})} - \mathbf{B}^{(\mathrm{I})}) = 0. \tag{11b}$$

* This is so because although the waves can be expressed in both regions supposing $\Phi=0$, nevertheless from the Lorentz condition it follows that a discontinuity of $\mathbf{A}$ at the boundary corresponds to a change of Φ; therefore if we want to describe the waves on both sides of S in a form $\Phi=0$ then we have to construct the field in II matching it to that in I and then we have to submit the latter solution to a transformation of gauge to make $\Phi^{\mathrm{II}}=0$.

* The relations

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}} \quad \text{and} \operatorname{div}\mathbf{B} = 0$$

are valid in the region of the boundary. From the latter relations follows that the tangential component of $\mathbf{E}$ must be continuous — a discontinuity of this component would leed to infinitely large values of $\dot{\mathbf{B}}$ thus the field would change rapidly until the break in the tangential component of $\mathbf{E}$ vanishes. From $\operatorname{div}\mathbf{B}=0$ it follows that the normal component of $\mathbf{B}$ must be continuous in any state of the field.

With the help of (3) and (4), (10a, b) can be written

$$\mathbf{K}(\varepsilon_{II}\mathbf{E}^{II}-\varepsilon_{I}\mathbf{E}^{I}) = 0 \tag{11c}$$

$$\mathbf{K}\times\left(\frac{\mathbf{B}^{II}}{\mu_{II}}-\frac{\mathbf{B}^{I}}{\mu_{I}}\right) = 0. \tag{11d}$$

Thus (11a—d) give the boundary conditions on the surface S.

The equation of the surface S can be written in a parameter representation

$$\mathbf{r} = \mathbf{K}\times\mathbf{s} \quad \text{if } \mathbf{r} \text{ a point of } S, \tag{12}$$

where $\mathbf{s}$ is arbitrary. If we want the field on both sides of S to satisfy any kind of boundary condition identically in t then it is necessary that the arguments of the wave functions appearing in the fields on both sides of S should be identical. Thus two waves may satisfy a boundary condition if

$$\mathbf{k}(\mathbf{K}\times\mathbf{s})-\omega t+\varphi = \mathbf{k}'(\mathbf{K}\times\mathbf{s})-\omega' t+\varphi' \tag{13}$$

where we have made use of (12); $\mathbf{k}$, ω, t and, $\mathbf{k}'$, ω', φ' respectively, are the parameters of the wave to be matched at the surface. From (13) it follows that

$$\omega = \omega' \quad \varphi = \varphi' \tag{14}$$

and therefore

$$\mathbf{k}(\mathbf{K}\times\mathbf{s}) = \mathbf{k}'(\mathbf{K}\times\mathbf{s}) \tag{15}$$

for all values of $\mathbf{s}$. Since

$$\mathbf{k}(\mathbf{K}\times\mathbf{s}) = \mathbf{s}(\mathbf{k}\times\mathbf{K})$$

(15) is fulfilled then and only then if

$$\mathbf{k}\times\mathbf{K} = \mathbf{k}'\times\mathbf{K}. \tag{16}$$

From (16) it follows that $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{K}$ lie in one plane. Thus the plane defined by the propagation vectors $\mathbf{k}$ and $\mathbf{k}'$ of two matched waves is perpendicular to the plane S.

Furthermore it follows from (16) that

$$k\sin\vartheta = k'\sin\vartheta' \tag{17}$$

where ϑ and ϑ' are the angles between the vectors $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{K}$; thus in accordance with

$$\frac{\sin\vartheta}{\sin\vartheta'} = \frac{V}{V'}.$$

If $V=V_{I}$, $V'=V_{II}$, then (10) gives the Snellius law; considering, however, two waves on the same side of S, then we have $V=V'$, and $\vartheta\neq\vartheta'$ can only be satisfied if

$$\vartheta' = \pi-\vartheta;$$

thus we obtain the law of reflection. We obtained thus the connection between the wave vectors of waves which match on the surface S.

To match the fields on both sides of S the amplitudes of the fields have also to be matched.

Considering one wave in each of the regions we obtain an overdetermined set of equations—therefore the matching is only possible if at least three waves are considered. This corresponds to the fact that one wave approaching the surface *e.g.* coming through I, this wave produces a reflected wave returning into I and a refracted wave autcoming into II. The amplitude of the vector potentials can be denoted, using the suffies 0, 1, 2, thus

$\mathbf{A}_0$, $\mathbf{k}_0$ describe the incident wave,

$\mathbf{A}_1$, $\mathbf{k}_1$ describe the reflected wave,

$\mathbf{A}_2$, $\mathbf{k}_2$ describe the refracted wave.

As the vector potentials are perpendicular to the directions of propagation, we can express them in terms of components in the directions $\mathbf{M}$ and $\mathbf{k}_m \times \mathbf{M}$, thus

$$\mathbf{A}_m = a_m \mathbf{M} + b_m(\mathbf{k}_m \times \mathbf{M}).$$

The boundary condition contains the electric field strength its amplitudes can be written, since $\omega/c = k_m/n_m$ as

$$\mathbf{E}_m = \frac{k_m}{n_m}\left(a_m \mathbf{M} + b_m(\mathbf{k}_m \times \mathbf{M})\right). \tag{18}$$

The amplitudes of the magnetic field strength are

$$\mathbf{B}_m = \mathbf{k}_m \times \mathbf{A}_m .$$

We find thus easily

$$\mathbf{B}_m = -n_m k_m b_m \mathbf{M} + a_m(\mathbf{k}_m \times \mathbf{M}). \tag{19}$$

The boundary condition for components of the field strength; in accord with (11*a*—*d*) can be written

$$\begin{aligned} \mathbf{K}(\varepsilon_2 \mathbf{E}_2 - \varepsilon_1 \mathbf{E}_1) &= \mathbf{K}\varepsilon_0 \mathbf{E}_0, \\ \mathbf{K}(\mathbf{B}_2 - \mathbf{B}_1) &= \mathbf{K}\mathbf{B}_0, \\ \mathbf{K} \times (\mathbf{E}_2 - \mathbf{E}_1) &= \mathbf{K} \times \mathbf{E}_0, \\ \mathbf{K} \times \left(\frac{\mathbf{B}_2}{\mu_2} - \frac{\mathbf{B}_1}{\mu_1}\right) &= \mathbf{K} \times \frac{\mathbf{B}_0}{\mu_0}. \end{aligned} \tag{20}$$

Giving the values of a_0, b_0 *i.e.* intensity and polarization of the incident beam, (20) gives six equations for the four coefficients a_m, b_m, $m = 1, 2$. The system is, however, not overdetermined as we find that among the six equations obtained, two pairs are identical; indeed, introducing (18) and (19) into (20) there are only four different ones. The result of a short calculation shows that (20) reduces to

$$\begin{aligned} \varepsilon_2 b_2 - \varepsilon_1 b_1 &= \varepsilon_0 b_0 \\ (\mathbf{k}_2\mathbf{K})b_2 - (\mathbf{k}_1\mathbf{K})b_1 &= (\mathbf{k}_0\mathbf{K})b_0 \end{aligned} \tag{21}$$

and

$$a_2 - a_1 = a_0$$
$$\frac{\mathbf{k}_2\mathbf{K}}{\mu_2} a_2 - \frac{\mathbf{k}_1\mathbf{K}}{\mu_1} a_1 = \frac{\mathbf{k}_0\mathbf{K}}{\mu_0} a_0. \tag{22}$$

We see thus that the equations separate automatically into two sets, one containing the amplitudes of the components polarized in the direction of **M**, thus components polarized parallel to S. The other set of equations gives the components polarized perpendicular to **M**. We see thus that the consequent calculation leads automatically to the separation of the two states of polarization.

The explicite solutions of (21) and (22) can be written as follows,

$$b_1 = -\frac{(\varepsilon_2\mathbf{k}_0 - \varepsilon_0\mathbf{k}_2)\mathbf{K}}{(\varepsilon_2\mathbf{k}_1 - \varepsilon_1\mathbf{k}_2)\mathbf{K}} \cdot b_0$$

$$b_2 = -\frac{(\varepsilon_1\mathbf{k}_0 - \varepsilon_0\mathbf{k}_1)\mathbf{K}}{(\varepsilon_2\mathbf{k}_1 - \varepsilon_1\mathbf{k}_2)\mathbf{K}} \cdot b_0$$

$$a_1 = -\frac{\left(\frac{\mathbf{k}_0}{\mu_0} - \frac{\mathbf{k}_2}{\mu_2}\right)\mathbf{K}}{\left(\frac{\mathbf{k}_1}{\mu_1} - \frac{\mathbf{k}_2}{\mu_2}\right)\mathbf{K}} \cdot a_0 \tag{23}$$

$$a_2 = -\frac{\left(\frac{\mathbf{k}_0}{\mu_0} - \frac{\mathbf{k}_1}{\mu_1}\right)\mathbf{K}}{\left(\frac{\mathbf{k}_1}{\mu_1} - \frac{\mathbf{k}_2}{\mu_2}\right)\mathbf{K}} \cdot a_0.$$

The expressions (23) are equivalent with the well-known expressions of Fresnel. We note, that one should insert into (22)

$$\varepsilon_0 = \varepsilon_1 = \varepsilon_{\mathrm{I}} \quad \varepsilon_2 = \varepsilon_{\mathrm{II}}$$

$$\mu_0 = \mu_1 = \mu_{\mathrm{I}} \quad \mu_2 = \mu_{\mathrm{II}}$$

where ε_{I}, $\varepsilon_{\mathrm{II}}$, μ_{I} and μ_{II} are the dielectric constants and permeabilities in the regions I and II.

ЧЁТКАЯ ФОРМУЛИРОВКА ФОРМУЛ ФРЕНЕЛЯ

Л. Яноши

Показано, что хорошо известные формулы Френеля можно получить простым путём при последовательном использовании векторного формализма. Главная цель работы — осветить некоторые общие вопросы, связанные с теорией Максвелла, и, в частности, улучшить с дидактической точки зрения метод получения формул.