



Monotone iterative technique for $(k, n - k)$ conjugate boundary value problems

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Abstract. In this paper, a comparison result for $(k, n - k)$ conjugate boundary value problems is established. By using the monotone iterative technique and the method of upper and lower solutions, we investigate the existence of extremal solutions for a nonlinear differential equation with $(k, n - k)$ conjugate boundary value problems. As an application, an example is presented to illustrate the main results.

Keywords: $(k, n - k)$ conjugate boundary value problems, monotone iterative technique, comparison result.

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1 Introduction

We consider the existence of solution of the following $(k, n - k)$ conjugate boundary value problems for nonlinear ordinary differential equations, using the method of upper and lower solutions and its associated monotone iterative technique

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = f(t, x(t)), & 0 < t < 1, & n \geq 2, & 1 \leq k \leq n - 1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, & 0 \leq i \leq k - 1, & 0 \leq j \leq n - k - 1, \end{cases} \quad (1.1)$$

where $n \geq 2$ and $k \geq 1$ are fixed integers.

The subject of $(k, n - k)$ conjugate boundary value problems for nonlinear ordinary differential equations derives from its theoretical challenge, and have close relationship with oscillation theory (see [4] for more details). Recently, many people paid attention to existence result of solution of $(k, n - k)$ conjugate boundary value problems, such as [1,2,5–7,9,10,12–20], by means of some fixed point theorems.

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The method of upper and lower solutions coupled with the monotone iterative technique plays a very important role in investigating the existence of solutions to ordinary differential equation problems, for example [3, 8, 11]. However, as far as we know, there are no papers dealing with the existence of solutions for $(k, n - k)$ conjugate boundary value problems, by means of the lower and upper solutions method.

The aims of this paper are to establish comparison result for $(k, n - k)$ conjugate boundary value problems and to investigate the existence of extremal solutions of problem (1.1).

The rest of this paper is organized as follows: in Section 2, we present some useful preliminaries and lemmas. The main results are given in Section 3. In Section 4, examples are presented to illustrate the main results.

2 Preliminaries and lemmas

Let $C[0, 1]$ denote the Banach space of real-valued continuous function with norm $\|x\| = \max_{t \in [0, 1]} |x(t)|$.

Throughout this paper, we shall use the following notation:

$$G(t, s) = \begin{cases} \frac{1}{(k-1)!(n-k-1)!} \int_0^{t(1-s)} u^{k-1}(u+s-t)^{n-k-1} du, & 0 \leq t \leq s \leq 1, \\ \frac{1}{(k-1)!(n-k-1)!} \int_0^{s(1-t)} u^{n-k-1}(u+t-s)^{k-1} du, & 0 \leq s \leq t \leq 1. \end{cases}$$

It is well known from the papers [10, 17] that $G(t, s)$ is the Green's function of the following homogeneous boundary value problem:

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = 0, & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, & 0 \leq i \leq k-1, \quad 0 \leq j \leq n-k-1. \end{cases}$$

Lemma 2.1 ([14, 19]). *The function $G(t, s)$ defined as above has the following properties:*

$$\begin{aligned} G(t, s) &\leq \beta s^{n-k}(1-s)^k, \quad 0 \leq t, s \leq 1, \\ \frac{\beta}{n-1} g(t) s^{n-k}(1-s)^k &\leq G(t, s) \leq \alpha g(t) s^{n-k-1}(1-s)^{k-1}, \quad 0 \leq t, s \leq 1, \end{aligned}$$

where

$$\begin{aligned} \beta &= \frac{1}{(k-1)!(n-k-1)!}, & g(t) &= t^k(1-t)^{n-k}, \\ \alpha &= \frac{1}{\min\{k, n-k\}(k-1)!(n-k-1)!}. \end{aligned}$$

In the rest of this paper, we also make the following assumptions:

(H₁) $\emptyset \neq I^+ \cup I^- \subset \{0, 1, \dots, k-1\}$, where $i \in I^+$ (or $i \in I^-$) means that the following $(k, n - k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = 0, & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x(0) = x'(0) = \dots = x^{(i-1)}(0) = x^{(i+1)}(0) = \dots = x^{(k-1)}(0) = 0, \\ x^{(i)}(0) = 1, \quad x^{(j)}(1) = 0, & 0 \leq j \leq n-k-1 \end{cases}$$

has a unique nonnegative (or nonpositive) solution $I_i(t)$ with $|I_i(t)| \geq \frac{t^k(1-t)^{n-k}}{n!}$, $t \in [0, 1]$.

(H₂) $\emptyset \neq J^+ \cup J^- \subset \{0, 1, \dots, n - k - 1\}$, where $j \in J^+$ (or $j \in J^-$) means that the following $(k, n - k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = 0, & 0 < t < 1, & n \geq 2, & 1 \leq k \leq n - 1, \\ x^{(i)}(0) = 0, & 0 \leq i \leq k - 1, & x^{(j)}(1) = 1, \\ x(1) = x'(1) = \dots = x^{(j-1)}(1) = x^{(j+1)}(1) = \dots = x^{(n-k-1)}(1) = 0 \end{cases}$$

has a unique nonnegative (nonpositive) solution $J_j(t)$ with $|J_j(t)| \geq \frac{t^k(1-t)^{n-k}}{n!}$, $t \in [0, 1]$.

Remark 2.2. It follows from (H_1) and (H_2) that for any $a_i, b_j \in \mathbb{R}$ ($0 \leq i \leq k - 1$, $0 \leq j \leq n - k - 1$) such that

$$a_i = 0, \quad \text{if } i \notin I^+ \cup I^-$$

and

$$b_j = 0, \quad \text{if } j \notin J^+ \cup J^-,$$

the $(k, n - k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = 0, & 0 < t < 1, & n \geq 2, & 1 \leq k \leq n - 1, \\ x^{(i)}(0) = a_i, & x^{(j)}(1) = b_j, & 0 \leq i \leq k - 1, & 0 \leq j \leq n - k - 1 \end{cases}$$

has a unique solution $\psi(t) = \sum_{i=0}^{k-1} a_i I_i(t) + \sum_{j=0}^{n-k-1} b_j J_j(t)$, in which we may take $I_i(t) = J_j(t) \equiv 0$ for $i \notin I^+ \cup I^-$ and $j \notin J^+ \cup J^-$. Moreover, if

$$a_i \geq 0, \quad \text{if } i \in I^+; \quad a_i \leq 0, \quad \text{if } i \in I^-$$

and

$$b_j \geq 0, \quad \text{if } j \in J^+; \quad b_j \leq 0, \quad \text{if } j \in J^-$$

hold, $\psi(t)$ becomes a nonnegative function.

Remark 2.3. We point out from examples below that the assumptions (H_1) and (H_2) appear naturally in the study involving $(k, n - k)$ conjugate boundary value problem.

Example 2.4. When $n = 3$, $k = 1$, the unique solution of

$$x'''(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(1) = c$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cJ_1(t),$$

where

$$I_0(t) = 1 - t^2 \geq 0, \quad J_0(t) = -t^2 + 2t \geq 0, \quad J_1(t) = -t(1 - t) \leq 0, \quad t \in [0, 1].$$

Example 2.5 ([15]). When $n = 4$, $k = 2$, the unique solution of

$$x^{(4)}(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(0) = c, \quad x'(1) = d$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cI_1(t) + dJ_1(t),$$

where

$$\begin{aligned} I_0(t) &= 2t^3 - 3t^2 + 1 \geq 0, & J_0(t) &= -2t^3 + 3t^2 \geq 0, \\ I_1(t) &= t^3 - 2t^2 + t \geq 0, & J_1(t) &= t^3 - t^2 \leq 0, \quad t \in [0, 1]. \end{aligned}$$

Example 2.6. When $n = 5, k = 3$, the unique solution of

$$x^{(5)}(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(0) = c, \quad x'(1) = d, \quad x''(0) = e$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cI_1(t) + dJ_1(t) + eI_2(t),$$

where

$$\begin{aligned} I_0(t) &= 3t^4 - 4t^3 + 1 \geq 0, & J_0(t) &= -3t^4 + 4t^3 \geq 0, \\ I_1(t) &= t(2t + 1)(1 - t)^2 \geq 0, & J_1(t) &= t^3 - t^4 \leq 0, \\ I_2(t) &= \frac{1}{2}t^2(1 - t)^2 \geq 0, & t &\in [0, 1]. \end{aligned}$$

Remark 2.7. Under assumptions $(H_1), (H_2)$, we give the definition of lower and upper solution for $(k, n - k)$ conjugate boundary value problem.

Definition 2.8. $u \in C^n[0, 1]$ is called a lower solution of $(k, n - k)$ conjugate boundary value problem if

$$\begin{cases} (-1)^{n-k}u^{(n)}(t) \leq f(t, u(t)), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n - 1, \\ u^{(i)}(0) \leq 0, \text{ if } i \in I^+; \quad u^{(i)}(0) \geq 0, \text{ if } i \in I^-; \quad u^{(i)}(0) = 0, \text{ if } i \notin I^+ \cup I^-; \\ u^{(j)}(1) \leq 0, \text{ if } j \in J^+; \quad u^{(j)}(1) \geq 0, \text{ if } j \in J^-; \quad u^{(j)}(1) = 0, \text{ if } j \notin J^+ \cup J^-. \end{cases}$$

Analogously, $v \in C^n[0, 1]$ is called an upper solutions of $(k, n - k)$ conjugate boundary value problem if the above inequalities are reversed.

For example, u is a lower solution of $(3, 2)$ conjugate boundary value problem if

$$\begin{cases} u^{(5)}(t) \leq f(t, u(t)), & 0 < t < 1, \\ u(0) \leq 0, \quad u'(0) \leq 0, \quad u''(0) \leq 0; \\ u(1) \leq 0, \quad u'(1) \geq 0. \end{cases}$$

Now we consider the linear $(k, n - k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = -Mx(t) + \sigma(t), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n - 1, \\ x^{(i)}(0) = a_i, \quad x^{(j)}(1) = b_j, & 0 \leq i \leq k - 1, \quad 0 \leq j \leq n - k - 1 \end{cases} \quad (2.1)$$

where M is a nonnegative constant and $\sigma \in C[0, 1], a_i, b_j \in \mathbb{R}$.

Lemma 2.9. *If*

$$\alpha MB(n, n) < 1, \quad (2.2)$$

where α is given in Lemma 2.1 and $B(t, s)$ denotes the Beta function, then (2.1) has a unique solution x , which can be expressed by

$$x(t) = \psi(t) + \int_0^1 Q(t, s)\psi(s)ds + \int_0^1 H(t, s)\sigma(s)ds, \quad (2.3)$$

where $\psi(t)$ is given in Remark 2.2,

$$G_1(t, s) = -MG(t, s), \quad Q(t, s) = \sum_{m=1}^{+\infty} G_m(t, s), \quad (2.4)$$

$$G_m(t, s) = (-M)^m \int_0^1 \cdots \int_0^1 G(t, r_1)G(r_1, r_2) \cdots G(r_{m-1}, s)dr_1 \cdots dr_{m-1},$$

and

$$H(t, s) = G(t, s) + \int_0^1 Q(t, \tau)G(\tau, s)d\tau.$$

All functions $G_n(t, s)$, $H(t, s)$, $Q(t, s)$ are continuous on $[0, 1] \times [0, 1]$ and the series on the right-hand side of (2.4) converges uniformly on $[0, 1] \times [0, 1]$.

Proof. It follows from the paper [10] that $x \in C^n[0, 1]$ is a solution of (2.1) if and only if $x \in C[0, 1]$ is a solution of the following operator equation

$$x + Tx = \varphi \quad (2.5)$$

with operator $T: C[0, 1] \rightarrow C[0, 1]$ given by

$$(Tx)(t) = M \int_0^1 G(t, s)x(s)ds,$$

and

$$\varphi(t) = \psi(t) + \int_0^1 G(t, s)\sigma(s)ds. \quad (2.6)$$

We shall prove $r(T) < 1$, where $r(T)$ denotes the spectral radius of operator T . Actually, for $x \in C[0, 1]$, by Lemma 2.1, we have

$$\begin{aligned} |Tx(t)| &\leq M \int_0^1 G(t, s)|x(s)|ds \\ &\leq \alpha M t^k (1-t)^{n-k} \int_0^1 s^{n-k-1} (1-s)^{k-1} ds \|x\| \\ &= \alpha MB(k, n-k) \|x\| t^k (1-t)^{n-k}. \end{aligned}$$

Hence, we have

$$\begin{aligned} |T^2x(t)| &\leq M \int_0^1 G(t, s)|Tx(s)|ds \\ &\leq \alpha^2 M^2 B(k, n-k) \|x\| t^k (1-t)^{n-k} \int_0^1 s^{n-1} (1-s)^{n-1} ds \\ &= \alpha^2 M^2 B(k, n-k) B(n, n) \|x\| t^k (1-t)^{n-k}. \end{aligned}$$

By the induction method, we have

$$|T^m x(t)| \leq \alpha^m M^m B(k, n-k) B^{m-1}(n, n) \|x\| t^k (1-t)^{n-k},$$

which implies that $\|T^m\| \leq \alpha^m M^m B(k, n-k) B^{m-1}(n, n)$. It follows from $r(T) = \lim_{m \rightarrow \infty} \|T^m\|^{1/m}$ that

$$r(T) \leq \alpha M B(n, n) < 1.$$

This yields that the unique solution of operator equation (2.5) is given by

$$x = (I + T)^{-1} \varphi = (I - T + T^2 + \cdots + (-1)^m T^m + \cdots) \varphi.$$

Substituting (2.6) into the above equality, we get (2.3) and the proof is complete. \square

Lemma 2.10. *Suppose that $x \in C^n[0, 1]$ satisfies*

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) \geq -Mx(t), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) \geq 0, \text{ if } i \in I^+; \quad x^{(i)}(0) \leq 0, \text{ if } i \in I^-; \quad x^{(i)}(0) = 0, \text{ if } i \notin I^+ \cup I^-, \\ x^{(j)}(1) \geq 0, \text{ if } j \in J^+; \quad x^{(j)}(1) \leq 0, \text{ if } j \in J^-; \quad x^{(j)}(1) = 0, \text{ if } j \notin J^+ \cup J^-, \end{cases}$$

where the nonnegative constant M satisfies (2.2),

$$B(k, n-k) \left[M\alpha\beta + \frac{M^3\alpha^2\beta^2 B(n, n) B(k+1, n-k+1)}{1 - M^2\alpha^2\beta^2 B^2(n, n)} \right] < \frac{\beta}{n-1}, \quad (2.7)$$

$$MN\alpha + \frac{NM^3\alpha^2\beta B(n, n) B(k, n-k)}{1 - M^2\alpha^2\beta^2 B^2(n, n)} < \frac{1}{n!}, \quad (2.8)$$

in which

$$N = \max \left\{ \int_0^1 s^{n-k-1} (1-s)^{k-1} y(s) ds : y \in \{|I_i|, i \in I^+ \cup I^-\} \cup \{|J_j|, j \in J^+ \cup J^-\} \right\}.$$

Then $x(t) \geq 0$ for $t \in [0, 1]$.

Proof. Let $\sigma(t) = (-1)^{n-k} x^{(n)}(t) + Mx(t)$ and

$$a_i = x^{(i)}(0), \quad 0 \leq i \leq k-1; \quad b_j = x^{(j)}(1), \quad 0 \leq j \leq n-k-1.$$

Then $\sigma(t) \geq 0$ and

$$\begin{cases} a_i \geq 0, \text{ if } i \in I^+; \quad a_i \leq 0, \text{ if } i \in I^-; \quad a_i = 0, \text{ if } i \notin I^+ \cup I^-; \\ b_j \geq 0, \text{ if } j \in J^+; \quad b_j \leq 0, \text{ if } j \in J^-; \quad b_j = 0, \text{ if } j \notin J^+ \cup J^-. \end{cases}$$

By Lemma 2.9, (2.3) holds in which $\psi(t) \geq 0$ for $t \in [0, 1]$. It follows from the expression of $G_m(t, s)$ that $G_m(t, s) \leq 0$ when m is odd and $G_m(t, s) \geq 0$ when m is even. Thus, we obtain

for $m = 3, 5, \dots$, by using Lemma 2.1,

$$\begin{aligned}
 G_m(t, s) &= -M^m \int_0^1 \cdots \int_0^1 G(t, r_1)G(r_1, r_2) \cdots G(r_{m-2}, r_{m-1})G(r_{m-1}, s)dr_1 \cdots dr_{m-1} \\
 &\geq -M^m \int_0^1 \cdots \int_0^1 \left(\alpha g(t)r_1^{n-k-1}(1-r_1)^{k-1} \right) \cdot \left(\alpha r_1^k(1-r_1)^{n-k}r_2^{n-k-1}(1-r_2)^{k-1} \right) \cdots \\
 &\quad \times \left(\alpha r_{m-2}^k(1-r_{m-2})^{n-k}r_{m-1}^{n-k-1}(1-r_{m-1})^{k-1} \right) \cdot \left(\beta s^{n-k}(1-s)^k \right) dr_1 \cdots dr_{m-1} \\
 &= -M^m \alpha^{m-1} \beta g(t) s^{n-k} (1-s)^k \int_0^1 r_1^{n-1} (1-r_1)^{n-1} dr_1 \\
 &\quad \times \int_0^1 r_2^{n-1} (1-r_2)^{n-1} dr_2 \cdots \int_0^1 r_{m-2}^{n-1} (1-r_{m-2})^{n-1} dr_{m-2} \\
 &\quad \times \int_0^1 r_{m-1}^{n-k-1} (1-r_{m-1})^{k-1} dr_{m-1} \\
 &= -M^m \alpha^{m-1} \beta g(t) s^{n-k} (1-s)^k B^{m-2}(n, n) B(k, n-k).
 \end{aligned}$$

Consequently, we have

$$\begin{aligned}
 H(t, s) &= G(t, s) + \int_0^1 Q(t, \tau)G(\tau, s)d\tau = G(t, s) + \sum_{m=1}^{+\infty} \int_0^1 G_m(t, \tau)G(\tau, s)d\tau \\
 &\geq G(t, s) - M \int_0^1 G(t, \tau)G(\tau, s)d\tau + \sum_{m=1}^{+\infty} \int_0^1 G_{2m+1}(t, \tau)G(\tau, s)d\tau \\
 &\geq \frac{\beta}{n-1} g(t) s^{n-k} (1-s)^k - M \alpha \beta g(t) s^{n-k} (1-s)^k \int_0^1 \tau^{n-k-1} (1-\tau)^{k-1} d\tau \\
 &\quad - \sum_{m=1}^{+\infty} M^{2m+1} \alpha^{2m} \beta^2 g(t) s^{n-k} (1-s)^k B^{2m-1}(n, n) B(k, n-k) \int_0^1 \tau^{n-k} (1-\tau)^k d\tau \\
 &= g(t) s^{n-k} (1-s)^k \left[\frac{\beta}{n-1} - M \alpha \beta B(k, n-k) \right. \\
 &\quad \left. - \sum_{m=1}^{+\infty} M^{2m+1} \alpha^{2m} \beta^2 B^{2m-1}(n, n) B(k, n-k) B(k+1, n-k+1) \right].
 \end{aligned}$$

and for $y \in \{I_i, i \in I^+\} \cup \{-I_i, i \in I^-\} \cup \{J_j, j \in J^+\} \cup \{-J_j, j \in J^-\}$,

$$\begin{aligned}
 y(t) &+ \int_0^1 Q(t, s)y(s)ds \\
 &\geq y(t) - M \int_0^1 G(t, s)y(s)ds + \sum_{m=1}^{+\infty} \int_0^1 G_{2m+1}(t, s)y(s)ds \\
 &\geq \frac{g(t)}{n!} - M \alpha g(t) \int_0^1 s^{n-k-1} (1-s)^{k-1} y(s)ds + \sum_{m=1}^{+\infty} \int_0^1 G_{2m+1}(t, s)y(s)ds \\
 &\geq \frac{g(t)}{n!} - M \alpha g(t) \int_0^1 s^{n-k-1} (1-s)^{k-1} y(s)ds \\
 &\quad - \sum_{m=1}^{+\infty} M^{2m+1} \alpha^{2m} \beta B^{2m-1}(n, n) B(k, n-k) g(t) \int_0^1 s^{n-k} (1-s)^k y(s)ds \\
 &\geq \frac{g(t)}{n!} - M N \alpha g(t) - N \sum_{m=1}^{+\infty} M^{2m+1} \alpha^{2m} \beta B^{2m-1}(n, n) B(k, n-k) g(t) \\
 &= g(t) \left[\frac{1}{n!} - M N \alpha - N \sum_{m=1}^{+\infty} M^{2m+1} \alpha^{2m} \beta B^{2m-1}(n, n) B(k, n-k) \right].
 \end{aligned}$$

Thus, by (2.8), we have that $x(t) \geq 0$ for $t \in [0, 1]$, and the lemma is proved. \square

3 Main results

In this section, we prove the existence of extremal solutions of differential equation (1.1).

Theorem 3.1. *Let $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$; v_0, w_0 be lower and upper solutions of (1.1) such that $v_0(t) \leq w_0(t)$ on $[0, 1]$. Suppose further that there exists $M > 0$ such that*

$$f(t, x) - f(t, y) \geq -M(x - y), \quad (3.1)$$

whenever $v_0(t) \leq y \leq x \leq w_0(t)$ and M satisfies (2.2), (2.7) and (2.8). Then there exist monotone sequences $\{v_m(t)\}, \{w_m(t)\}$ which converge uniformly on $[0, 1]$ to the extremal solutions of problem (1.1) in the order interval $[v_0, w_0] = \{u \in C[0, 1] : v_0(t) \leq u(t) \leq w_0(t), t \in [0, 1]\}$.

Proof. For any $\eta \in [v_0, w_0]$, we consider the linear differential equation

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = -Mx(t) + f(t, \eta(t)) + M\eta(t), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, & 0 \leq i \leq k-1, \quad 0 \leq j \leq n-k-1. \end{cases} \quad (3.2)$$

By Lemma 2.9, (3.2) has a unique solution $x(t) = \int_0^1 H(t, s)[f(s, \eta(s)) + M\eta(s)]ds$ in $C[0, 1]$. Define the mapping A by $A\eta = x$ with operator $A : [v_0, w_0] \rightarrow C[0, 1]$ given by

$$(A\eta)(t) = \int_0^1 H(t, s)[f(s, \eta(s)) + M\eta(s)]ds$$

and use it to construct the sequences $\{v_m(t)\}, \{w_m(t)\}$. Let us prove that

- (i) $v_0 \leq Av_0, Aw_0 \leq w_0$;
- (ii) A is a monotone operator on $[v_0, w_0]$.

To prove (i), set $Av_0 = v_1$, where v_1 is the unique solution of (3.2) with $\eta = v_0$. Setting $p = v_1 - v_0$, we see that

$$\begin{cases} (-1)^{n-k} p^{(n)}(t) \geq -Mp(t), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ p^{(i)}(0) \geq 0, \text{ if } i \in I^+; p^{(i)}(0) \leq 0, \text{ if } i \in I^-; p^{(i)}(0) = 0, \text{ if } i \notin I^+ \cup I^-, \\ p^{(j)}(1) \geq 0, \text{ if } j \in J^+; p^{(j)}(1) \leq 0, \text{ if } j \in J^-; p^{(j)}(1) = 0, \text{ if } j \notin J^+ \cup J^-. \end{cases}$$

This shows, by Lemma 2.10, that $p(t) \geq 0$ on $[0, 1]$ and hence $v_0 \leq Av_0$. Similarly, we can show that $Aw_0 \leq w_0$.

To prove (ii), let $\eta_1, \eta_2 \in [v_0, w_0]$ such that $\eta_1 \leq \eta_2$. Suppose that $x_1 = A\eta_1$, and $x_2 = A\eta_2$. Set $p = x_2 - x_1$ so that

$$\begin{cases} (-1)^{n-k} p^{(n)}(t) \geq -Mp(t), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ p^{(i)}(0) = p^{(j)}(1) = 0, & 0 \leq i \leq k-1, \quad 0 \leq j \leq n-k-1, \end{cases} \quad (3.3)$$

here we have used the condition (3.1). By Lemma 2.10, (3.3) implies that $A\eta_1 \leq A\eta_2$ proving (ii).

Now let $v_m = Av_{m-1}$, $w_m = Aw_{m-1}$, $m = 1, 2, \dots$. From the foregoing arguments, we conclude that

$$v_0 \leq v_1 \leq \dots \leq v_m \leq \dots \leq \dots w_m \leq \dots \leq w_1 \leq w_0. \quad (3.4)$$

Obviously the sequences $\{v_m\}$, $\{w_m\}$ are uniformly bounded on $[0, 1]$, and by (3.1), we have

$$\begin{aligned} f(t, v_0(t)) + Mv_0(t) &\leq f(t, v_m(t)) + Mv_m(t) \\ &\leq f(t, w_m(t)) + Mw_m(t) \leq f(t, w_0(t)) + Mw_0(t), \quad m \in \mathbb{N}, t \in [0, 1]. \end{aligned}$$

This together with the continuity of $H(t, s)$ on $[0, 1] \times [0, 1]$ imply that $\{v_m\}_{m=2}^\infty = \{Av_m\}_{m=1}^\infty$ and $\{w_m\}_{m=2}^\infty = \{Aw_m\}_{m=1}^\infty$ are two sequentially compact sets. As a result, there exist subsequences $\{v_{m_i}\}$, $\{w_{m_i}\}$ that converge uniformly on $[0, 1]$. In view of (3.4), it also follows that the entire sequences $\{v_m\}$, $\{w_m\}$ converge uniformly and monotonically to their limit functions $v^*(t)$, $w^*(t)$ respectively, that is,

$$\lim_{m \rightarrow \infty} v_m(t) = v^*(t), \quad \lim_{m \rightarrow \infty} w_m(t) = w^*(t), \quad \text{uniformly on } [0, 1].$$

It is now easy to show that v^*, w^* are solutions of conjugate boundary value problem (1.1), using the corresponding integral equation

$$x(t) = (A\eta)(t) = \int_0^1 H(t, s)[f(s, \eta(s)) + M\eta(s)]ds$$

for (3.2).

Next, we prove that v^*, w^* are extremal solutions of (1.1) in $[v_0, w_0]$. In fact, we assume that x is any solution of (1.1). That is,

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = f(t, x(t)), & 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, & 0 \leq i \leq k-1, \quad 0 \leq j \leq n-k-1. \end{cases}$$

By (3.1) and Lemma 2.10, it is easy by induction to show that

$$v_m \leq x \leq w_m, \quad m = 1, 2, 3, \dots \quad (3.5)$$

Now, letting $m \rightarrow \infty$ in (3.5), we have $v^* \leq x \leq w^*$. That is, v^* and w^* are extremal solutions of (1.1) in $[v_0, w_0]$. \square

4 Examples

Consider the following $(2, 2)$ conjugate boundary value problems:

$$\begin{cases} x^{(4)}(t) = \frac{1}{5}(t^2 - x(t))^3 - \frac{1}{5}t^9, & 0 < t < 1, \\ x(0) = x'(0) = x(1) = x'(1) = 0. \end{cases} \quad (4.1)$$

Let $f(t, x) = \frac{1}{5}(t^2 - x)^3 - \frac{1}{5}t^9$. Obviously, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. Take $w_0(t) = t^2 - 3t^3/4$, $v_0(t) = 0$, then $v_0(t) \leq w_0(t)$ for $t \in [0, 1]$ and we have

$$\begin{cases} w_0^{(4)}(t) = 0 \geq -\frac{37}{320}t^9 = \frac{1}{5}(t^2 - w_0(t))^3 - \frac{1}{5}t^9, & 0 < t < 1, \\ w_0(0) = w_0'(0) = 0, w_0(1) = \frac{1}{4} \geq 0, w_0'(1) = -\frac{1}{4} \leq 0, \end{cases}$$

$$\begin{cases} v_0^{(4)}(t) = 0 \leq \frac{t^6 - t^9}{5} = \frac{1}{5}(t^2 - v_0(t))^3 - \frac{1}{5}t^9, & 0 < t < 1, \\ v_0(0) = v_0'(0) = v_0(1) = v_0'(1) = 0. \end{cases}$$

Consequently, by Definition 2.8 and Example 2.5, v_0, w_0 are lower and upper solutions of (4.1) respectively. If $v_0(t) \leq v \leq u \leq w_0(t)$, we have

$$f(t, u) - f(t, v) = \frac{1}{5}(t^2 - u)^3 - \frac{1}{5}v(t^2 - v)^3 \geq -\frac{3}{5}(u - v).$$

It is clear that $M = \frac{3}{5}$, $\alpha = \frac{1}{2}$, $\beta = 1$, $n = 4$, $k = 2$,

$$N = \max \left\{ \int_0^1 s(1-s)y(s)ds : y \in \{2t^3 - 3t^2 + 1, -2t^3 + 3t^2, t^3 - 2t^2 + t, t^2 - t^3\} \right\} = \frac{1}{12},$$

and so, it is easy to show that inequalities (2.2), (2.7) and (2.8) are satisfied.

By Theorem 3.1, problem (4.1) has extremal solutions in $[v_0, w_0]$.

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