On linked products of groups

By B. H. NEUMANN and HANNA NEUMANN in Sale (Cheshire, England)

To Ladislaus Rédei for his 60th birthday

1. Introduction

Linked products were first introduced in a recent paper [5] by JAMES WIEGOLD and the second author; a number of questions were left unanswered there. We propose to answer some of them in this note.

The products considered are generalizations of GOLOVIN's regular products: the group G is a regular product of its subgroups A and B, if A and B generate G and are retracts (that is, images under idempotent endomorphisms) of G; or, equivalently, the normal closure of A in G meets B trivially, and the normal closure of B in G meets A trivially. In the case of a linked product we ask for a group G generated by A and B in such a way that, although A and B have still only the unit element in common in G, mapping A onto the trivial group induces a prescribed homomorphism of B onto a factor group B/Y, and mapping B onto the trivial group induces a prescribed homomorphism of A onto A/X; in other words, we call G a *linked product of A and B with kernels X and Y* if G is generated by A and B, if A and B have only the unit element in common, but the normal closure of A in G meets B in Y, and the normal closure of B in G meets A in X. We think of A and its normal subgroup X, and of B and its normal subgroup Y, as given.

The question immediately arises whether it is always possible to construct a linked product of given groups A and B with given kernels X and Y. It was shown in [5], Example 6.2, that this is certainly not possible when A and B are both of order 2, and of the kernels X and Y one is trivial and the other not. But the positive results obtained in [5] strongly suggested that this case is, in fact, the only exception. We still can not prove this conjecture, but we make somme further progress towards it. The "unsymmetrical case", where just one of the kernels is trivial, proved the more resistant case in [5]. One of our results here reduces it in many cases — we conjecture:

in all but one case — to the "symmetrical case", where both kernels are non-trivial (§3). (We are not concerned with the case that both kernels are trivial: this is, of course, just the case of GOLOVIN's regular products.) Using results of [5] based on a theorem of WIELANDT [6], we establish in particular the existence of linked products of arbitrary finite groups A and B with arbitrarily prescribed normal subgroups X and Y as kernels, apart, of course, from the one exceptional case described above. Moreover the proof of the reduction theorem shows that these linked products of finite groups can be chosen finite. This confirms another conjecture made in [5].

It is fairly obvious that linked products, where they exist, are not uniquely determined by their constituents A and B and the kernels X and Y. In §5 we give some indication just how widely they may differ: If A and B are finite and X and Y non-trivial proper normal subgroups of A and B, we show — subject to some restrictions on the orders and indices of X and Y arising out of the exceptional case — that there exist finite linked products of A and B in which the kernels X and Y generate a simple group, but there also are finite linked products of A and B in which X and Y generate their direct product.

In §4 we prepare the ground for these constructions by restating some known facts on the embedding of group amalgams.

2. Notation

Groups are denoted by capital letters, their elements by small letters. We write a^b for $b^{-1}ab$, and $[a, b]$ for the commutator $a^{-1}b^{-1}ab$. The unit element of all groups is denoted by 1; the trivial group is always denoted by E. Small Greek letters stand for homomorphisms of groups. Capital German letters are used for group amalgams, that is for set-theoretical unions of given groups intersecting pairwise in given subgroups, with multiplication defined — in the natural way — for those and only those pairs of elements that belong to one and the same constituent group of the amalgam.

If the group G is generated by the set M, we write $G = \mathrm{gp}(M)$; similarly $G = \mathrm{gp}(A, B)$ means that G is generated by its subgroups A, B. If $G = \mathrm{gp}(A, B)$, then the group generated by all commutators $[a, b]$, $a \in A$, $b \in B$, is normal in G (cf. GOLOVIN [1]). It is denoted by $[A, B]$ and called the *cartesian subgroup* of G. If $G = \mathrm{gp}(A, B) = [A, B]$, we call G *self-cartesian*.

The normal closure in G of a set M, that is the least normal subgroup of G containing M, is denoted by M^G. If again $G = \mathrm{gp}(A, B)$, then the normal closure of A in G is $A^G = A \cdot [A, B]$ (cf. GOLOVIN [1]).

Finally we denote the order of a group G by $|G|$.

3. A reduction theorem

In [5], certain reduction theorems were obtained which deduced the existence of a linked product of A and B with kernels X and Y from that of a linked product of X and Y with kernels X and Y — if both X and Y are non-trivial — or from a linked product of X and B with kernels X and E in the unsymmetrical case. Here we prove a reduction theorem which operates, as it were, in the opposite direction: it deduces the existence of a linked product of A and B with kernels X and Y from that of certain linked products of A and B with kernels A and B:

Theorem 3.1. *Let there be a self-cartesian linked product G of A and B with kernels A and B, and let X and Y be arbitrary normal subgroups of A and B, respectively. Then there exists a linked product of A and B with kernels X and Y.*

Proof. Let φ be the canonic epimorphism of A onto $A_1 = A/X$, and ψ the canonic epimorphism of B onto $B_1 = B/Y$. We form the direct product

$$G_0 = A_1 \times G \times B_1;$$

its elements are the triplets $(a\varphi, g, b\psi)$ where a, g, b range over A, G, B, respectively. The triplets

$$a_0 = (a\varphi, a, 1)$$

form a group A_0 which is clearly isomorphic to A; the subgroup X_0 of A_0 corresponding to X consists of the triples $x_0 = (1, x, 1)$. Similarly, the triplets

$$b_0 = (1, b, b\psi)$$

form a group B_0 isomorphic to B, and the triplets $y_0 = (1, y, 1)$ form the subgroup Y_0 of B_0 that corresponds to Y in B.

We prove that G_0 is a linked product of A_0 and B_0 with kernels X_0 and Y_0. Firstly, any element of G_0 common to A_0 and B_0 has first and third components 1, and as $A \cap B = E$, also middle component 1; hence in G_0,

$$A_0 \cap B_0 = E.$$

Secondly, we find the cartesian subgroup $[A_0, B_0]$ of $\mathrm{gp}(A_0, B_0)$. A typical commutator $[a_0, b_0]$ is of the form

$$[a_0, b_0] = (1, [a, b], 1),$$

and as the commutators $[a, b]$ are assumed to generate G, the commutators $[a_0, b_0]$ generate $E \times G \times E$. We identify this subgroup of G_0 with G and then have

$$[A_0, B_0] = G.$$

But A_0 and G generate A_1 (similarly identified with $A_1 \times E \times E \leqq G_0$), and B_0 and G generate B_1; thus A_0 and B_0 between them generate all three direct factors of G_0, and

$$G_0 = \mathrm{gp}(A_0, B_0).$$

Finally

$$A_0^{G_0} = A_0 [A_0, B_0] = A_0 G = A_1 \times G.$$

Thus $A_0^{G_0} \cap B_0 = (A_1 \times G) \cap B_0$, and this consists of all elements $(a\varphi, g, 1)$ that are simultaneously of the form $(1, b, b\psi)$. It follows that $a\varphi = 1, g = b$, and $b\psi = 1$; hence $b = y \in Y$, and the elements of the intersection are just the elements

$$y_0 = (1, y, 1).$$

Thus

$$A_0^{G_0} \cap B_0 = Y_0.$$

A symmetrical argument shows that

$$A_0 \cap B_0^{G_0} = X_0,$$

and the theorem follows. The following is an immediate consequence of the proof, as G_0 clearly is finite when G is.

Corollary 3.11. *If G is finite, the linked product also can be taken finite.*

Corollary 3.12. *If the linked product G of A and B with kernels A and B is not only self-cartesian, but simple, then in G_0 — constructed as in the proof of the theorem — every non-trivial subgroup C_0 of A_0 has the property*

$$C_0^{G_0} \cap B_0 = Y_0,$$

and symmetrically, for every $D_0 \leqq B_0, D_0 \neq E$,

$$A_0 \cap D_0^{G_0} = X_0.$$

If both A and B are of order 2, generated by a and b, respectively, then the condition of the theorem cannot be satisfied. For in this case, as $a^2 = b^2 = 1$, the group generated by a and b is dihedral, of order $2n$ or of infinite order; in either case the commutator $[a, b] = (ab)^2$ is an element of the cyclic subgroup of index 2, and therefore cannot generate the whole group. This is, of course, in accordance with the fact that in this case there is no linked product of A and B with kernels A and E.

Again we conjecture that this is the only exception, that is, that any two groups A and B not both of which are of order 2 possess a self-cartesian linked product with kernels A and B. We cannot prove this in general.

If, however, A and B are finite and not both of order 2, then by a theorem of WIELANDT [6] A and B can be so embedded in a finite simple group G, that G is generated by A and B and their intersection in G is trivial; but clearly a simple group is self-cartesian (with respect to any two non-trivial subgroups that generate it). Thus we have:

Theorem 3.2. *Given two finite groups A and B with normal subgroups X and Y, where $E \leqq X \leqq A$ and $E \leqq Y \leqq B$; then there is a finite linked product of A and B with kernels X and Y, unless $|A| = |B| = |X| \cdot |Y| = 2$.*

It may be remarked that here we have the situation in which Corollary 3.12 applies.

Here X and Y will not in general themselves generate a simple group. In order to establish the existence of linked products with this additional property, we have to go back to the procedure used in [5], §3, of building up the linked product of A and B from a linked product of the kernels X and Y. In order to ensure that these linked products also are finite when A and B are finite, we need some facts on embeddings of amalgams of groups.

4. Two lemmas

Let $\mathfrak{A}$ be an amalgam of finitely many finite groups. We assume that $\mathfrak{A}$ is embeddable in a group; therefore the generalized free product F of $\mathfrak{A}$ exists. F is characterized by the facts that it embeds the amalgam $\mathfrak{A}$, is generated by it, and that every homomorphism of $\mathfrak{A}$ into a group can be continued to a homomorphism of F into that group. We further assume[1]) that $\mathfrak{A}$ can be embedded in a finite group P, say, which we may assume to be generated by $\mathfrak{A}$.

Lemma 4.1. [3] *If φ is a homomorphism of $\mathfrak{A}$ into a finite group D, then there is an embedding θ of $\mathfrak{A}$ in a finite group Q generated by $\mathfrak{A}\theta$ and a homomorphism ψ of Q into D such that*

$$\varphi = \theta\psi.$$

For the sake of completeness we here briefly indicate the proof (cf. [3], § 2, where the lemma is proved in a slightly more general situation).

Let $D_1 = \mathrm{gp}(\mathfrak{A}\varphi)$. Then $D_1 \cong F/M$, where F is the generalized free product of $\mathfrak{A}$ and M a normal subgroup of finite index in F. Similarly $P \cong F/N$ where N also has finite index in F. Then $M \cap N$ also is a normal

[1]) It is not known whether this is really an additional assumption, or whether a finite amalgam that is embeddable in a group is also embeddable in a finite group; cf. [2], § 5.

subgroup of finite index in F, and we put $Q = F/M \cap N$. The canonic epimorphism F onto Q induces a homomorphism θ of $\mathfrak{A}$ into Q, and this is a monomorphism because it can be further multiplied by an epimorphism of Q onto P so as to result in the given embedding of $\mathfrak{A}$ in P. We take as ψ the canonic homomorphism of Q onto its factor group $Q/(M/M \cap N) \cong F/M \cong D_1$ followed by an isomorphism onto D_1.

The amalgams to which we are going to apply this lemma are sufficiently simple that we can easily check the validity of the embeddability assumptions made for Lemma 4.1. We use the following known fact ([4], Corollary 15.2).

Lemma 4.2. *An amalgam of two finite groups is embeddable in a finite group.*

5. Some special types of linked product

Let A and B be groups containing the non-trivial normal subgroups X and Y, respectively. Let Z be a linked product of X and Y with kernels X and Y. We form the amalgam $\mathfrak{A}$ of the groups A, B, and Z amalgamating E between A and B, X between A and Z, and Y between B and Z. It is fairly easily seen (and shown in detail in [5], § 3) that the free product of A, B, Z with these amalgamations exists and is a linked product of A and B with kernels X and Y.

If X and Y are finite, we know that such a linked product Z of X and Y with kernels X and Y exists and can be taken finite; and that, moreover, it can be taken as a simple group unless both X and Y have order 2 (WIELANDT [6]). But even if A and B are also finite, the free product will be an infinite group as at least one of A and B contains X, or Y, properly. To construct a finite linked product also in this case, we use Lemma 4.1.

It was stated already that the amalgam $\mathfrak{A}$ of A, B, and Z is embeddable in a group. To see that it is embeddable in a finite group, we first note that, by Lemma 4.2, the subamalgam formed by A and Z amalgamating X is embeddable in a finite group, A_1, say. Similarly the subamalgam formed by B and Z is embeddable in a finite group, B_1, say. We consider the amalgam $\mathfrak{B}$ of A_1 and B_1 amalgamating Z. It contains $\mathfrak{A}$ as a subamalgam. But, again by Lemma 4.2, $\mathfrak{B}$ is embeddable in a finite group, and so, therefore, is $\mathfrak{A}$. Finally, $\mathfrak{A}$ possesses a homomorphism φ into the direct product $D = A/X \times B/Y$; for mapping $A \to A/X$ and $B \to B/Y$ canonically induces automatically the mapping of Z on E. The assumptions of Lemma 4.1 are therefore satisfied, and we deduce:

Lemma 5.1. *The amalgam* $\mathfrak{A}$ *of* A, B, *and* Z *can be embedded in a finite group* Q *generated by it, such that there exists a homomorphism* ψ *of* Q *onto* $D = A/X \times B/Y$ *which maps* A *in* Q *onto* $A\psi = A/X$ *and* B *in* Q *onto* $B\psi = B/Y$, *canonically.*

We now show that this group Q is a linked product of A and B with kernels X and Y. Q is clearly generated by A and B, as it is generated by $\mathfrak{A}$, and the constituent Z of $\mathfrak{A}$ is generated by $X \leqq A$ and $Y \leqq B$. Also $A \cap B = E$ in $\mathfrak{A}$, and therefore in Q. Finally $A^Q \geqq X^Z \geqq Y$, and therefore $A^Q \cap B \geqq Y$. To show that this intersection cannot contain Y properly, it suffices to exhibit one normal subgroup K of Q which contains A and meets B exactly in Y. Now Q possesses a homomorphism ψ onto $D = A/X \times B/Y$; and D can be mapped homomorphically onto B/Y by the retraction that maps A/X onto E. The product of these homomorphisms is a homomorphism β of Q onto B/Y which maps A onto E and B onto B/Y canonically. The kernel K of β contains A and intersects B in Y, as required. Similarly one shows that $B^Q \cap A = X$, which completes the argument. We have therefore:

Theorem 5.2. *If* A *and* B *are finite groups, if* X *and* Y *are non-trivial normal subgroups of* A *and* B, *respectively, and if not both of* X *and* Y *have order* 2, *then there exists a finite linked product of* A *and* B, *with kernels* X *and* Y, *in which* X *and* Y *generate a simple group.*

In order to obtain the other extreme, a linked product in which X and Y centralize each other, we first use the methods of § 3 to prove:

Lemma 5.3. *If* A/X *and* Y *are finite, non-trivial groups and not both of order* 2, *then there exists a linked product of* A *and* Y *with kernels* E *and* Y *in which* X *and* Y *centralize each other.*

Proof. Let φ be the canonic epimorphism of A onto $A\varphi = A/X$. As $A\varphi$ and Y do not both have order 2, there exists a self-cartesian linked product G of $A\varphi$ and Y with kernels $A\varphi$ and Y. We form the direct product

$$H_1 = A \times G,$$

and consider in it the group A_0 consisting of all elements $a_0 = (a, a\varphi)$. Again A_0 is isomorphic to A, and we show that H_1 is a linked product of A_0 and Y of the required kind; here Y is thought of as identified with the group consisting of all $y_0 = (1, y)$.

As $[a_0, y] = [(a, a\varphi), (1, y)] = (1, [a\varphi, y])$, and as $[A\varphi, Y] = G$, we have (upon identifying $E \times G \leqq H_1$ with G)

$$[A_0, Y] = G.$$

But A_0 and G generate $A \times E$, and therefore H_1, so that $H_1 = \mathrm{gp}(A_0, Y)$.

Further $A_0^{H_1} = A_0 \cdot [A_0, Y] = A_0 \cdot G = H_1 \geqq Y$, but

$$Y^{H_1} = (Y^G)^{H_1} = G^{H_1} = G,$$

and so $Y^{H_1} \frown A_0$ consists of those elements $(a, a\varphi)$ for which $a = 1$, that is, $Y^{H_1} \frown A_0 = E$. Thus H_1 is a linked product of A_0 and Y with kernels E and Y; and if the subgroup X_0 of A_0 consisting of all elements $(x, 1)$ is identified with X in A, it is clear that together with Y in G it generates the direct product $X \times Y$.

The lemma puts us in a position to construct the kind of linked product we want:

Theorem 5.4. *If A and B are finite groups with non-trivial proper normal subgroups X and Y, respectively, and if neither A/X and Y, nor B/Y and X are simultaneously of order 2, then there exists a finite linked product of A and B, with kernels X and Y, in which X and Y centralize each other.*

Proof. Construct, by Lemma 5.3, a linked product H_1 of A and Y, with kernels E and Y, in which X and Y centralize each other. Symmetrically, also by Lemma 5.3, construct a linked product H_2 of X and B, with kernels X and E, in which X and Y also centralize each other.

The subgroup generated by X and Y is their direct product in both H_1 and H_2, and we denote it by the same letter, T, say, in both. Let $\mathfrak{A}$ be the amalgam of H_1 and H_2 amalgamating T. Then $\mathfrak{A}$ possesses a finite embedding, by Lemma 4.2. Also, by mapping Y onto E and A identically, H_1 is mapped homomorphically onto A. By further mapping A canonically onto A/X we obtain, therefore, a homomorphism φ_1 of H_1 onto A/X in which A is mapped canonically. Similarly there is a homomorphism φ_2 of H_2 onto B/Y which maps B canonically. The two homomorphisms φ_1 and φ_2 agree on $T = X \times Y$, which is mapped on E by both. If, therefore, we define the mapping φ of $\mathfrak{A}$ into $D = A/X \times B/Y$ by $\varphi = \varphi_1$ on H_1 and $\varphi = \varphi_2$ on H_2, then φ maps the amalgam $\mathfrak{A}$ homomorphically into D. By Lemma 4.1, there exists a finite group Q embedding the amalgam $\mathfrak{A}$ and generated by it, such that Q possesses a homomorphism ψ mapping it onto D in such a way that A and B are mapped as by φ, that is canonically onto A/X and B/Y, respectively.

Again we can show now that Q is a linked product of A and B with kernels X and Y; for Q is generated by H_1 and H_2, that is by A, Y, B, and X, and thus by A and B. Also

$$A \frown B \leqq H_1 \frown H_2 = T = X \times Y,$$

and so $A \frown B = (A \frown T) \frown (B \frown T) = X \frown Y = E$.

Finally $A^Q \geqq A^{H_1} \geqq Y$, and so $A^Q \cap B \geqq Y$. But, as before, Q possesses a homomorphism ψ onto $A/X \cap B/Y$, and therefore also a homomorphism β mapping A on E and B canonically onto B/Y. The kernel of β is a normal subgroup of Q which contains A and intersects B exactly in Y, and so it follows that $A^Q \cap B = Y$. Similarly one shows that $B^Q \cap A = X$. Finally X and Y generate the direct product $T = X \times Y$ in Q, so that Q is a linked product of the required kind, and the theorem follows.

References

[1] O. N. Golovin, Nilpotent products of groups, *Mat. Sbornik*, (N. S.) **27 (69)** (1950), 427—454; *Amer. Math. Soc. Translations* (2nd Ser.), **2** (1956), 89—115.

[2] B. H. Neumann and Hanna Neumann, A contribution to the embedding theory of group amalgams, *Proc. London Math. Soc.*, (3) **3** (1953), 243—256.

[3] B. H. Neumann and Hanna Neumann, Partial endomorphisms of finite groups, *J. London Math. Soc.*, **29**, (1954), 434—440.

[4] B. H. Neumann, An essay on free products of groups with amalgamations, *Phil. Trans. Roy. Soc. London,* (A) **246**, (1954), 503—554.

[5] Hanna Neumann and James Wiegold, Linked products and linked embeddings of groups, *Math. Zeitschr.*, **73** (1960), 1—19.

[6] Helmut Wielandt, Einbettung zweier Gruppen in eine einfache Gruppe, *Math. Zeitschr.*, **73** (1960), 20—21.

THE UNIVERSITY, MANCHESTER,
THE MANCHESTER COLLEGE OF SCIENCE AND TECHNOLOGY

(Received January 31, 1960)