A theorem on diophantine approximation with application to Riemann zeta-function

By P. TURÁN in Budapest

To the sixtieth birthday of my friend Prof. L. Rédei

1. In a recent paper[1]) I proved among others the following theorem. If for $n > n_0$ none of the Dirichlet-polynomials

(1) $$U_n(s) = \sum_{\nu \leq n} \nu^{-s} \qquad (s = \sigma + it)$$

vanishes in a half-strip

(1.2) $$\sigma \geqq 1 + \frac{\log^3 n}{\sqrt{n}}, \quad \gamma_n \leqq t \leqq \gamma_n + e^{n^3}$$

with a suitable real γ_n, then RIEMANN's conjecture is true.

Also a sort of converse theorems was proved in the above quoted paper. The aim of the present note is to improve the above quoted theorem by proving that *Riemann's conjecture follows even from the weaker assumption that for $n > n_0$ the polynomials $U_n(s)$ do not vanish in the half-strip*

(1.3) $$\sigma \geqq 1 + \frac{\log^3 n}{\sqrt{n}}, \quad \gamma_n \leqq t \leqq \gamma_n + e^{n^{\frac{3}{2}}}$$

with a suitable real γ_n.

Probably the half-strip (1.3) could be replaced in the theorem by the half-strip

(1.4) $$\sigma \geqq 1 + \frac{\log^3 n}{\sqrt{n}}, \quad \gamma_n \leqq t \leqq \gamma_n + e^{c_1 n}$$

with a suitable c_1, where c_1 — and later $c_2, c_3, \ldots$ — stand for positive numerical constants.

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1) P. TURÁN, Nachtrag zu meiner Abhandlung "On some approximative Dirichlet polynomials in the theory of zeta-function of Riemann", *Acta Math. Acad. Sci. Hung.*, 10 (1959), 277—298.

Again an infinite number of exceptional polynomials $U_n(s)$ vanishing in *every* half-strip of the form (1. 3), could have been admitted, supposing that the number of such indices not exceeding x is $o(\log x)$ for $x \to \infty$. We shall omit this as well as the similar theorems for

$$C_n(s) = \sum_{\nu \leqq n} \left(1 - \frac{\nu}{n+1}\right) \nu^{-s}, \quad V_n(s) = \sum_{\nu \leqq n} (-1)^{\nu+1} \nu^{-s},$$
$$W_n(s) = \sum_{\nu \leqq n} (-1)^{\nu} (2\nu - 1)^{-s}$$

and the proof that $U_n(s)$ does not vanish for $n > c_2$ in the domain

(1. 5) $$\sigma \geqq 1, \quad c_3 \leqq t \leqq e^{c_4 \log^{\frac{8}{7}} n}.$$

But I do emphasize again as I did in my above quoted paper two facts. First that in order to verify the condition in (1. 3) it would suffice to prove

(1. 6) $$\overline{\lim_{T \to \infty}} \frac{N_n(T)}{T} < e^{-n^{\frac{3}{2}}}$$

where $N_n(T)$ stands for the number of zeros of $U_n(s)$ for

(1. 7) $$\sigma \geqq 1 + \frac{\log^3 n}{\sqrt{n}}, \quad 0 \leqq t \leqq T$$

and for which to prove e. g. the inequality

$$N_n(T) < c_5 \frac{T}{n}$$

is easy. Secondly though the proofs work "essentially" also for all functions $f(s)$ which are representable for $\sigma > 1$ by an absolutely convergent Dirichlet-series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with positive monotonical coefficients a_n and also by an Euler-product

$$\prod_p \frac{1}{1 - \frac{b_p}{p^s}} \qquad (b_p \text{ real}),$$

these three properties already *characterize* $\zeta(s)$ up to a translation of s, perhaps surprisingly for the first minute, owing to the existing big literature of the characterisation-problem.

2. The improvement is furnished by a lemma which belongs to the theory of diophantine approximation; a theory to which RÉDEI made valuable contributions. This will be the

Lemma. *If* $2=p_1<p_2<\cdots<p_N$ *stand for the first* N *primes,* $d, \beta_1, \beta_2, \ldots, \beta_N$ *for arbitrary real numbers and* ω *is an integer* $\geqq 4$, *then there is a* t_0 *with*

$$d \leqq t_0 \leqq d+e^{17\omega N\log^2 N} \tag{2.1}$$

such that for $\nu=1,2,\ldots,N$ *the inequalities*

$$|t_0 \log p_\nu-\beta_\nu-e_\nu| \leqq \frac{1}{\omega} \qquad (e_\nu \text{ integers}) \tag{2.2}$$

hold[2]), *if only* $N>c_6(>e^{31}), N>\omega$.

The half-strip (1.3) could be replaced by the one in (1.4) if the interval in (2.1) could have been replaced by

$$d \leqq t_0 \leqq d+\omega^N$$

say (which would be essentially best-possible).

For the proof of the lemma we make first some preparations. Let

$$k=[\log N]\ (\geqq 30), \quad m=[e^2\omega]\ (>4), \tag{2.3}$$

$$A=\int_0^1\left(\frac{\sin m\pi x}{\sin \pi x}\right)^{2k} dx, \tag{2.4}$$

and

$$P(x)=\frac{1}{A}\cdot\left(\frac{\sin m\pi x}{\sin \pi x}\right)^{2k}. \tag{2.5}$$

Since after FEJÉR's formula we have

$$\frac{1}{m}\left(\frac{\sin m\pi x}{\sin \pi x}\right)^2=\sum_{\nu=-(m-1)}^{m-1}\left(1-\frac{|\nu|}{m}\right)e^{2\pi i\nu x},$$

we get

$$P(x)=\sum_{\nu=-(m-1)k}^{(m-1)k} a'_\nu e^{2\pi i\nu x} \tag{2.6}$$

with positive a'_ν satisfying

$$a'_{-\nu}=a'_\nu \quad\text{and}\quad a'_0=1. \tag{2.7}$$

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2) In the previous form the exponent in (2.1) was $3(N\omega)^2\log^2 N\omega$.

It is easy to verify from (2.6) and (2.5) that

(2.8) $$\sum_{\nu=-(m-1)k}^{(m-1)k} a'_\nu z^\nu = \frac{1}{A} z^{-(m-1)k} \left(\frac{z^m-1}{z-1}\right)^{2k}.$$

Finally we shall need the simple inequality from (2.4)

$$A > \int_0^1 \left(\frac{\sin m\pi x}{\pi x}\right)^{2k} dx = \frac{1}{\pi} m^{2k-1} \int_0^{m\pi} \left(\frac{\sin t}{t}\right)^{2k} dt >$$

$$> \frac{m^{2k-1}}{\pi} \int_0^{k^{-\frac{1}{8}}} \left(\frac{\sin t}{t}\right)^{2k} dt > \frac{m^{2k-1}}{3\pi\sqrt{k}}$$

i.e.

(2.9) $$A > \frac{m^{2k-1}}{3\pi\sqrt{k}}$$

(and obviously $\geqq 2$).

3. Let

(3.1) $$T = e^{17\omega N \log^3 N}$$

and modifying an idea of H. Bohr and B. Jessen[3]) devised by them for the proof of Kronecker's theorem we consider the function

(3.2) $$K_N(t) = \prod_{j=1}^{N} P(t \log p_j - \beta_j)$$

with fixed β_j. If the lemma would be false then for all $d \leqq t \leqq d+T$ for a suitable index $\nu = \nu(t)$ we would have

$$|t \log p_\nu - \beta_\nu - e_\nu| > \frac{1}{\omega}$$

for all integer e_ν and thus, using also (2.3), (2.9) and (2.5),

(3.3) $$(0 \leqq) P(t \log p_\nu - \beta_\nu) < \frac{1}{A} \frac{1}{\sin^{2k}\frac{\pi}{\omega}} < \frac{3\pi\sqrt{k}}{m^{2k-1}} \cdot \frac{1}{\sin^{2k}\frac{\pi}{\omega}} <$$

$$< \frac{3\pi e^2 \omega \sqrt{\log N}}{\left(\frac{2m}{\omega}\right)^{2k}} < \frac{3\pi e^6 \omega \sqrt{\log N}}{N^4} < \frac{3\pi e^6 \sqrt{\log N}}{N^3} < \frac{1}{N^2},$$

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[3]) H. Bohr—B. Jessen, Zum Kroneckerschen Satz, *Rendiconti del Circolo Mat. Palermo*, **57** (1933), 123—129.

if c_6 is sufficiently large. Hence for this t we had

$$K_N(t) < \frac{1}{N^2} \prod_{\substack{j=1 \\ j \neq \nu}}^{N}{}' P(t \log p_j - \beta_j) \overset{\text{def}}{=} \frac{1}{N^2} K_{N\nu}(t)$$

and thus owing to the nonnegativity of $P(x)$

$$K_N(t) < \frac{1}{N^2} \sum_{\nu=1}^{N} K_{N\nu}(t)$$

would hold throughout $[0, T]$. Integrating we would obtain

$$J_N \overset{\text{def}}{=} \int_d^{d+T} K_N(t)\,dt < \frac{1}{N^2} \sum_{\nu=1}^{N} \int_d^{d+T} K_{N\nu}(t)\,dt \overset{\text{def}}{=} \frac{1}{N^2} \sum_{\nu=1}^{N} J_{N\nu}. \tag{3. 4}$$

4. In order to deduce a contradiction from (3. 4) we have to estimate J_N and the $J_{N\nu}$'s. To do it simultaneously let

$$q_1, q_2, \ldots, q_r \qquad 1 \leqq r \leqq N$$

be r different primes, $\gamma_1, \gamma_2, \ldots, \gamma_r$ real,

$$G_r(t) = \prod_{j=1}^{r} P(t \log q_j - \gamma_j) \tag{4. 1}$$

and

$$\int_d^{d+T} G_r(t)\,dt = H_r. \tag{4. 2}$$

Then (2. 6) gives owing to the rational independence of the $\log q_j$'s and (2. 7)

$$G_r(t) = 1 + \sum_{\substack{-(m-1)k \leqq \nu_j \leqq (m-1)k \\ j=1,2,\ldots,r}} a'_{\nu_1} a'_{\nu_2} \ldots a'_{\nu_r} e^{-2\pi i(\nu_1\gamma_1 + \cdots + \nu_r\gamma_r)} e^{2\pi i t \log\left(q_1^{\nu_1} q_2^{\nu_2} \cdots q_r^{\nu_r}\right)}$$

and thus with a ϑ, $-\frac{1}{\pi} \leqq \vartheta \leqq \frac{1}{\pi}$,

$$H_r = T + \vartheta \sum_{\substack{-(m-1)k \leqq \nu_j \leqq (m-1)k \\ j=1,2,\ldots,r}} \frac{a'_{\nu_1} a'_{\nu_2} \ldots a'_{\nu_r}}{\left|\log\left(q_1^{\nu_1} q_2^{\nu_2} \ldots q_r^{\nu_r}\right)\right|} \overset{\text{def}}{=} T + \vartheta Z. \tag{4. 3}$$

In order to obtain an upper bound for Z we consider first with an l, $1 \leqq l \leqq r$, the partial sum

$$Z_{j_1 j_2 \ldots j_l} \qquad (1 \leqq j_1 < \cdots < j_l \leqq r)$$

of Z, consisting of the terms where exactly the summation variables

$\nu_{j_1}, \nu_{j_2}, \ldots, \nu_{j_l}$ have values different from 0. Hence owing to (2.7) we have

$$(4.4)\qquad Z_{j_1 j_2 \ldots j_l} = \sum_{\substack{1 \leqq \varkappa_i \leqq (m-1)k \\ (i=1,\ldots,l)}} a'_{\varkappa_1} a'_{\varkappa_2} \ldots a'_{\varkappa_l} \sum_{\substack{\varepsilon_i = \pm 1 \\ (i=1,\ldots,l)}} \frac{1}{|\log (q_{j_1}^{\varepsilon_1 \varkappa_1} q_{j_2}^{\varepsilon_2 \varkappa_2} \ldots q_{j_l}^{\varepsilon_l \varkappa_l})|}.$$

Since for integer $a > b \geqq 1$

$$\left|\log \frac{a}{b}\right| = \left|\log \frac{b}{a}\right| = \log\left(1 + \frac{a-b}{b}\right) > \frac{1}{2b},$$

we get for the inner sum in (4.4) at once the upper bound

$$2(1+q_{j_1}^{\varkappa_1})(1+q_{j_2}^{\varkappa_2})\ldots(1+q_{j_l}^{\varkappa_l}).$$

Putting this into (4.4) we get the inequality

$$Z_{j_1 j_2 \ldots j_l} < 2\prod_{\mu=1}^{l}\left\{\sum_{1\leqq \varkappa_\mu \leqq (m-1)k} a'_{\varkappa_\mu}(1+q_{j_\mu}^{\varkappa_\mu})\right\} < 2\prod_{\mu=1}^{l}\left\{2\sum_{1\leqq \varkappa_\mu \leqq (m-1)k} a'_{\varkappa_\mu} q_{j_\mu}^{\varkappa_\mu}\right\}.$$

and thus

$$Z < 2\prod_{\mu=1}^{r}\left\{1 + 2\sum_{1\leqq \varkappa_\mu \leqq (m-1)k} a'_{\varkappa_\mu} q_\mu^{\varkappa_\mu}\right\} < 2\prod_{\mu=1}^{r} 2\left\{\sum_{-(m-1)k\leqq \varkappa_\mu \leqq (m-1)k} a'_{\varkappa_\mu} q_\mu^{\varkappa_\mu}\right\}.$$

Using the identity (2.8) this gives

$$Z < 2\left(\frac{2}{A}\right)^r \frac{1}{(q_1 q_2 \ldots q_r)^{(m-1)k}} \prod_{\mu=1}^{r}\left(\frac{q_\mu^m - 1}{q_\mu - 1}\right)^{2k}$$

and roughly

$$(4.5)\qquad Z < 2(q_1 q_2 \ldots q_r)^{(m-1)k} \prod_{\mu=1}^{r} \frac{1 - \frac{1}{q_\mu^m}}{1 - \frac{1}{q_\mu}} < 20\,(q_1 q_2 \ldots q_r)^{(m-1)k} e^{\sum_{\mu=1}^{r} q_\mu^{-1}}.$$

Applying it with

$$r = N, \quad (q_1, q_2, \ldots, q_r) = (p_1, p_2, \ldots, p_N), \quad \gamma_j = \beta_j$$

resp.

$$r = N-1, \quad (q_1, q_2, \ldots, q_r) = (p_1, p_2, \ldots, p_{\nu-1}, p_{\nu+1}, \ldots, p_N), \quad \gamma_j = \beta_i$$

$(\nu = 1, 2, \ldots, N)$ we get from (3.4), (4.3) and (4.5)

$$T - 7(p_1 p_2 \ldots p_N)^{(m-1)k} e^{\sum_{\nu=1}^{N} p_\nu^{-1}} < \frac{1}{N^2}\left\{NT + 7N(p_1 p_2 \ldots p_N)^{(m-1)k} e^{\sum_{\nu=1}^{N} p_\nu^{-1}}\right\}$$

i. e. roughly

$$\frac{1}{2} T < 11\,(p_1 p_2 \ldots p_N)^{(m-1)k} e^{\sum_{\nu=1}^{N} p_\nu^{-1}}.$$

But as well-known, choosing c_6 sufficiently large, it follows

$$\frac{1}{2} T < 11 \log^2 N \cdot e^{2N \log N \cdot (m-1)k} < 11 \log^2 N e^{2c^2 \omega N \log^2 N}.$$

Hence, if c_6 is large enough,

$$T < 22 \log^2 N \cdot e^{16 \omega N \log^2 N} < e^{17 \omega N \log^2 N},$$

in contradiction to (3. 1). Hence the lemma is proved.

5. Since the other parts of the proof are unchanged, as it is given in my above quoted paper, a sketch of it will suffice, for the sake of completeness. Let $\lambda(\nu)$ stand for LIOUVILLE's symbol, further for $n > c_7$

(5. 1)
$$G_n(s) = \sum_{\nu \leqq n} \lambda(\nu) \nu^{-s}$$

and

(5. 2)
$$\delta = \frac{\log^4 n}{\sqrt{n}};$$

we shall use the well-known estimation

(5. 3)
$$|B(x)| \overset{\text{def}}{=} |\sum_{\nu \leqq x} \lambda(\nu)| < x e^{-c_8 \sqrt{\log x}}.$$

With this c_8 (5. 3) gives easily that

(5. 4)
$$|G_n(s)| < c_9$$

in the domain

$$\sigma \geqq 1 - \frac{c_8}{3} \frac{1}{\sqrt{\log n}}, \quad |t| \leqq 1,$$

if only n sufficiently large. Supposing now that $G_n(s)$ has a real zero σ_0 between $(1 + 2\delta)$ and $\left(1 + 3 \frac{\log \log n}{\log n}\right)$ and putting

(5. 5)
$$G_n(s) = \sum_{l=1}^{\infty} d_l \, (s - \sigma_0)^l,$$

CAUCHY's coefficient-estimation, applied to the circle

$$|s - \sigma_0| \leqq \frac{c_8}{8 \sqrt{\log n}}$$

gives from (5. 4) the estimation

(5. 6)
$$|d_l| < (c_9 \log n)^{\frac{l}{2}}.$$

Further from

$$|d_1| = \left| \sum_{\nu \leqq n} \frac{\lambda(\nu) \log \nu}{\nu^{\sigma_0}} \right|,$$

from (5. 3) and simple properties of $\zeta(s)$ we get the lower bound

$$(5.7)\qquad |d_1| > \frac{1}{3}.$$

From (5. 5), (5. 6) and (5. 7) we get the estimation

$$(5.8)\qquad |G_n(s)| > \frac{\delta}{4}$$

on the circle $|s-\sigma_0| = \delta$. Application of the lemma with

$$N = \pi(n)\left(< 2\frac{n}{\log n}\right), \quad \beta_1 = \beta_2 = \cdots = \beta_{\pi(n)} = \frac{1}{2}, \quad \omega = \left[\frac{50 \log^2 n}{\delta}\right] + 1$$

gives to every real d the existence of a τ_d with

$$d \leqq \tau_d \leqq d + e^{17^{\frac{51\sqrt{n}}{\log^2 n}} \cdot 2^{\frac{n}{\log n}} \cdot \log^2 n} \leqq d + \left(e^{n^{\frac{3}{2}}} - 1\right)\frac{1}{2\pi}$$

(if c_7 is large enough) such that

$$\left|\tau_d \log p - \frac{1}{2} - e_p\right| < \frac{\delta}{50 \log^2 n} \qquad (e_p \ \text{ integer})$$

for all $p \leqq n$. From this one can deduce that if c_7 is large enough than for $n > c_7$ and $\sigma \geqq 1$ we have

$$(5.9)\qquad |G_n(s) - U_n(s + 2\pi i \tau_d)| < \frac{\pi}{25}\delta.$$

Then by an adaptation of a reasoning of BOHR one can deduce from our assumption (1. 3) that for $n > c_7$ the inequality

$$(5.10)\qquad \sum_{\nu \leqq n} \lambda(\nu)\nu^{-1-2\frac{\log^4 n}{\sqrt{n}}} \geqq 0$$

holds. Using (5. 3) this gives easily the inequality

$$(5.11)\qquad L(x) \overset{\text{def}}{=} \sum_{\nu \leqq x} \frac{\lambda(\nu)}{\nu} > -c_{10}\frac{\log^4 x}{\sqrt{x}} > -x^{-\frac{1}{2}+\varepsilon}$$

for $x > c_{11}(\varepsilon)$, ε arbitrarily small positive. Since for $\sigma > 1$ the identity

$$(5.12)\qquad \int_1^\infty \frac{L(x) + x^{-\frac{1}{2}+\varepsilon}}{x^s}\,dx = \frac{\zeta(2s)}{(s-1)\zeta(s)} + \frac{1}{s - \frac{1}{2} - \varepsilon}$$

holds, (5. 11) gives owing to a theorem of LANDAU that the "outstanding" singularity of the right hand side of (5. 12) is on the real axis. But this proves the theorem.

(Received April 26, 1960)