Remarks to the theory of semi-modular lattices

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Dedicated to Professor L. Rédei on his 60th birthday

1. By a theorem of G. BIRKHOFF ([1], p. 105) a semi-modular lattice of finite length is complemented if and only if its greatest element is the join of atoms. The „only if" part of this theorem does not depend on the semi-modularity: it holds obviously for all atomic lattices too. The „if" part may be also considerably generalized: such a generalization is given in our Theorem 1.

In the first half of the proof of Theorem 1 we make use of semi-complements which are represented as joins of atoms. This fact makes one interested in semi-complements of this type. Here, considering a semi-modular lattice L, an arbitrary element e and a countable set P of atoms of L, we give in Theorem 2 a condition which is sufficient for the join of each finite subset of P to be a semi-complement of e.

Finally, using this Theorem, we generalize Theorem 2 of our earlier paper [6].

2. We begin with some definitions; for the concepts not mentioned in this section, see [1].

By an *upper-directed set* S we mean a partly ordered set having the following property: given $a, b \in S$, there exists some $c \in S$ satisfying $c \geqq a$ and $c \geqq b$.

A lattice L is called *upper-continuous* if: (i) L is complete; (ii) for each upper-directed subset $\{s_\delta\}_{\delta \in \Delta}$ and for each element t of L,

$$\left(\bigvee_{\delta \in \Delta} s_\delta\right) \cap t = \bigvee_{\delta \in \Delta} (s_\delta \cap t).$$

A lattice L will be called *semi-modular* if it satisfies the following condition due to S. MACLANE and R. CROISOT[1]): if a and b are incomparable

1) For lattices of finite length this condition is equivalent to that of [1], p. 100; see [2], p. 99, Théorème 4.

elements of L and x any element of L such that $a \cap b < x < a$, then there exists an element t such that $a \cap b < t \leqq b$ and $(x \cup t) \cap a = x$.

Further, if a lattice has a least or a greatest element, then we shall it denote by o and i, respectively.

Finally, for the definition of semi-complements, proper semi-complements and semi-complemented lattices, see [5].

3. In this section we shall make use of two lemmas.

Lemma 1. *Let L be an upper-continuous lattice, $\{p_\gamma\}_{\gamma \in \Gamma}$ a set of atoms and e an arbitrary element of L. If $e \cap \bigvee_{\gamma \in \Gamma_0} p_\gamma = o$ for each finite subset Γ_0 of Γ, then $e \cap \bigvee_{\gamma \in \Gamma} p_\gamma = o$ too.*

This is a corollary of Hilfssatz 1.7 of Chapter I of [3].

Lemma 2. *Let $p_1, \ldots, p_n$ be arbitrary atoms of a semi-modular lattice with least element o. Then the length of the interval $[o, p_1 \cup \cdots \cup p_n]$ is at most n.*

Proof. By Propriété 2 on page 90 of [2], all distinct elements of the set

$$o, p_1, p_1 \cup p_2, \ldots, p_1 \cup \cdots \cup p_n$$

form a maximal chain C between o and $p_1 \cup \cdots \cup p_n$. The length of C is at most n. Thus, Lemma 2 is implied by the Corollary of Theorem 1 in [4].

Now we prove

Theorem 1. *Let L be an upper-continuous semi-modular lattice with greatest element i. If there exists a set $\{p_\gamma\}_{\gamma \in \Gamma}$ of atoms in L such that $\bigvee_{\gamma \in \Gamma} p_\gamma = i$, then L is complemented and atomic*[2]).

Proof. In the first half of the proof we follow the way given in [3], pp. 78—79. Nevertheless, for the sake of completeness we give a full discussion.

Let a denote an arbitrary element of L different from o and i. Consider the family $\mathfrak{M}$ of all index sets $\Delta (\subseteq \Gamma)$ having the property

$$a \cap \bigvee_{\delta \in \Delta} p_\delta = o.$$

Then, firstly, $\mathfrak{M}$ is non-empty, because — by $a \neq i$ — $a \cap p_\gamma = o$ for some $\gamma \in \Gamma$. Next, let $\mathfrak{N}$ be a (non-empty) subchain of $\mathfrak{M}$ and $\Delta_{\mathfrak{N}}$ denote the (set-theoretical) union of all sets Δ belonging to $\mathfrak{N}$. We show $\Delta_{\mathfrak{N}} \in \mathfrak{M}$. Indeed, consider an arbitrary finite subset Λ of $\Delta_{\mathfrak{N}}$. Then, by the definition

[2]) For upper-continuous modular lattices, see [3], Chapter III, Satz 2.1.

of $\varDelta_{\mathfrak{N}}$, there exists a set $\varDelta'$ in $\mathfrak{N}$ which includes $\varDelta$. Consequently,

$$a \frown \bigvee_{\delta \in \varDelta} p_\delta \leqq a \frown \bigvee_{\delta \in \varDelta'} p_\delta = o.$$

Hence, by Lemma 1,

$$a \frown \bigvee_{\delta \in \varDelta_{\mathfrak{N}}} p_\delta = o;$$

in other words, $\varDelta_{\mathfrak{N}} \in \mathfrak{N}$.

By the preceding paragraph, the Zorn Lemma may be applied for $\mathfrak{N}$. It follows that there exists a maximal subset $\varDelta^*$ of Γ, i. e. a maximal set $\varDelta^*$ with the property

$$a \frown \bigvee_{\delta \in \varDelta^*} p_\delta = o. \tag{1}$$

We show that just the element

$$b = \bigvee_{\delta \in \varDelta^*} p_\delta \tag{2}$$

is a complement of a; by our assumption on $\{p_\gamma\}_{\gamma \in \Gamma}$, it suffice to prove that $a \smile b \geqq p_\gamma$ for all $\gamma \in \Gamma$. Clearly, $b \neq o$.

Suppose $p = p_{\gamma_0} \nleqq a \smile b$ for some $\gamma_0 \in \Gamma$. Then $a \smile b$ and p are incomparable elements and

$$(a \smile b) \frown p = o. \tag{3}$$

On the other hand, $a \smile b \neq b$, because $a \smile b = b$ would imply, by (2) and (1), $o = a \frown b = a$, in contradiction to our assumption $a \neq o$. Hence

$$(a \smile b) \frown p < b < a \smile b.$$

It follows, by the semi-modularity of L, that there exists an element t such that

$$(a \smile b) \frown p < t \leqq p \tag{4}$$

and

$$(b \smile t) \frown (a \smile b) = b. \tag{5}$$

But, by (3) the inequalities (4) have no solution other than $t = p$. Therefore (5) implies

$$(b \smile p) \frown (a \smile b) = b.$$

From this equation we get, with respect to (2) and (1),

$$o = a \frown b = a \frown (a \smile b) \frown (b \smile p) = a \frown (b \smile p) =$$
$$= a \frown \Big(\bigvee_{\delta \in \varDelta^*} p_\delta \smile p\Big) = a \frown \Big(\bigvee_{\delta \in \varDelta^*} p_\delta \smile p_{\gamma_0}\Big),$$

which contradicts the maximality of $\varDelta^*$. Thus the first assertion of our theorem is proved.

Now we prove the second assertion. By definition,

$$a \cap \bigvee_{\gamma \in \Gamma} p_\gamma = a \cap i = a \neq o.$$

Therefore, by Lemma 1,

$$a \cap \bigvee_{\gamma \in \Gamma_0} p_\gamma \neq o$$

for some finite subset Γ_0 of Γ. Denote $c = \bigvee_{\gamma \in \Gamma_0} p_\gamma$. Then, by Lemma 2, the interval $[o, c]$ is of finite length. Consequently, there exists an atom q in $[o, c]$ such that $q \leqq a \cap c \leqq a$. This completes the proof of Theorem 1.

4. Now we deal with our above-mentioned theorems concerning semi-complements in semi-modular lattices.

Theorem 2. *Let e be an arbitrary element of a semi-modular lattice L with least element o. If the (finite or infinite) sequence $P = \{p_1, p_2, \dots\}$ satisfies the condition*

$$(6) \qquad (e \cup p_1 \cup \cdots \cup p_{k-1}) \cap p_k = o \qquad (k = 1, 2, \dots),$$

then the join of every finite subset of P is a semi-complement of e.

Proof. Denote

$$(7) \qquad e \cap (p_1 \cup \cdots \cup p_r) = d_r \qquad (r = 1, 2, \dots).$$

Clearly, it suffices to show $d_r = o$ for each r.

Let us consider a d_r with fixed r. Since by (7) $e \geqq d_r$, we have

$$(e \cup p_1 \cup \cdots \cup p_{k-1}) \cap p_k \geqq (d_r \cup p_1 \cup \cdots \cup p_{k-1}) \cap p_k$$

for $k = 1, \dots, r$. Hence, by our assumption (6),

$$p_k \nleqq d_r \cup p_1 \cup \cdots \cup p_{k-1} \qquad (k = 1, \dots, r).$$

This implies, together with (7),

$$(8) \qquad d_r < d_r \cup p_1 < \cdots < d_r \cup p_1 \cup \cdots \cup p_r = p_1 \cup \cdots \cup p_r.$$

On the other hand, by Lemma 2, the length of the interval $[o, p_1 \cup \cdots \cup p_r]$ does not exceed r. Thus from (8) we conclude $d_r = o$.

As a corollary of Theorem 2 we get

Theorem 3. *Let L be a semi-complemented semi-modular atomic lattice. If an element $e (\neq o)$ of L has no complement, then to each non-negative integer ν there exists a semi-complement of e whose height is equal to ν.*

Proof. Being L semi-complemented, e has a proper semi-complement x_1. Since L is atomic too, there exists an atom p_1 in L such that $p_1 \leqq x_1$. Then $e \cap p_1 \leqq e \cap x_1 = o$. If e has no complement, then $e \cup p_1$ ($\neq i$ and thus it) has also a proper semi-complement x_2. Again, there exists an atom p_2 such that $p_2 \leqq x_2$ and $(e \cup p_1) \cap p_2 = o$; and so on. Applying now Theorem 2, we get Theorem 3.

References

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