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Symmetric periodic solutions for a class of differential delay equations with distributed delay

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Abstract. We consider the nonlinear distributed delay equation

$$x'(t) = f\left[\int_{t-1}^{t-d} g(x(s))\,ds\right], \quad d \in [0,1),$$

where g and f are smooth, bounded, and odd and satisfy a positive and a negative feedback condition, respectively. Using elementary fixed point theory we prove the existence of a nontrivial periodic solution of period $2+2d$ satisfying certain symmetries, given certain growth conditions on f and g near zero.

Keywords: differential delay equations, distributed delay, periodic solutions.

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1 Introduction

We consider the following autonomous nonlinear real-valued differential equation with distributed delay:

$$x'(t) = f\left[\int_{t-1}^{t-d} g(x(s))\,ds\right], \quad d \in [0,1). \tag{1.1}$$

We shall impose the following hypotheses on f and g:

$$(H)\begin{cases} g(0) = f(0) = 0; \\ g \text{ and } f \text{ are bounded and } C^1\text{, with bounded derivative;} \\ g \text{ and } f \text{ are odd;} \\ xg(x) > 0 \text{ for all } x \neq 0 \text{ (positive feedback), and } xf(x) < 0 \text{ for all } x \neq 0 \text{ (negative} \\ \text{feedback).} \end{cases}$$

In this paper we describe conditions on f and g that guarantee the existence of a certain nontrivial periodic solution of (1.1). This solution has period $2+2d$. Throughout this paper we shall write $m = (1+d)/2$, and accordingly shall frequently write $4m$ for the period of our periodic solution.

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We define the following additional translation and symmetry conditions, where x is a function whose domain includes $[0, 4m]$:

$(T1)$ $x(0) = 0$;

$(T2)$ x is nondecreasing on $[0, m]$;

$(S1)$ $x(2m + t) = -x(t)$ for all $t \in [0, 2m]$;

$(S2)$ $x(2m + t) = -x(2m - t)$ for all $t \in [0, 2m]$.

The symmetry conditions $(S1)$, $(S2)$ above are convenient for our purposes. As we show in the next section, though (Lemma 2.3), any $4m$-periodic function $x \colon \mathbb{R} \to \mathbb{R}$ satisfying $(S1)$ and $(S2)$ also satisfies the somewhat more conventional-looking symmetries

$$(S)\ \ x(t + 2m) = -x(t) \ \text{ and } \ x(-t) = -x(t) \ \text{ for all } t \in \mathbb{R}.$$

We now state our main theorem.

Theorem 1.1. *Let $d \in [0, 1)$ be given, and assume that hypotheses (H) hold. Suppose that*

$$2|f'(0)||g'(0)| \left(\frac{2m}{\pi}\right)^2 \cos\left(\frac{\pi}{2m} d\right) > 1.$$

Then (1.1) has a periodic solution of period $2 + 2d = 4m$ that satisfies $(T1)$, $(T2)$, $(S1)$, and $(S2)$.

In Section 2 we lay the groundwork for the proof of Theorem 1.1; we prove the theorem in Section 3.

The lineariztion of (1.1) at 0 is

$$x'(t) = -\gamma \int_{t-1}^{t-d} x(s)\, ds,$$

where $\gamma = |f'(0)||g'(0)|$. The corresponding characteristic equation is

$$\lambda = -\gamma \int_{-1}^{-d} e^{\lambda s}\, ds. \tag{1.2}$$

In [6] it is proven that, when $d \in [1/2, 1)$, Equation (1.2) has roots with positive real part if and only if

$$2\gamma \left(\frac{2m}{\pi}\right)^2 \cos\left(\frac{\pi}{2m} d\right) > 1.$$

(The proof given in [6] seems to extend to the $d \in [0, 1)$ case; we do not supply the details here, since we do not actually need this result in what follows.) Accordingly, the hypotheses of Theorem 1.1 being satisfied suggests instability of the equilibrium solution of (1.1) at 0, and makes it reasonable to suspect that there is a nontrivial periodic solution. Indeed, Theorem 2 of [6] proves the existence of a nontrival periodic solution of (1.1) when the hypotheses of Theorem 1.1 are satisfied, in the case that f is linear and $d \in [1/2, 1)$. (The map g is not assumed odd in [6], and the period of the solution, while bounded below, is not given explictly. In the current work, the assumption that f and g are both odd is crucial to our ability to obtain an explicit expression for the period.)

Also related to the current work is [1], where the authors study solutions of the system

$$x_1'(t) = \int_{-1}^{0} f_1(x_2(t+\theta))\, d\theta$$
$$x_2'(t) = \int_{-1}^{0} f_2(x_1(t+\theta))\, d\theta$$

that satisfy certain symmetries and obtain, as a special case, criteria on g for the existence of a periodic solution of (1.1) in the case that $f = -id$ and $d = 0$.

Beyond proving Theorem 1.1, a chief motivation for this work is to suggest the possibility of strong parallels between the dynamics of (1.1) and those of the much better-known delay equation

$$x'(t) = f(x(t-1)). \tag{1.3}$$

Numerical simulations of (1.1) suggest that, in several instances, periodic solutions of the type described in Theorem 1.1 attract large sets of solutions, including solutions whose initial conditions have several zeros per unit time; the analogous phenomenon for Equation (1.3) is well-known. Accordingly, it seems that the periodic solution of Equation (1.1) that we investigate in this paper should be regarded as "slowly oscillating". We emphasize, though, that we have not yet formulated a satisfactory non-increasing "oscillation speed" for (1.1); such a formulation would be useful for developing connections between Equations (1.1) and (1.3).

When $d \in [1/2, 1)$, it is possible to identify a particular subset X of initial conditions whose continuations can reasonably be called "slowly oscillating"; Theorem 2 of [6], mentioned above, employs the approach of finding a fixed point for an appropriately defined Poincaré map on X. When $d \in [0, 1/2)$, though, even the definition of a forward-invariant set of "slowly oscillating" solutions for Equation (1.1) does not seem obvious. Accordingly, the approach we take to proving Theorem 1.1 is somewhat different from the Poincaré map approach.

To see one connection between Equations (1.1) and (1.3) we recall the paper [5], where it is proven that, for f smooth, bounded, and odd with negative feedback, if $|f'(0)| > \pi/2$ then Equation (1.3) has a nontrivial periodic solution p of period 4 satisfying $p(0) = 0$ and the symmetries

$$p(t) = -p(t+2) \text{ and } p(-t) = -p(t),$$

which are just the symmetries (S) with $d = 1$. Furthermore, if we write

$$c = \frac{\pi^2}{8m^2 \cos(\pi d/(2m))}$$

for the threshold value of $|f'(0)||g'(0)|$ in Theorem 1.1, we have

$$\lim_{d \to 1} (1-d)c = \frac{\pi}{2}$$

(this observation was also made in [6]). The above-mentioned result from [5] can therefore be viewed, heuristically, as a limiting version of Theorem 1.1 as $d \to 1$.

To draw another connection between Equations (1.1) and (1.3), we can consider the following "model" version of (1.1) with step feedback,

$$x'(t) = -\operatorname{sign}\left[\int_{t-1}^{t-d} \operatorname{sign}(x(s))\, ds\right], \tag{1.4}$$

and compare it to the equation

$$x'(t) = -\operatorname{sign}(x(t-1)), \tag{1.5}$$

a model version of (1.3) that is essentially completely understood (see, e.g., [3]). In particular, it is known that Equation (1.5) has a countable set of periodic solutions; that all of these solutions are unstable except a single "slowly oscillating" one; and that this slowly oscillating periodic solution attracts most other solutions in a suitable sense. It turns out that Equation (1.4) has an analogous countable set of periodic solutions, all but one of which are unstable. We do not provide the details here, but content ourselves with exhibiting, in Section 4, the single stable periodic solution of Equation (1.4), and describing some of its domain of attraction. This stable periodic solution is a counterpart of the solution of Equation (1.1) described in Theorem 1.1, and our work in Section 4 will also yield some heuristic insight into this latter solution — in particular, into its apparent stability.

2 The map $F\colon \Omega \to \Omega$

Throughout this section, we shall assume that the hypotheses (H) hold, and we shall continue to write $m = (1+d)/2$. Note that $d < m < 1$.

We begin by collecting some simple consequences of the symmetries introduced before the statement of Theorem 1.1.

Lemma 2.1. *If $x\colon [0,4m] \to \mathbb{R}$ satisfies $(S1)$ and $(S2)$, then x also satisfies*

$(S3)$ $x(m+t) = x(m-t)$ *for all* $t \in [0,m]$*;*

$(S4)$ $x(3m+t) = x(3m-t)$ *for all* $t \in [0,m]$*;*

$(S5)$ $x(4m-t) = -x(t)$ *for all* $t \in [0,4m]$*.*

Proof. If $x\colon [0,4m] \to \mathbb{R}$ satisfies $(S1)$ and $(S2)$, then for $t \in [0,m]$ we have

$$x(m+t) = x(2m-(m-t)) \overset{(S2)}{=} -x(2m+m-t) \overset{(S1)}{=} x(m-t).$$

The proof that $x(3m+t) = x(3m-t)$ is similar.

If $t \in [0,2m]$ we have

$$x(4m-t) = x(2m+(2m-t)) \overset{(S2)}{=} -x(2m-(2m-t)) = -x(t);$$

if $t \in [2m,4m]$ we have

$$x(4m-t) = x(2m-(t-2m)) \overset{(S2)}{=} -x(2m+(t-2m)) = -x(t).$$

This completes the proof. □

We define the following function $\tau\colon \mathbb{R} \to [0,4m)$:

$$\tau(t) = t \bmod[4m] = t - 4m \cdot \operatorname{floor}(t/(4m)).$$

(In what follows, we are going to study a particular subset Ω of $C([0,4m],\mathbb{R})$ whose elements extend to continuous $4m$-periodic functions on $\mathbb{R}$. The function τ is, loosely speaking, a device to enable us to work in Ω while viewing its elements as $4m$-periodic functions.) We shall need the following observations about how the symmetries $(S1)$ and $(S2)$ interact with the function τ.

Lemma 2.2. *Suppose that* $x\colon [0, 4m] \to \mathbb{R}$ *satisfies* $(S1)$ *and* $(S2)$*. Then given any* $t \in \mathbb{R}$ *we have*

$$x(\tau(t+2m)) = -x(\tau(t)).$$

If $t \in \mathbb{R}$ *and* k *and* j *are integers such that* $2k \equiv 2j$ *modulo 4, then*

$$x(\tau(2km + t)) = -x(\tau(2jm - t)).$$

Proof. If $\tau(t) < 2m$ then $\tau(t+2m) = \tau(t) + 2m$ and the first equality follows immediately from $(S1)$.

If $\tau(t) \geq 2m$, then $\tau(t+2m) = \tau(t) + 2m - 4m = \tau(t) - 2m$, and the first equality follows again from $(S1)$.

Suppose that $2k$ and $2j$ are congruent modulo 4. Then, given any t, $2km - t$ and $2jm + t$ can be written, respectively, as $2k'm - t'$ and $2j'm + t'$, where $2k'$ and $2j'$ are congruent modulo 4 and $t' \in [0, 2m]$. If $2j'$ is congruent to 0 modulo 4 we have

$$\tau(2jm - t) = \tau(2j'm - t') = 4m - t' = 4m - \tau(2k'm + t') = 4m - \tau(2km + t);$$

and if $2j'$ is congruent to 2 modulo 4 we have

$$\tau(2jm - t) = \tau(2j'm - t') = 2m - t' = 4m - (2m + t') = 4m - \tau(2k'm + t') = 4m - \tau(2km + t).$$

The second part of the lemma now follows from symmetry $(S5)$ (recall Lemma 2.1). □

The following very simple lemma shows that $(S1)$ and $(S2)$ imply the symmetries (S). A typical $4m$-periodic function satisfying $(T1)$, $(T2)$, $(S1)$, and $(S2)$ is pictured in Figure 2.1.

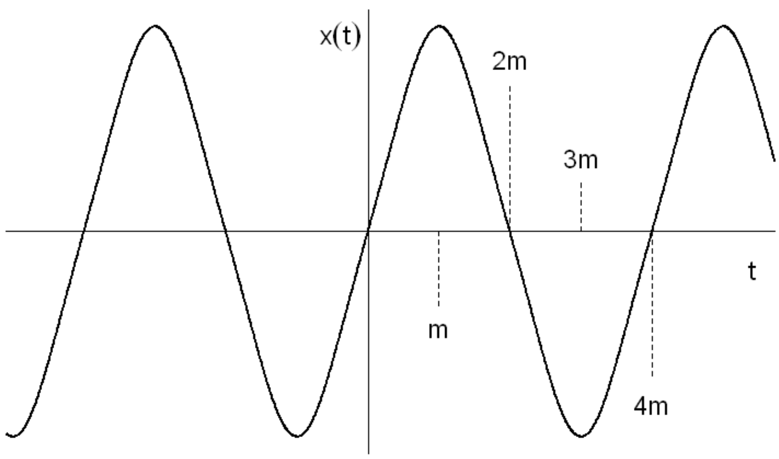

Figure 2.1: A typical $4m$-periodic function satisfying $(T1)$, $(T2)$, $(S1)$, and $(S2)$.

Lemma 2.3. *Suppose that* $x\colon \mathbb{R} \to \mathbb{R}$ *is a* $4m$*-periodic function satisfying* $(S1)$ *and* $(S2)$*. Then* x *also satisfies*

$$(S)\quad x(t+2m) = -x(t) \ \textit{for all } t \in \mathbb{R} \ \textit{ and } \ x(-t) = -x(t) \ \textit{for all } t \in \mathbb{R}.$$

Proof. Since x is assumed to be periodic with period $4m$,

$$x(s) = x(\tau(s)) \text{ for all } s \in \mathbb{R}.$$

Therefore, applying Lemma 2.2, for all $t \in \mathbb{R}$ we have

$$x(t+2m) = x(\tau(t+2m)) = -x(\tau(t)) = -x(t).$$

Any $s \geq 0$ can be written in the form $s = 2km + t$, where $k \in \mathbb{Z}_+$ and $t \in [0, 2m)$. In this case $-s = -2km - t$. Since $2k$ is congruent to $-2k$ modulo 4, Lemma 2.2 implies

$$x(-s) = x(\tau(-2km - t)) = -x(\tau(2km + t)) = -x(s).$$

□

We write $C = C([-1,0], \mathbb{R})$ for the space of continuous real-valued functions on $[-1,0]$, equipped with the sup norm. Under hypotheses (H), the functions f and g have bounded derivatives and hence are both Lipschitz; we write ℓ_f and ℓ_g for their respective Lipshitz constants. The map

$$C \ni x \mapsto f\left[\int_{-1}^{-d} g(x(s))\,ds\right]$$

has Lipshitz constant $\ell_f \ell_g (1-d)$. The existence and uniqueness of solutions of (1.1) therefore follows from standard theory for delay equations (see, for example, [4]). In particular, given any $x_0 \in C([-1,0], \mathbb{R})$, x_0 has a unique continuation $x\colon [-1,\infty) \to \mathbb{R}$ that satisfies (1.1) for all $t > 0$.

Let us suppose that a solution $x\colon J \to \mathbb{R}$ of (1.1) is given, where $J = [-1,\infty)$ or $J = \mathbb{R}$. For any $t - 1 \in J$, define

$$y(t) = \int_{t-1}^{t-d} g(x(s))\,ds.$$

The basic observation is that x and y together solve the system

$$x'(t) = f(y(t)); \;\; y'(t) = g(x(t-d)) - g(x(t-1)), \; t-1 \in J.$$

Now suppose that $x\colon \mathbb{R} \to \mathbb{R}$ is in fact a $4m$-periodic solution of (1.1) satisfying the symmetries (S). Then y is defined for all time, and since (using (S) and the oddness of g)

$$-g(x(t-1)) = g(-x(t-1)) = g(x(t-1+2m)) = g(x([t-1]+[1+d])) = g(x(t+d)),$$

we actually have that x and y solve the system

$$x'(t) = f(y(t)), \quad y'(t) = g(x(t-d)) + g(x(t+d)). \tag{2.1}$$

This observation motivates the construction that we now undertake.

Let us write $C[0,4m]$ for the Banach space of continuous real-valued functions on $[0,4m]$, equipped with the sup norm, and $C_k[0,4m]$ for the subset of $C[0,4m]$ consisting of functions with Lipschitz constant at most k. We now define the following subset of $C_{\|f\|}[0,4m]$, where $\|f\| = \sup_{s\in\mathbb{R}} |f(s)|$:

$$\Omega = \left\{ x \in C_{\|f\|}[0,4m] \; : \; \left\{ \begin{array}{l} (T1)\; x(0) = 0; \\ (T2)\; x \text{ is nondecreasing on } [0,m]; \\ (S1)\; x(2m+t) = -x(t) \text{ for all } t \in [0,2m]; \\ (S2)\; x(2m+t) = -x(2m-t) \text{ for all } t \in [0,2m]. \end{array} \right. \right\}$$

Observe that Ω is closed and convex and, by the Ascoli–Arzelà theorem, compact. Since $(T1)$ and $(S1)$ together imply that $x(0) = x(4m)$, the functions $x(\tau(t-d))$ and $x(\tau(t+d))$ are continuous on $[0,4m]$. Thus if $x \in \Omega$, then $\xi(t) = x(\tau(t))$ is a continuous $4m$-periodic function; by Lemma 2.3, ξ satisfies the symmetries (S).

We now define the map $G\colon \Omega \to C[0,4m]$ as follows: $y = G(x)$ satisfies $y(m) = 0$ and

$$y'(t) = g(x(\tau(t-d))) + g(x(\tau(t+d)))$$

for all $t \in (0,4m)$. (Since $x(\tau(t-d))$ and $x(\tau(t+d))$ are continuous on $[0,4m]$, y is continuously differentiable on $(0,4m)$ and continuous on $[0,4m]$.) We also define the map $H\colon C[0,4m] \to C[0,4m]$ as follows: $u = H(y)$ satisfies $u(0) = 0$ and $u'(t) = f(y(t))$ for all $t \in (0,4m)$. Finally, we define the map $F\colon \Omega \to C[0,4m]$ by $F = H \circ G$.

We spend the rest of this section establishing some facts about the map F: in particular, that F is a continuous self-mapping of Ω and that nonzero fixed points of F correspond to nontrivial $4m$-periodic solutions of (1.1) that satisfy conditions $(T1)$, $(T2)$, $(S1)$, and $(S2)$.

Lemma 2.4. *The functions G, H, and F are all Lipschitz continuous.*

Proof. Given x_1 and x_2 in Ω, let us write $y_1 = G(x_1)$, $y_2 = G(x_2)$, $u_1 = H(y_1)$, and $u_2 = H(y_2)$. As above we take ℓ_f and ℓ_g to be Lipschitz constants for f and g, respectively. Then (crudely) for any $t \in [0,4m]$ we have

$$|y_1(t) - y_2(t)| \leq 2\ell_g(4m)\|x_1 - x_2\|$$

and

$$|u_1(t) - u_2(t)| \leq \ell_f(4m)\|y_1 - y_2\| \leq \ell_f\ell_g 32m^2\|x_1 - x_2\|.$$

Thus we see that G, H, and F are all Lipschitz continuous. □

Proposition 2.5. *F maps Ω to itself.*

Proof. Let $x \in \Omega$ be given, and write $y = G(x)$ and $u = H(y) = F(x)$. $u(0) = 0$ by assumption, and u clearly has Lipschitz constant $\|f\|$. It remains to show that u satisfies $(T2)$, $(S1)$, and $(S2)$. We proceed in several steps.

Step 1: y' satisfies the symmetries $(S1)$ and $(S2)$ (in what follows, we take the derivatives of y at 0 and $4m$ to be appropriately one-sided). Using that g is odd and applying the first part of Lemma 2.2, for all $t \in [0,2m]$ we have

$$\begin{aligned} y'(2m+t) &= g(x(\tau(2m+t-d))) + g(x(\tau(2m+t+d))) \\ &= g(-x(\tau(t-d))) + g(-x(\tau(t+d))) \\ &= -\left[g(x(\tau(t-d))) + g(x(\tau(t+d)))\right] = -y'(t). \end{aligned}$$

Similarly, for all $t \in [0,2m]$ we have (applying the second part of Lemma 2.2) that

$$\begin{aligned} y'(2m+t) &= g(x(\tau(2m+t-d))) + g(x(\tau(2m+t+d))) \\ &= g(-x(\tau(2m-t+d))) + g(-x(\tau(2m-t-d))) \\ &= -\left[g(x(\tau(2m-t+d))) + g(x(\tau(2m-t-d)))\right] = -y'(2m-t). \end{aligned}$$

Thus $y'(t)$ satisfies $(S1)$ and $(S2)$. Lemma 2.1 now shows that, for $t \in [0,m]$,

$$y'(m+t) = y'(m-t) \quad \text{and} \quad y'(3m+t) = y'(3m-t).$$

Observe that, since x satisfies $(S2)$, $d < m$, and g is odd, we have $y'(2m) = 0$. Since y' satisfies $(S1)$, we in fact have $y'(0) = y'(2m) = y'(4m) = 0$.

Step 2: y satisfies the following symmetries:

i) $y(2m+t) = y(2m-t)$ for all $t \in [0, 2m]$;

ii) $y(m+t) = -y(m-t)$ and $y(3m+t) = -y(3m-t)$ for all $t \in [0, m]$.

For $t \in [0, 2m]$ we have (applying the symmetries of $y'(t)$ and the assumption that $y(m) = 0$)

$$\begin{aligned} y(2m+t) &= \int_m^{2m+t} y'(s)\, ds \\ &= \int_m^{2m-t} y'(s)\, ds + \int_{2m-t}^{2m} y'(s)\, ds + \int_{2m}^{2m+t} y'(s)\, ds \\ &= \int_m^{2m-t} y'(s)\, ds = y(2m-t). \end{aligned}$$

This proves i). In particular, we see that $y(0) = y(4m)$ and that $y(m) = y(3m) = 0$.

To prove the first part of ii) see that, for $t \in [0, m]$,

$$\begin{aligned} y(m+t) &= \int_m^{m+t} y'(s)\, ds = \int_0^t y'(m+s)\, ds \\ &= \int_0^t y'(m-s)\, ds = -\int_m^{m-t} y'(s)\, ds = -y(m-t). \end{aligned}$$

The proof that $y(3m+t) = -y(3m-t)$ for all $t \in [0, m]$ is similar.

Step 3: $y(m+t) \geq 0$ for all $t \in (0, m]$. To see this, we write

$$\begin{aligned} y(m+t) &= \int_0^t y'(m+s)\, ds \\ &= \int_0^t g(x(\tau(m+s+d)))\, ds + \int_0^t g(x(\tau(m+s-d)))\, ds. \end{aligned}$$

We break the left-hand integral just above into three pieces and the right-hand integral into two pieces to get the following expression:

$$\begin{aligned} & y(m+t) = \\ & \int_0^{m-d} g(x(\tau(m+s+d)))\, ds + \int_{m-d}^{2m-2d} g(x(\tau(m+s+d)))\, ds + \int_{2m-2d}^{t} g(x(\tau(m+s+d)))\, ds \\ & + \int_0^{2d-2m+t} g(x(\tau(m+s-d)))\, ds + \int_{2d-2m+t}^{t} g(x(\tau(m+s-d)))\, ds. \end{aligned}$$

Regrouping (the first two integrals on the first line together, the last integral on the first line and the first integral on the second line together, and the second integral on the second line by itself) and rewriting the limits of integration we get

$$\begin{aligned} y(m+t) = & \int_{m+d}^{2m} g(x(\tau(s)))\, ds + \int_{2m}^{3m-d} g(x(\tau(s)))\, ds \\ & + \int_{3m-d}^{t+d+m} g(x(\tau(s)))\, ds + \int_{m-d}^{t+d-m} g(x(\tau(s)))\, ds \\ & + \int_{t+d-m}^{t+m-d} g(x(\tau(s)))\, ds. \end{aligned}$$

The two integrals in the first line above sum to zero since x satisfies $(S2)$ (note that $2m - (m + d) = m - d = (3m-d) - 2m$), and the two integrals in the second line sum to zero as well by the first part of Lemma 2.2. The last integral is nonnegative: if $t + d - m \geq 0$ this is immediate (since $(t+d-m, t+m-d) \subset (0, 2m)$, and x is nonnegative on the latter interval — keep in mind that we are assuming that $t \in (0, m]$); if $t+d-m < 0$ we have

$$\int_{t+d-m}^{t+m-d} g(x(\tau(s)))\, ds = \int_{t+d-m}^{m-t-d} g(x(\tau(s)))\, ds + \int_{m-t-d}^{m+t-d} g(x(\tau(s)))\, ds,$$

and the first integral on the right is zero by the symmetry of x and the oddness of g, while the second integral on the right is nonnegative since x is nonnegative on $[0, 2m]$. This completes Step 3.

Step 4: The sign of u'. Applying the already-established symmetries of y, from Step 3 we now obtain that y is nonpositive $[0, m]$, nonnegative on $[m, 3m]$, and nonpositive on $[3m, 4m]$. Since $yf(y) < 0$ for all nonzero y, we see that u is

- nondecreasing on $[0, m]$;
- nonincreasing on $[m, 3m]$;
- nondecreasing on $[3m, 4m]$.

In particular, we see that u satisfies $(T2)$.

Step 5: u satisfies $(S1)$ and $(S2)$. For $t \in [0, 2m]$ we have (using the symmetries of y and the oddness of f):

$$\begin{aligned}
u(2m+t) &= \int_0^{2m+t} f(y(s))\, ds \\
&= \int_0^{m} f(y(s))\, ds + \int_m^{2m} f(y(s))\, ds + \int_{2m}^{2m+t} f(y(s))\, ds \\
&= \int_{2m}^{2m+t} f(y(s))\, ds = \int_{2m-t}^{2m} f(y(s))\, ds = \int_{m-t}^{m} f(y(m+s))\, ds \\
&= -\int_{m-t}^{m} f(y(m-s))\, ds = \int_t^0 f(y(s))\, ds = -\int_0^t f(y(s))\, ds = -u(t).
\end{aligned}$$

Similarly, for $t \in [0, 2m]$ (using that $u(2m) = 0$, which follows from the just-established $(S1)$) we have

$$\begin{aligned}
u(2m+t) &= \int_0^{2m+t} f(y(s))\, ds \\
&= \int_0^{2m} f(y(s))\, ds + \int_{2m}^{2m+t} f(y(s))\, ds \\
&= \int_0^{2m} f(y(s))\, ds + \int_{2m-t}^{2m} f(y(s))\, ds \\
&= u(2m) + (u(2m) - u(2m-t)) = -u(2m-t).
\end{aligned}$$

This shows that u has the desired symmetries, and completes the proof of the proposition. □

It is clear that $F(0) = 0$. Our computations in Step 3 of the above proof showed that

$$y(2m) = y(m+m) = \int_d^{2m-d} g(x(\tau(s)))\, ds.$$

If $x \in \Omega \setminus \{0\}$, then the integral above is strictly positive and so $y(2m) > 0$; it follows that $y(0) < 0$ and that $u'(t) > 0$ for t near 0. We conclude the following.

Corollary 2.6. *Given $x \in \Omega \setminus \{0\}$, $F(x)$ is strictly positive on $(0, m]$ (and so in particular is nonzero).*

In what follows we shall need the following more detailed information on the shape of $F(x)$ when $\|x\|$ is small.

Lemma 2.7. *Suppose that there is a number $c > 0$ such that both f and g are monotonic on the interval $[-c, c]$. Then there is a constant $\kappa > 0$, depending on f, g, and d, such that if $x \in \Omega$ and $\|x\| \leq \kappa$, then $u = F(x)$ is concave down on $[0, m]$ — that is, $u'(t)$ is nonincreasing on $(0, m)$.*

Note that, since f and g are assumed to be C^1 with negative and positive feedback, respectively, the conditions of the above lemma are satisfied if $f'(0)$ and $g'(0)$ are both nonzero (and so in particular if the hypotheses of Theorem 1.1 are satisfied).

Proof. Given $x \in \Omega$, write $y = G(x)$ and $u = F(x)$. Since G is Lipschitz, there is a $\kappa > 0$ such that $\|x\| \leq \kappa$ implies both $\|x\| \leq c$ and $\|y\| \leq c$. When considering the action of F on such x, we may assume that f and g are monotonic.

Suppose, then, that g is nondecreasing and f is nonincreasing. Recall that

$$y'(t) = g(x(\tau(t-d))) + g(x(\tau(t+d))).$$

We claim that $y'(t) \geq 0$ for all $t \in (0, m)$. Assuming the claim, we have that y is nondecreasing on $(0, m)$. Since $u'(t) = f(y(t))$ and f is nonincreasing, we then see that u' is nonincreasing on $(0, m)$, as desired.

We now prove the claim. Since $x(\tau(-d)) = -x(\tau(d))$ by Lemma 2.2 and g is odd, we have $y'(0) = 0$. As t moves from 0 to $m - d$, $x(\tau(t-d))$ and $x(\tau(t+d))$ are nondecreasing and so, since g is nondecreasing, $y'(t)$ is nondecreasing too — and so in particular is nonnegative.

We now consider two cases.

If $d \leq 1/3$, then $d \leq m - d$ and so, for $t \in [m-d, m)$, we have $0 \leq t - d < t + d < 2m$ and

$$x(\tau(t-d)) = x(t-d) \geq 0, \quad x(\tau(t+d)) = x(t+d) \geq 0.$$

In this case $y'(t) \geq 0$.

If $d > 1/3$, then $m - d < d$ and for $t \in [m-d, d]$ we have

$$m \leq t + d < t + d + (m-d) \leq t - d + 2m \leq 2m,$$

and so (since x is nonincreasing on $[m, 2m]$) we have $x(t+d) \geq x(t-d+2m)$. But Lemma 2.2 now yields

$$-x(\tau(t-d)) = x(\tau(t-d+2m)) = x(t-d+2m) \leq x(t+d) = x(\tau(t+d)),$$

and since g is nondecreasing and odd we see that $y'(t) \geq 0$ for all $t \in [m-d, d]$. For $t \in [d, m)$, we have $0 \leq t - d < t + d < 2m$ and

$$x(\tau(t-d)) = x(t-d) \geq 0, \quad x(\tau(t+d)) = x(t+d) \geq 0;$$

in this case $y'(t) \geq 0$. This proves the claim. □

We have established that F is a continuous self-mapping of the compact convex set Ω, and so are in a good position to apply standard fixed point theorems. The following proposition establishes the connection between nontrivial fixed points of F and solutions of Equation (1.1).

Proposition 2.8. *If x is any nonzero fixed point of F, and ξ is the $4m$-periodic extension of x to all of $\mathbb{R}$, then ξ is a nontrivial solution of (1.1) satisfying $(T1)$, $(T2)$, $(S1)$, $(S2)$.*

Proof. That ξ satisfies $(T1)$, $(T2)$, $(S1)$, $(S2)$ is obvious since these are properties of the restriction of ξ to $[0, 4m]$, which is just $x \in \Omega$.

Write $y = G(x)$. From the proof of Proposition 2.5 we know that $y(0) = y(4m)$. Thus the right-hand derivative of $x = H(y)$ at 0 is equal to the left-hand derivative of $x = H(y)$ at $4m$, and we conclude that ξ is continuously differentiable everywhere.

For all $t \in \mathbb{R}$ we define

$$w(t) = \int_{t-1}^{t-d} g(\xi(s))\, ds.$$

Observe that w is $4m$-periodic.

We wish to show that $w(t) = y(t)$ for all $t \in [0, 4m]$. For this will establish that $w(t)$ is the $4m$-periodic extension of $y(t)$, and it follows that

$$\xi'(t) = \xi'(\tau(t)) = x'(\tau(t)) = f(y(\tau(t))) = f(w(\tau(t))) = f(w(t))$$

for all t — that is, that ξ solves Equation (1.1).

First observe that $w(m) = 0$: for since $d = 2m - 1$,

$$w(m) = \int_{m-1}^{m-d} g(\xi(s))\, ds = \int_{m-1}^{m-2m+1} g(\xi(s))\, ds = \int_{m-1}^{1-m} g(\xi(s))\, ds = 0$$

(since ξ satisfies $-\xi(t) = \xi(-t)$ for all t — recall Lemma 2.3). Lemma 2.3 also yields that

$$w'(t) = g(\xi(t-d)) - g(\xi(t-1)) = g(\xi(t-d)) + g(\xi(t-1+2m)) = g(\xi(t-d)) + g(\xi(t+d))$$

for all t (recall our derivation of the system (2.1)). Thus for $t \in (0, 4m)$ we have

$$\begin{aligned} y'(t) &= g(x(\tau(t-d))) + g(x(\tau(t+d))) \\ &= g(\xi(\tau(t-d))) + g(\xi(\tau(t+d))) \\ &= g(\xi(t-d)) + g(\xi(t+d)) = w'(t). \end{aligned}$$

Since $w(m) = 0 = y(m)$ and $w'(t) = y'(t)$ for all $t \in (0, 4m)$, we see that $w(t) = y(t)$ for all $t \in [0, 4m]$. This completes the proof. □

Remark 2.9. By Proposition 2.8, the proof of Theorem 1.1 will be complete if we show that, under the hypotheses of the theorem, $F\colon \Omega \to \Omega$ has a nonzero fixed point x. We do this in the next section by applying Browder's ejective fixed point principle. There are, of course, many other hypotheses that one can formulate for f and g that allow one to prove the existence of a nontrivial fixed point of $F\colon \Omega \to \Omega$. For example, if one imposes a condition (admittedly stringent) that f and g are close enough to step functions away from 0, it is not hard to exhibit a closed, convex subset of Ω that does not include 0 and that is mapped to itself by F; the desired result then follows from Schauder's theorem.

3 Proof of Theorem 1.1

0 is a fixed point of F; we wish to show that F has another fixed point in Ω. To do this we make use of Browder's ejective fixed point principle, which we now recall. Suppose that K is a compact, convex, infinite-dimensional subset of some Banach space and that $\phi\colon K \to K$ is

continuous. A fixed point z_0 of ϕ in K is called *ejective* if there is an open subset $U \subset K$ about z_0 such that, for any $z \in U \setminus \{z_0\}$, there is a positive integer $n(z)$ for which $\phi^{n(z)}(z) \notin U$. The ejective fixed point principle (Theorem 1 in [2]) states:

With notation as above, $\phi\colon K \to K$ has at least one fixed point that is not ejective.

If, therefore, we show that 0 is an ejective fixed point of $F\colon \Omega \to \Omega$ under the hypotheses of Theorem 1.1, the theorem will be proven.

To mitigate the notational complexity of what follows, for this section we shall, slightly abusively, identify any point $x \in \Omega$ with its $4m$-periodic extension — so we write $x(\tau(t)) = x(t)$ for all t. In this section we shall also employ some standard facts about real trigonometric Fourier series; these can be found, for example, in [7].

Let us now choose $x \in \Omega$, and let us assume moreover that x is C^1-smooth (this will be the case, for example, if $x \in F(\Omega)$). Since x is odd, the Fourier series of x consists only of sine terms; since x is smooth, the Fourier series for x converges uniformly to x on $[0, 4m]$. Therefore we can write

$$x(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n}{2m} t\right), \tag{3.1}$$

where the nth Fourier coefficient a_n is given by the formula

$$a_n = \frac{1}{2m} \int_0^{4m} x(t) \sin\left(\frac{\pi n}{2m} t\right)\, dt.$$

Lemma 3.1. *In the above series, $a_n = 0$ for all even n.*

Proof. If n is even, then for any $t \in [0, 2m]$ we have

$$\sin\left(\frac{\pi n}{2m}(t + 2m)\right) = \sin\left(\frac{\pi n}{2m} t\right).$$

Since $x(2m + t) = -x(t)$ for all such t, we see that the integral defining a_n will equal 0. □

For the rest of the paper we shall write $a(x)$ for the first Fourier coefficient a_1 defined above. Note that, for $x \in \Omega$, the symmetries shared by $x(\cdot)$ and $\sin(\frac{\pi}{2m}\cdot)$ yield that

$$a(x) = 4\left[\frac{1}{2m} \int_0^{m} x(t) \sin\left(\frac{\pi}{2m} t\right)\, dt\right]. \tag{3.2}$$

Formula (3.2) and Corollary 2.6 now yield

Lemma 3.2. *If $x \in \Omega \setminus \{0\}$, $a(F(x)) > 0$.*

The following lemma relates the size of $a(x)$ to the size of $\|x\|$. The rough idea is that, because $x \in \Omega$ has the same general shape as $\sin(\frac{\pi}{2m}\cdot)$, $a(x)$ cannot be too different from $\|x\|$.

Lemma 3.3. *Suppose that $x \in C[0, 4m]$. Then*

$$a(x) = \frac{1}{2m} \int_0^{4m} x(t) \sin\left(\frac{\pi}{2m} t\right)\, dt \le \frac{4}{\pi}\|x\|.$$

Furthermore, if $x \in \Omega$ and x is concave down on $(0, m)$ (that is, if x' is nonincreasing on $(0, m)$), then

$$\frac{8}{\pi^2}\|x\| \le a(x).$$

Proof. Since $x(t) \leq \|x\|$ for all $t \in [0, m]$, we have

$$a(x) \leq \frac{2}{m} \int_0^m \|x\| \sin\left(\frac{\pi}{2m} t\right) \, dt = \frac{2\|x\|}{m} \frac{2m}{\pi} = \frac{4}{\pi} \|x\|.$$

On the other hand, if $x \in \Omega$ is concave down on $(0, m)$ then $x(t) \geq \frac{t}{m}\|x\|$ for all $t \in [0, m]$ and so applying (3.2) we have

$$\begin{aligned} a(x) &\geq \frac{2\|x\|}{m^2} \int_0^m t \sin\left(\frac{\pi}{2m} t\right) \, dt \\ &= \frac{2\|x\|}{m^2} \left[\frac{-t \cos\left(\frac{\pi}{2m} t\right)}{\pi/(2m)} + \frac{\sin\left(\frac{\pi}{2m} t\right)}{(\pi/(2m))^2} \right]_{t=0}^{t=m} \\ &= \frac{2\|x\|}{m^2} \frac{4m^2}{\pi^2} = \frac{8}{\pi^2} \|x\|. \end{aligned}$$

This completes the proof. □

The above lemma allows us to conclude the following.

Lemma 3.4. *Given $\delta > 0$, write*

$$\mathcal{U}(\delta) = \{\, x \in \Omega \setminus \{0\} \;:\; \|x\| < \delta \,\}.$$

Suppose that there are numbers $\delta > 0$ and $\gamma > 1$ such that $x \in F(\mathcal{U}(\delta)) \cap \mathcal{U}(\delta)$ implies that

$$a(F(x)) \geq \gamma a(x).$$

Then 0 is an ejective fixed point of F — in particular, given any $x \in \mathcal{U}(\delta)$, $\|F^n(x)\| \geq \delta$ for some positive integer n.

Proof. Suppose that $x \in \mathcal{U}(\delta)$ and imagine that $F^n(x) \in \mathcal{U}(\delta)$ for all $n \in \mathbb{N}$. Then, since $F^n(x)$ is smooth for all $n \in \mathbb{N}$, applying Lemma 3.3 we have

$$\|F^n(x)\| \geq \frac{\pi}{4} a(F^n(x)) \geq \frac{\pi}{4} \gamma^{n-1} a(F(x)).$$

Since $x \neq 0$, $F(x) \neq 0$ (Corollary 2.6), and so by Lemma 3.2 we have $a(F(x)) > 0$. We now see that the expression on the right above goes to ∞ as $n \to \infty$ — a contradiction. □

We spend the rest of the section showing that, under the hypotheses of Theorem 1.1, the conditions of Lemma 3.4 hold. Our approach is essentially to linearize F about 0.

Let us write $\beta = g'(0) > 0$ and $-\alpha = f'(0) < 0$, and let us further write

$$g(v) = \beta v + g_e(v) \;\text{ and }\; f(v) = -\alpha v + f_e(v).$$

Since g and f are assumed differentiable, given any $\epsilon > 0$ there is some $\delta > 0$ such that $|v| \leq \delta$ implies both that $|g_e(v)| \leq \epsilon |v|$ and that $|f_e(v)| \leq \epsilon |v|$.

Now, given $x \in \Omega$, we have

$$\begin{aligned} G(x)(t) &= \int_m^t g(x(s-d)) + g(x(s+d)) \, ds \\ &= \beta \int_m^t x(s-d) + x(s+d) \, ds + \int_m^t g_e(x(s-d)) + g_e(x(s+d)) \, ds. \end{aligned}$$

Let us write

$$G_L(x)(t) = \beta \int_m^t x(s-d) + x(s+d)\, ds;$$
$$G_e(x)(t) = \int_m^t g_e(x(s-d)) + g_e(x(s+d))\, ds.$$

Similarly, we can write

$$\begin{aligned} F(x(t)) = H(G(x(t))) &= \int_0^t f(G_L(x)(s) + G_e(x(s)))\, ds \\ &= -\alpha \int_0^t G_L(x)(s)\, ds - \alpha \int_0^t G_e(x)(s)\, ds + \int_0^t f_e(G(x)(s))\, ds. \end{aligned}$$

Finally, we write

$$F_L(x)(t) = -\alpha \int_0^t G_L(x)(s)\, ds; \;\; F_e(x)(t) = -\alpha \int_0^t G_e(x)(s)\, ds + \int_0^t f_e(G(x)(s))\, ds.$$

Elementary estimates now yield the following lemma.

Lemma 3.5. *Given any $\epsilon > 0$, there is some $\delta > 0$ such that $\|x\| < \delta$ implies that*

$$\|F_e(x)\| < \epsilon \|x\|.$$

We now compute $F_L(x)$ for $x \in F(\Omega)$ (in particular, for x smooth). Writing

$$x(t) = \sum_{n \text{ odd}} a_n \sin\left(\frac{\pi n}{2m} t\right),$$

we have

$$G_L(x)(t) = \beta \int_m^t \left[\sum_{n \text{ odd}} a_n \sin\left(\frac{\pi n}{2m}(s-d)\right) + \sum_{n \text{ odd}} a_n \sin\left(\frac{\pi n}{2m}(s+d)\right) \right] ds.$$

Since the convergence of the sums is uniform we may first combine the sums term-by-term and then integrate term-by-term to obtain

$$\begin{aligned} G_L(x)(t) &= \sum_{n \text{ odd}} \beta a_n \int_m^t \sin\left(\frac{\pi n}{2m}(s-d)\right) + \sin\left(\frac{\pi n}{2m}(s+d)\right) ds \\ &= -\sum_{n \text{ odd}} 2\beta \cos\left(\frac{\pi n}{2m} d\right) \frac{2m}{\pi n} a_n \cos\left(\frac{\pi n}{2m} t\right). \end{aligned}$$

Now computing

$$F_L(x)(t) = -\alpha \int_0^t G_L(x)(s)\, ds$$

we obtain

$$F_L(x)(t) = \sum_{n \text{ odd}} 2\alpha\beta \cos\left(\frac{\pi n}{2m} d\right) \left(\frac{2m}{\pi n}\right)^2 a_n \sin\left(\frac{\pi n}{2m} t\right).$$

Notice that the Fourier coefficient of the first term is

$$a(x) \times 2\alpha\beta \left(\frac{2m}{\pi}\right)^2 \cos\left(\frac{\pi}{2m} d\right) =: a(x) \times \eta.$$

If the hypotheses of Theorem 1.1 are satisfied, then $\eta > 1$.

We now prove Theorem 1.1 by showing that, when $\eta > 1$, the hypothesis of Lemma 3.4 holds. Applying Lemmas 3.5 and 2.7, given any $\epsilon > 0$, we can choose $\delta > 0$ such that $x \in F(\mathcal{U}(\delta)) \cap \mathcal{U}(\delta)$ implies that

- $\|F_e(x)\| \le \epsilon \|x\|$; and
- x is concave down on $(0, m)$ (and so the second estimate in Lemma 3.3 applies to x).

Applying Lemma 3.3 and the fact that $a(\cdot)$ is linear, we now have, for $x \in F(\mathcal{U}(\delta)) \cap \mathcal{U}(\delta)$,

$$a(F(x)) = a(F_L(x) + F_e(x)) = a(F_L(x)) + a(F_e(x)) \ge \eta a(x) - \frac{4}{\pi}\|F_e(x)\|$$
$$\ge \eta a(x) - \frac{4}{\pi}\epsilon\|x\| \ge \eta a(x) - \frac{4}{\pi}\epsilon\frac{\pi^2}{8}a(x) = \left(\eta - \epsilon\frac{\pi}{2}\right)a(x).$$

Now choose δ (and hence ϵ) ϵ small enough so that

$$\gamma := \left(\eta - \epsilon\frac{\pi}{2}\right) > 1;$$

we have established that $x \in F(\mathcal{U}(\delta)) \cap \mathcal{U}(\delta)$ implies that $a(F(x)) \ge \gamma a(x)$. Lemma 3.4 now implies that 0 is an ejective fixed point of $0 \in \Omega$; the proof of Theorem 1.1 is complete.

Remark 3.6. In the $d = 0$ case, periodic solutions of (1.1) satisfying the symmetries (S) correspond to solutions of the system

$$x'(t) = f(y(t)), \;\; y'(t) = g(x(t)) - g(x(t-1)) = 2g(x(t)).$$

We suspect that in this $d = 0$ case, under hypotheses (H), the basic approach in [5] to establishing existence of nontrivial periodic solutions can be imitated much more closely. We have not carried through the details.

4 The "slowly oscillating" solution of Equation (1.4)

Numerical simulations suggest that the periodic solution proven to exist in Theorem 1.1 is frequently stable. The proof we have given, however, offers little indication of the reason for this stability. By way of providing some heuristic insight into the dynamics of Equation (1.1), in this brief concluding section we prove an analog of Theorem 1.1 for Equation (1.4) in the case that $d \in (0, 1)$. (Most of what we will say here applies to the $d = 0$ case as well, but some technical difficulties arise in the $d = 0$ case — for example, not all continuous initial conditions are continuable as solutions — that we wish to avoid for the sake of brevity.)

As in Section 2 we write $C = C([-1, 0], \mathbb{R})$ for the set of real-valued continuous functions on $[-1, 0]$, equipped with the sup norm. If x is a continuous function whose domain includes $[t-1, t]$, we write x_t for the member of C given by

$$x_t(s) = x(t+s), \; s \in [-1, 0].$$

By a solution of (1.4) we mean a solution of the corresponding integral equation

$$x(t) = x(0) + \int_0^t -\operatorname{sign}\left[\int_{s-1}^{s-d} \operatorname{sign}(x(u))\,du\right] ds, \;\; t > 0.$$

For $d \in (0, 1)$, existence and uniqueness of solutions of Equation (1.4) is straightforward: given $x_0 \in C$, the function

$$t \mapsto -\operatorname{sign}\left[\int_{t-1}^{t-d} \operatorname{sign}(x_0(s))\,ds\right]$$

is defined and Lebesgue measurable for all $t \in [0, d]$, and so the continuation x of x_0 as a solution of the above integral equation — and hence of Equation (1.4) — is uniquely defined on $[0, d]$. The solution can then be continued by steps on $[0, \infty)$. The solution is differentiable almost everywhere, and where it is differentiable it satisfies Equation (1.4) as usually written.

(It is useful to formulate a verbal description of the feedback mechanism embodied in (1.4): $x'(t) = 1$ if the restriction of x to $[t-1, t-d]$ is negative most of the time, and $x'(t) = -1$ if the restriction of x to $[t-1, t-d]$ is positive most of the time.)

We now state and prove the main result of the section. This result can be viewed as the counterpart to Theorem 1.1 for Equation (1.4); but the simplicity of the equation allows us to explicitly describe a portion of the domain of attraction of the periodic solution.

Proposition 4.1. *Let $d \in (0, 1)$, and write $m = (1 + d)/2$. Suppose that $x_0 \in C$ with $x_0(0) = 0$ and $x_0(s) < 0$ for all $s \in [-m, 0)$. Write x for the continuation of x_0 as a solution of Equation (1.4).*

Then, for all $t \geq 0$, x coincides with a periodic solution of Equation (1.4) that has period $4m$ and satisfies the symmetries $(T1)$, $(T2)$, $(S1)$, and $(S2)$.

Figure 4.1 illustrates the solution discussed in Proposition 4.1.

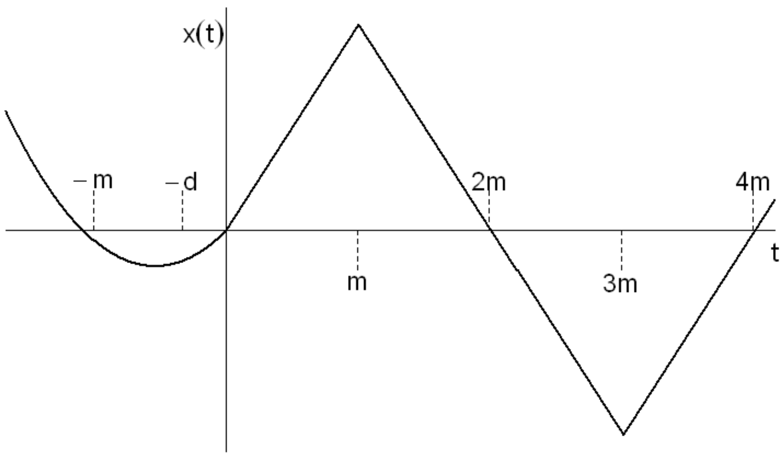

Figure 4.1: The solution discussed in Proposition 4.1.

Proof. With x as in the statement of the proposition, throughout the proof we write

$$y(t) = \int_{-1}^{-d} \operatorname{sign}(x(t+s))\, ds, \ t \geq 0.$$

$y(t)$ is defined for all $t \geq 0$, whether or not x is differentiable at t. If t is positive and $y(t) \neq 0$, then x is differentiable at t and we have $x'(t) = -\operatorname{sign}(y(t))$.

Observe that $1 - m = m - d = (1 - d)/2$. Thus

$$\operatorname{length}([-1, -d] \cap [-m, 0]) = (1 - d)/2,$$

and x is negative on strictly more than half of the interval $[-1, -d]$. Thus $y(0) < 0$.

Claim: $y(t) < 0$ for all $t \in (0, m)$. Assume this claim for the moment. Then $x'(t) = 1$ for all $t \in (0, m)$ and $x(t) = t$ for all $t \in [0, m]$. Observe that

$$m - 1 = \frac{d-1}{2} > -m.$$

Therefore the portion of $[m-1, m-d]$ for which $x(t)$ is negative is exactly $[m-1, 0)$ and the portion of $[m-1, m-d]$ for which $x(t)$ is positive is exactly $(0, m-d]$; each of these two intervals has length $(1-d)/2$. Thus $y(m) = 0$. Now write $z = \inf\{\, t \geq m \ :\ x(t) = 0 \,\}$. Since

$$\text{length}([t-1, t-d] \cap [-m, 0]) < \frac{1-d}{2}$$

for all $t \in (m, z]$ and $x(t) > 0$ for all $t \in (0, z)$, we have that $x'(t) = -1$ for all $t \in (m, z)$. We conclude that $z = 2m$. The proposition now follows by symmetry, since $-x_z$ satisfies the hypotheses of the proposition, and the odd feedback for Equation (1.4) yields that $x(2m + t) = -x(t)$ for all $t \geq 0$.

It remains to prove the claim. The main observation, which we shall use repeatedly, is that if

$$\text{length}([t-1, t-d] \cap [-m, 0]) > \frac{1-d}{2}$$

then it is guaranteed that x is negative on more than half of the interval $[t-1, t-d]$, and so $y(t) < 0$.

As t moves from 0 to m, $t - d$ crosses the point 0 (and so leaves the interval $[-m, 0]$) at time d, while $t - 1$ crosses then point $-m$ (and so enters the interval $[-m, 0]$) at time $1 - m$. It is convenient to consider separately the two cases where these two crossings occur in opposite orders.

Case 1: $d \in (0, 1/3]$. In this case we have $d \leq 1 - m$.

For $t \in (0, d]$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = \frac{1-d}{2} + t > \frac{1-d}{2}$$

and so $y(t) < 0$; for $t \in [d, 1-m]$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = \frac{1-d}{2} + d = m > \frac{1-d}{2}$$

and so $y(t) < 0$; for $t \in [1-m, m)$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = m - (t - (1-m)) = 1 - t > 1 - m = \frac{1-d}{2}$$

and so $y(t) < 0$.

Case 2: $d \in (1/3, 1)$. In this case we have $d > 1 - m$.

For $t \in (0, 1-m]$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = \frac{1-d}{2} + t > \frac{1-d}{2}$$

and so $y(t) < 0$; for $t \in [1-m, d]$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = \frac{1-d}{2} + (1-m) = 1 - d > \frac{1-d}{2}$$

and so $y(t) < 0$; for $t \in [d, m)$ we have

$$\text{length}([t-1, t-d] \cap [-m, 0]) = 1 - d - (t-d) = 1 - t > 1 - m = \frac{1-d}{2}$$

and so $y(t) < 0$.

This completes the proof. □

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