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Approximative solutions of difference equations

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Abstract. Asymptotic properties of solutions of difference equations of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n$$

are studied. Using the iterated remainder operator and fixed point theorems we obtain sufficient conditions under which for any solution y of the equation $\Delta^m y = b$ and for any real $s \leq 0$ there exists a solution x of the above equation such that $\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$ for any nonnegative integer $k \leq m$. Using a discrete variant of the Bihari lemma and a certain new technique we give also sufficient conditions under which for a given real $s \leq m-1$ all solutions x of the equation satisfy the condition $x = y + \mathrm{o}(n^s)$ where y is a solution of the equation $\Delta^m y = b$. Moreover, we give sufficient conditions under which for a given natural $k < m$ all solutions x of the equation satisfy the condition $x = y + u$ for a certain solution y of the equation $\Delta^m y = b$ and a certain sequence u such that $\Delta^k u = \mathrm{o}(1)$.

Keywords: difference equation, approximative solution, asymptotically polynomial solution, prescribed asymptotic behavior.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, the set of all integers and the set of real numbers, respectively. Let $m \in \mathbb{N}$. In this paper we consider the difference equation of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \tag{E}$$

$$n \in \mathbb{N}, \quad a_n, b_n \in \mathbb{R}, \quad f\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \sigma\colon \mathbb{N} \to \mathbb{N}, \quad \lim \sigma(n) = \infty.$$

We assume there is a given function $g\colon [0, \infty) \to [0, \infty)$ and a sequence w of real numbers such that

$$|f(n,t)| \leq g(|tw_n|) \;\text{ for }\; (n,t) \in \mathbb{N} \times \mathbb{R}. \tag{G}$$

By a solution of (E) we mean a sequence $x\colon \mathbb{N} \to \mathbb{R}$ satisfying (E) for all large n. If (E) is satisfied for all $n \in \mathbb{N}$ we say that x is a full solution of (E).

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The purpose of this paper is to study the asymptotic behavior of solutions of equation (E). In the study of solutions with prescribed asymptotic behavior some fixed point theorems are often used. Then there appear multiple sums of the form

$$\sum_{i_1=n}^{\infty}\sum_{i_2=i_1}^{\infty}\dots\sum_{i_m=i_{m-1}}^{\infty} x_{i_m}. \tag{R}$$

The reason is shown below. Let Z denote the space of all convergent to zero sequences. Then the operator

$$\Delta^m|_Z\colon Z \to \Delta^m(Z)$$

is bijective (it is a consequence of the equality $Z \cap \operatorname{Ker}\Delta^m = 0$). Moreover, if $x \in \Delta^m(Z)$ then the sum (R) is convergent and we may define a map

$$r^m\colon \Delta^m(Z) \to Z, \qquad r^m(x)(n) = \sum_{i_1=n}^{\infty}\sum_{i_2=i_1}^{\infty}\dots\sum_{i_m=i_{m-1}}^{\infty} x_{i_m}.$$

Then r^m (the iterated remainder operator) is a linear operator and $(-1)^m r^m$ is inverse to $\Delta^m|Z$. Hence for $x \in \Delta^m(Z)$ we have

$$\Delta^m((-1)^m r^m(x)) = x.$$

The last equality plays a crucial role in the application of fixed point theorems to the study of solutions of difference equations. Hence the operator r^m is very important. In Section 3, we establish some basic properties of this operator. It is easy to see that r^m is nondecreasing. This allows us to use the Knaster–Tarski fixed point theorem (see Section 6). The continuity of r^m is more subtle. The operator r^m is discontinuous (see Remark 4.6) but restrictions $r^m|S$ to some important sets S are continuous (see Lemma 4.5). This allows us to use the Schauder fixed point theorem (see Section 5). Multiple sums of the form (R) are used in many papers, see for example [15, 20, 24, 25, 43]. If the series

$$\sum_{n=0}^{\infty} n^{m-1}|x_n|$$

is convergent, then $x \in \Delta^m(Z)$ and we may rewrite the sum (R) in the more comfortable form

$$\sum_{k=0}^{\infty}\binom{m+k-1}{m-1}x_{n+k}$$

of a single sum (see Lemma 4.2). This is used to obtain fundamental properties of the operator r^m. The fact that $r^m(x) = \mathrm{o}(1)$ is often used to obtain results of type 'for a given sequence y there exists a solution x such that $x - y = \mathrm{o}(1)$', see, for example, [10, Theorem 1], [11, Theorem 1], [28, Theorem 1] or [39, Theorem 2.1]. If $s \in (-\infty, 0]$ and

$$\sum_{n=0}^{\infty} n^{m-1-s}|x_n| < \infty, \tag{1.1}$$

then $x \in \Delta^m(Z)$ and $r^m(x) = \mathrm{o}(n^s)$ (see Lemma 4.2). Using this fact one can obtain results of type 'for a given sequence y there exists a solution x such that $x - y = \mathrm{o}(n^s)$', see theorems in Section 3 of [29] and theorems in Sections 5 and 6 of this paper. Obviously, if $s < t \leq 0$ then the condition $x - y = \mathrm{o}(n^s)$ is more restrictive than $x - y = \mathrm{o}(n^t)$. Hence we obtain an approximative solution y and we may control the 'degree' of approximation.

In the study of asymptotic behavior of solutions of difference equations, asymptotically polynomial solutions play an important role. It is related to the fact that the solutions of the 'simplest' difference equation $\Delta^m x = 0$ are polynomial sequences. Analogously, if the difference $\Delta^m x$ is 'sufficiently small', then x is asymptotically polynomial. This effect is used in many papers, see for example Theorems 2 and 3 in [34] or Theorem 5 in [28]. The 'method of small difference' has been developed in [29]. It is based on [29, Theorem 2.1] which states that if $s \in (-\infty, m-1]$, $\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$ and $\Delta^m x_n = \mathrm{O}(a_n)$, then

$$x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s) = \{\varphi + u : \varphi \in \mathrm{Ker}\Delta^m,\ u_n = \mathrm{o}(n^s)\}. \tag{1.2}$$

In Lemma 3.11 we extend this result and, in Section 7, we use the 'method of small difference' to establish sufficient conditions under which all solutions of (E) are asymptotically polynomial.

Asymptotically polynomial solutions appear in the theory of both differential and difference equations. In particular, in the theory of second order equations, so called asymptotically linear solutions are considered. In the theory of differential equations, asymptotic linearity of solution x, usually means one of the following two conditions

$$x(t) = at + b + \mathrm{o}(1) \ \text{ or } \ x(t) = at + \mathrm{o}(t) \ \text{ as } \ t \to \infty. \tag{1.3}$$

In [32] the condition of the form $x(t) = at + \mathrm{o}(t^d)$ for certain $d \in (0,1)$ is also considered. In some papers in addition to (1.3), some properties of derivative x' are also considered. For example, in [32] Mustafa and Rogovchenko consider solutions x such that

$$x(t) = at + \mathrm{o}(t) \ \text{ and } \ x'(t) = a + \mathrm{o}(1) \ \text{ as } \ t \to \infty.$$

Ehrnström in [13] considers solutions x such that

$$|x(t) - at - b| + |x'(t) - a| \to 0 \ \text{ as } \ t \to \infty. \tag{1.4}$$

A discrete analog of (1.4) may be written in the form

$$x_n = an + b + \mathrm{o}(1) \ \text{ and } \ \Delta x_n = a + \mathrm{o}(1). \tag{1.5}$$

We generalize (1.5) as follows. We say that a sequence x is regularly asymptotically polynomial if

$$x \in \mathrm{Pol}(m) + \Delta^{-k}\mathrm{o}(1) = \{\varphi + u : \varphi \in \mathrm{Ker}\Delta^{m+1},\ \Delta^k u = \mathrm{o}(1)\} \tag{1.6}$$

for some integers $m \geq -1$ and $k \in [0, m+1]$. By Remark 3.3, the condition (1.6) is equivalent to

$$\Delta^p x \in \mathrm{Pol}(m-p) + \mathrm{o}(n^{k-p}) \ \text{ for any } \ p \in \{0, 1, \dots, k\}.$$

When $k = m$ we obtain a special case. By Lemma 3.8 the condition $x \in \mathrm{Pol}(m) + \Delta^{-m}\mathrm{o}(1)$ is equivalent to the convergence of the sequence $\Delta^m x_n$ and to the condition

$$\lim_{n\to\infty} \frac{p!\Delta^{m-p} z_n}{n^p} = \lambda \tag{1.7}$$

for certain fixed real λ and any $p \in \{0, 1, \dots, m\}$. Convergence of the sequence $\Delta^m x_n$ is comparatively easy to verify and condition (1.7) appears in many papers, see for example [9, 14, 25, 30, 42] or the proof of Theorem 3.1 in [41]. Our 'small difference method' covers both the case of usually asymptotically polynomial sequences (1.2) and the case of regularly asymptotically polynomial sequences (1.6). Moreover, we extend the method to the case of 'forced' equations (see Lemma 3.11).

This paper is a continuation of the papers [25, 28] and [29]. The origin of our studies goes back to the results of Popenda and Werbowski [34], Hooker and Patula [19], Popenda [35], Drozdowicz and Popenda [11], Cheng and Patula [6], Popenda and Schmeidel [36] and Li and Cheng [22]. Our results show certain similarities to the results obtained in continuous case by Philos, Purnaras and Tsamatos in [33]. See also [2, 8, 17, 18, 21, 40] and papers on asymptotically linear solutions of second order differential equations: [12, 23, 31, 37]. Our methods also bear some similarities to the methods used in the studies of asymptotic behavior of solutions to difference equations of neutral type, see, for example, [15, 20, 24, 26, 27, 41, 43]. Some closely related results on difference and dynamic equations, including ones on approximative solutions, can be found in [4, 5], and [38].

The paper is organized as follows. In Section 2 we introduce notation and terminology. In Sections 3 and 4 we present some preliminary results. Section 3 is devoted to asymptotically polynomial sequences. We establish some fundamental properties of the spaces of asymptotically polynomial sequences and regularly asymptotically polynomial sequences. At the end of Section 3 we obtain Lemma 3.11 which is the base of our 'small difference method'. This method will be used in the proofs of Theorems 7.4 and 7.5.

In Section 4 we establish some properties of the iterated remainder operator. These results will be used in the proofs of Theorems 5.1, 5.2, 6.1 and 6.2. Moreover, using the Schauder's fixed point theorem we obtain a certain fixed point lemma (Lemma 4.7) which will be used in the proofs of Theorems 5.1 and 5.2. Using the Knaster–Tarski fixed point theorem we obtain another fixed point lemma (Lemma 4.9) which will be used in the proofs of Theorems 6.1 and 6.2.

The main results appear in Sections 5, 6 and 7. In Section 5, assuming the function f is continuous, we establish conditions under which for any $y \in \Delta^{-m}b$ such that $(y \circ \sigma)w = \mathrm{O}(1)$ there exists a solution (or full solution) x of (E) such that $x = y + \mathrm{o}(n^s)$. In Section 6 we obtain analogous results under the assumption that the function f is monotonic with respect to the second variable. In Section 7 we obtain the conditions under which all solutions of (E) are asymptotically polynomial or 'translated' asymptotically polynomial.

2 Notation and terminology

If $p, k \in \mathbb{Z}$, $p \le k$, then $\mathbb{N}(p)$, $\mathbb{N}(p,k)$ denote the sets defined by

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \dots, k\}.$$

The space of all sequences $x\colon \mathbb{N} \to \mathbb{R}$ we denote by SQ. For any $x \in \mathrm{SQ}$ we denote by $\bar{x}$ the sequence defined by

$$\bar{x}_n = f(n, x_{\sigma(n)}). \tag{2.1}$$

The Banach space of all bounded sequences $x \in \mathrm{SQ}$ with the norm

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$$

we denote by BS. If x, y in SQ, then xy denotes the sequence defined by pointwise multiplication

$$xy(n) = x_n y_n.$$

Moreover, $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$ for every n.
We use the symbols 'big O' and 'small o' in the usual sense but for $a \in \mathrm{SQ}$ we also regard $\mathrm{o}(a)$ and $\mathrm{O}(a)$ as subspaces of SQ. More precisely, let

$$\mathrm{o}(1) = \{x \in \mathrm{SQ} : x \text{ is convergent to zero}\}, \quad \mathrm{O}(1) = \{x \in \mathrm{SQ} : x \text{ is bounded}\}$$

and for $a \in \mathrm{SQ}$ let

$$\mathrm{o}(a) = a\mathrm{o}(1) = \{ax : x \in \mathrm{o}(1)\}, \qquad \mathrm{O}(a) = a\mathrm{O}(1) = \{ax : x \in \mathrm{O}(1)\}.$$

For $m \in \mathbb{N}(0)$ we define

$$\mathrm{Pol}(m-1) = \mathrm{Ker}\Delta^m = \{x \in \mathrm{SQ} : \Delta^m x = 0\}.$$

Then $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than m. Note that

$$\mathrm{Pol}(-1) = \mathrm{Ker}\Delta^0 = 0$$

is the zero space. For a subset X of SQ let

$$\Delta^m X = \{\Delta^m x : x \in X\}, \qquad \Delta^{-m} X = \{z \in \mathrm{SQ} : \Delta^m z \in X\}$$

denote respectively the image and the inverse image of X under the map $\Delta^m \colon \mathrm{SQ} \to \mathrm{SQ}$. If $b \in \mathrm{SQ}$, then $\Delta^{-m}\{b\}$ we also denote simply by $\Delta^{-m}b$. Now, we can define spaces of asymptotically polynomial sequences and regularly asymptotically polynomial sequences

$$\mathrm{Pol}(m-1) + \mathrm{o}(n^s), \qquad \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1),$$

where $s \in (-\infty, m-1]$ and $k \in \mathbb{N}(0, m-1)$. Moreover, we will also use the sets of 'translated' asymptotically polynomial sequences

$$\Delta^{-m}b + \mathrm{o}(n^s), \qquad \Delta^{-m}b + \Delta^{-k}\mathrm{o}(1).$$

Remark 2.1. Note that if y is an arbitrary element of $\Delta^{-m}b$, then $\Delta^{-m}b = y + \mathrm{Pol}(m-1)$ and so

$$\Delta^{-m}b + \mathrm{o}(n^s) = y + \mathrm{Pol}(m-1) + \mathrm{o}(n^s),$$
$$\Delta^{-m}b + \Delta^{-k}\mathrm{o}(1) = y + \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1).$$

Hence $\Delta^{-m}b + \mathrm{o}(n^s)$ and $\Delta^{-m}b + \Delta^{-k}\mathrm{o}(1)$ are affine subsets of the space SQ.

Now we define the spaces $\mathrm{S}(m)$ of m-times summable sequences and the remainder operator. Let

$$\mathrm{S}(0) = \mathrm{o}(1), \qquad \mathrm{S}(1) = \Big\{x \in \mathrm{SQ} : \text{ the series } \sum_{n=1}^{\infty} x_n \text{ is convergent}\Big\}.$$

For $x \in \mathrm{S}(1)$, we define the sequence $r(x)$ by the formula

$$r(x)(n) = \sum_{j=n}^{\infty} x_j.$$

Then $r(x) \in \mathrm{S}(0)$ and we obtain the remainder operator

$$r \colon \mathrm{S}(1) \to \mathrm{S}(0).$$

Obviously the operator r is linear. If $m \in \mathbb{N}$ then, by induction, we define the linear space $\mathrm{S}(m+1)$ and the operator

$$r^{m+1} \colon \mathrm{S}(m+1) \to \mathrm{S}(0)$$

by

$$\mathrm{S}(m+1) = \{x \in \mathrm{S}(m) : \ r^m(x) \in \mathrm{S}(1)\}, \qquad r^{m+1}(x) = r(r^m(x)).$$

The value $r^m(x)(n)$ we denote also by $r^m_n(x)$ or simply $r^m_n x$. Note that

$$r^m_n x = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}$$

for any $x \in \mathrm{S}(m)$ and any $n \in \mathbb{N}$.

A function $h \colon X \to Y$ of a topological space X to metric space Y is called locally bounded if for any $x \in X$ there exists a neighborhood U of x such that $h|_U$ is bounded.

Remark 2.2. If $X \subset \mathbb{R}$, then every continuous, every monotonic and every bounded function $h \colon X \to \mathbb{R}$ is locally bounded. Moreover, if X is closed, then h is locally bounded if and only if it is bounded on every bounded subset of X.

For $k \in \mathbb{N}(1)$ we use the factorial notation

$$n^{(k)} = n(n-1)\dots(n-k+1) \ \text{ with } \ n^{(0)} = 1.$$

We say that the equation (E) is of continuous type if the function f is continuous (we regard $\mathbb{N} \times \mathbb{R}$ as a metric subspace of the plane $\mathbb{R}^2$). If f is monotonic with respect to the second variable we say that (E) is of monotone type.

3 Asymptotically polynomial sequences

In this section we establish some basic properties of the spaces of asymptotically polynomial sequences. The main result of this section is Lemma 3.11 which will be used in the proofs of Theorems 7.4 and 7.5.

Lemma 3.1. *Assume $m \in \mathbb{N}(0)$, $k \in \mathbb{N}(0, m)$ and $x \in \mathrm{SQ}$. Then*

(a) $\Delta^m x \in \Delta^k \mathrm{o}(1) \iff x \in \mathrm{Pol}(m-1) + \Delta^{k-m}\mathrm{o}(1)$;

(b) $x \in \Delta^{-m}\mathrm{o}(1) \iff \Delta^p x \in \mathrm{o}(n^{m-p})$ *for every* $p \in \mathbb{N}(0, m)$;

(c) $\mathrm{Pol}(m-1) \subset \Delta^{-m}\mathrm{o}(1) \subset \mathrm{o}(n^m)$.

Proof. (a) If $\Delta^m x = \Delta^k u$, $u = \mathrm{o}(1)$, then choosing a sequence w such that $\Delta^{m-k} w = u$ we obtain

$$\Delta^m x = \Delta^k u = \Delta^k \Delta^{m-k} w = \Delta^m w.$$

Hence

$$x - w \in \mathrm{Ker}\Delta^m = \mathrm{Pol}(m-1) \ \text{ and } \ w \in \Delta^{k-m}\mathrm{o}(1).$$

Therefore

$$x = (x-w) + w \in \mathrm{Pol}(m-1) + \Delta^{k-m}\mathrm{o}(1).$$

Assume $x = u + w$, $u \in \mathrm{Pol}(m-1)$ and $w \in \Delta^{k-m}\mathrm{o}(1)$. Then

$$\Delta^m x = \Delta^m u + \Delta^m w = \Delta^m w = \Delta^k \Delta^{m-k} w \in \Delta^k \mathrm{o}(1).$$

(b) If $x \in \Delta^{-m}\mathrm{o}(1)$, then $\Delta^m x = \mathrm{o}(1)$ and

$$\frac{\Delta\Delta^{m-1}x_n}{\Delta n} = \Delta^m x_n = \mathrm{o}(1).$$

By the Stolz–Cesàro theorem $\Delta^{m-1}x_n = \mathrm{o}(n)$. Hence

$$\frac{\Delta\Delta^{m-2}x_n}{\Delta n^2} = \frac{n\Delta\Delta^{m-2}x_n}{n\Delta n^2} = \frac{\Delta^{m-1}x_n}{n}\frac{n}{\Delta n^2} \longrightarrow 0.$$

Again by the Stolz–Cesàro theorem $\Delta^{m-2}x_n = \mathrm{o}(n^2)$. Analogously $\Delta^{m-3}x_n = \mathrm{o}(n^3)$ and so on. Inverse implication is obvious.
(c) Obviously $\mathrm{Pol}(m-1) \subset \Delta^{-m}\mathrm{o}(1)$. Taking $p = 0$ in (b) we obtain $\Delta^{-m}\mathrm{o}(1) \subset \mathrm{o}(n^m)$. □

Remark 3.2. The inclusion

$$\Delta^{-m}\mathrm{o}(1) \subset \mathrm{o}(n^m)$$

is proper for any $m \in \mathbb{N}(1)$. For example, if $a_n = (-1)^n$, then

$$a \in \mathrm{o}(n^m), \quad \Delta^m a_n = 2^m(-1)^{m+n} \notin \mathrm{o}(1)$$

and so $a \notin \Delta^{-m}\mathrm{o}(1)$.

Remark 3.3. If $m \in \mathbb{N}(0)$ and $k \in \mathbb{N}(0, m)$, then by Lemma 3.1 we have

$$x \in \mathrm{Pol}(m) + \Delta^{-k}\mathrm{o}(1) \iff \Delta^p x \in \mathrm{Pol}(m-p) + \mathrm{o}(n^{k-p}) \text{ for any } p \in \mathbb{N}(0, k).$$

In the next two lemmas we describe elements of the spaces of asymptotically polynomial sequences and elements of the spaces of regularly asymptotically polynomial sequences.

Lemma 3.4. *Assume $m \in \mathbb{N}(0)$, $k \in \mathbb{N}(0, m)$ and $x \in \mathrm{SQ}$. Then*

$$x \in \mathrm{Pol}(m) + \mathrm{o}(n^k)$$

if and only if there exist constants $c_m, \dots, c_k$ and a sequence $w \in \mathrm{o}(n^k)$ such that

$$x_n = c_m n^m + c_{m-1}n^{m-1} + \cdots + c_k n^k + w_n.$$

Moreover, the constants $c_m, \dots, c_k$ and the sequence w are unique.

Proof. If $\mathrm{Pol}(m, k)$ denotes the subspace of $\mathrm{Pol}(m)$ generated by sequences

$$(n^m), (n^{m-1}), \dots, (n^k),$$

then

$$\mathrm{Pol}(m) + \mathrm{o}(n^k) = \mathrm{Pol}(m, k) + \mathrm{o}(n^k) \text{ and } \mathrm{Pol}(m, k) \cap \mathrm{o}(n^k) = 0.$$

Hence

$$\mathrm{Pol}(m) + \mathrm{o}(n^k) = \mathrm{Pol}(m, k) \oplus \mathrm{o}(n^k)$$

and we obtain the result. □

Lemma 3.5. *Assume $m \in \mathbb{N}(0)$, $k \in \mathbb{N}(0,m)$ and $x \in \mathrm{SQ}$. Then*

$$x \in \mathrm{Pol}(m) + \Delta^{-k}\mathrm{o}(1)$$

if and only if there exist constants $c_m, \dots, c_k$ and a sequence $w \in \mathrm{o}(n^k)$ such that

$$x_n = c_m n^m + c_{m-1} n^{m-1} + \cdots + c_k n^k + w_n$$

and $\Delta^p w_n = \mathrm{o}(n^{k-p})$ for any $p \in \mathbb{N}(0,k)$.

Proof. The result is a consequence of Lemma 3.4 and Lemma 3.1 (b). □

Remark 3.6. We can compare the spaces of asymptotically polynomial sequences. Let

$$P(m,k) = \mathrm{Pol}(m) + \mathrm{o}(n^k) \quad \text{and} \quad D(m,k) = \mathrm{Pol}(m) + \Delta^{-k}\mathrm{o}(1).$$

Then, using Lemma 3.1 and the fact that if $s, t \in \mathbb{R}$, then the condition

$$\mathrm{o}(n^s) \subset \mathrm{o}(n^t)$$

is equivalent to the condition $s \leq t$, we obtain a diagram

$$\begin{array}{ccccccccccc}
P(m,0) & \longrightarrow & P(m,1) & \longrightarrow & P(m,2) & \longrightarrow & \dots & \longrightarrow & P(m,m) & \longrightarrow & P(m,m+1) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
D(m,0) & \longrightarrow & D(m,1) & \longrightarrow & D(m,2) & \longrightarrow & \dots & \longrightarrow & D(m,m) & \longrightarrow & D(m,m+1)
\end{array}$$

where arrows denote inclusions. Note that $D(m,0) = \mathrm{Pol}(m) + \mathrm{o}(1) = P(m,0)$,

$$P(m,k) = \mathrm{o}(n^k) \quad \text{and} \quad D(m,k) = \Delta^{-k}\mathrm{o}(1) \quad \text{for} \quad k > m.$$

Now, we describe the elements of the space

$$D(m-1,k) = \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1)$$

in a different way than in Lemma 3.5.

Lemma 3.7. *Let $m \in \mathbb{N}(0)$, $k \in \mathbb{N}(0,m)$ and $z \in \mathrm{SQ}$. The following conditions are equivalent:*

(1) $z \in \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1)$.

(2) $\Delta^m z \in \Delta^{m-k}\mathrm{o}(1)$.

(3) $\Delta^k z \in \mathrm{Pol}(m-k-1) + \mathrm{o}(1)$.

(4) $\Delta^p z \in \mathrm{Pol}(m-p-1) + \Delta^{p-k}\mathrm{o}(1)$ *for certain* $p \in \mathbb{N}(0,k)$.

(5) $\Delta^p z \in \mathrm{Pol}(m-p-1) + \Delta^{p-k}\mathrm{o}(1)$ *for every* $p \in \mathbb{N}(0,k)$.

Proof. Implications (1) $\Rightarrow$ (4), (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (1), (5) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) are obvious. Assume (2) and let $p \in \mathbb{N}(0,k)$. Then

$$\Delta^{m-p}\Delta^p z = \Delta^m z \in \Delta^{m-k}\mathrm{o}(1) \quad \text{and} \quad (m-k) - (m-p) = p - k.$$

Hence, by Lemma 3.1, we have

$$\Delta^p z \in \mathrm{Pol}(m-p-1) + \Delta^{p-k}\mathrm{o}(1).$$

Therefore $(2) \Rightarrow (5)$. Assume

$$\Delta^p z \in \mathrm{Pol}(m-p-1) + \Delta^{p-k}\mathrm{o}(1) = \mathrm{Pol}(m-p-1) + \Delta^{(m-k)-(m-p)}\mathrm{o}(1).$$

Then, by Lemma 3.1, we have

$$\Delta^{m-p}\Delta^p z \in \Delta^{m-k}\mathrm{o}(1).$$

Hence $\Delta^m z \in \Delta^{m-k}\mathrm{o}(1)$. Therefore $(4) \Rightarrow (2)$ The proof is complete. □

In the next lemma we describe the elements of the space

$$D(m,m) = \mathrm{Pol}(m) + \Delta^{-m}\mathrm{o}(1).$$

Lemma 3.8. *Assume $z \in \mathrm{SQ}$ and $m \in \mathbb{N}(0)$. The following conditions are equivalent.*

(a) $\Delta^{m+1}z \in \Delta\mathrm{o}(1)$.

(b) *The sequence $\Delta^m z$ is convergent.*

(c) $z \in \mathrm{Pol}(m) + \Delta^{-m}\mathrm{o}(1)$.

(d) *There exists a constant $\lambda \in \mathbb{R}$ such that for any $p \in \mathbb{N}(0,m)$ we have*

$$\lim_{n\to\infty} \frac{p!\Delta^{m-p}z_n}{n^p} = \lim_{n\to\infty} \frac{p!\Delta^{m-p}z_n}{n^{(p)}} = \lambda.$$

Proof. Equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ easily follow from Lemma 3.7. Taking $p = 0$ in (d) we obtain $(d) \Rightarrow (b)$. Note that

$$\lim_{n\to\infty} \frac{\Delta^{m-p}z_n}{n^p} = \lim_{n\to\infty} \frac{\Delta^{m-p}z_n}{n^{(p)}} \lim_{n\to\infty} \frac{n^{(p)}}{n^p} = \lim_{n\to\infty} \frac{\Delta^{m-p}z_n}{n^{(p)}}.$$

Assume $\lim \Delta^m z_n = \lambda$. Then

$$\frac{\Delta\Delta^{m-1}z_n}{\Delta n} = \Delta^m z_n \to \lambda.$$

By the Stolz–Cesàro theorem $\lim n^{-1}\Delta^{m-1}z_n = \lambda$. Hence

$$\frac{\Delta\Delta^{m-2}z_n}{\Delta n^2} = \frac{n\Delta\Delta^{m-2}z_n}{n\Delta n^2} = \frac{\Delta^{m-1}z_n}{n}\frac{n}{\Delta n^2} \longrightarrow \frac{\lambda}{2}.$$

By the Stolz–Cesàro theorem $\lim n^{-2}\Delta^{m-2}z_n = \lambda/2 = \lambda/2!$. Similarly from the equality

$$\lim \frac{n^2}{\Delta n^3} = \frac{1}{3}$$

we obtain $\lim n^{-3}\Delta^{m-3}z_n = (\lambda/2)(1/3) = \lambda/3!$ and so on. We obtain the implication $(b) \Rightarrow (d)$. The proof is complete. □

The next two lemmas are used in the proof of Lemma 3.11.

Lemma 3.9. *Assume $s \in (-1,\infty)$, $m \in \mathbb{N}(1)$ and $\Delta^m x_n = \mathrm{o}(n^s)$. Then $x_n = \mathrm{o}(n^{s+m})$.*

Proof. See Lemma 2.1 in [29]. □

Lemma 3.10. *Assume u is a positive and nondecreasing sequence, $m \in \mathbb{N}(1)$ and*

$$\sum_{n=1}^{\infty} n^{m-1} u_n |a_n| < \infty.$$

Then there exists a sequence $w \in \mathrm{o}(u^{-1})$ such that $\Delta^m w = a$.

Proof. See Lemma 2.3 in [29]. □

The following lemma is a base of our 'small difference method' which we use in Section 7. This lemma extends Theorem 2.1 of [29].

Lemma 3.11. *Assume $a, b, x \in \mathrm{SQ}$, $m \in \mathbb{N}(1)$, $s \in (-\infty, m-1]$, $k \in \mathbb{N}(0, m-1)$, $c = |a| + |b|$ and*

$$\Delta^m x \in \mathrm{O}(a) + b.$$

Then

(a) *if* $\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$, *then* $x \in \Delta^{-m} b + \mathrm{o}(n^s)$,

(b) *if* $\sum_{n=1}^{\infty} n^{m-1-s}|c_n| < \infty$, *then* $x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s)$,

(c) *if* $\sum_{n=1}^{\infty} n^{m-1-k}|a_n| < \infty$, *then* $x \in \Delta^{-m} b + \Delta^{-k}\mathrm{o}(1)$,

(d) *if* $\sum_{n=1}^{\infty} n^{m-1-k}|c_n| < \infty$, *then* $x \in \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1)$.

Proof. (a) Assume $\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$. Let $s \leq 0$. The condition $\Delta^m x - b \in \mathrm{O}(a)$ implies

$$\sum_{n=1}^{\infty} n^{m-1-s} |\Delta^m x_n - b| < \infty.$$

Let $u_n = n^{-s}$. By Lemma 3.10, there exists a sequence $w = \mathrm{o}(n^s)$ such that $\Delta^m w = \Delta^m x - b$. Then $\Delta^m(x - w) = b$. Hence $x - w \in \Delta^{-m} b$ and

$$x = x - w + w \in \Delta^{-m} b + \mathrm{o}(n^s).$$

Let $s \in (0, m-1]$. Choose $k \in \mathbb{N}(1, m-1)$ such that $k - 1 < s \leq k$. Then

$$\sum_{n=1}^{\infty} n^{m-k-1} n^{k-s} |\Delta^m x - b| < \infty$$

and by Lemma 3.10 there exists $w = \mathrm{o}(n^{s-k})$ such that $\Delta^{m-k} w = \Delta^m x - b$. Choose $z \in \mathrm{SQ}$ such that $\Delta^k z = w$. Since $s - k > -1$, so by Lemma 3.9 we obtain $z = \mathrm{o}(n^s)$. Moreover

$$\Delta^m z = \Delta^{m-k} \Delta^k z = \Delta^{m-k} w = \Delta^m x - b \quad \text{and} \quad x = x - z + z \in \Delta^{-m} b + \mathrm{o}(n^s).$$

(b) If $\sum_{n=1}^{\infty} n^{m-1-s}(|a_n| + |b_n|) < \infty$, then $\Delta^m x \in 0 + \mathrm{O}(|a| + |b|)$ and by (a) we obtain

$$x \in \Delta^{-m} 0 + \mathrm{o}(n^s) = \mathrm{Pol}(m-1) + \mathrm{o}(n^s).$$

(c) Assume $\sum_{n=1}^{\infty} n^{m-1-k}|a_n| < \infty$. The condition $\Delta^m x - b \in \mathrm{O}(a)$ implies

$$\sum_{n=1}^{\infty} n^{m-k-1} |\Delta^m x_n - b| < \infty.$$

By Lemma 3.10, there exists a sequence $w = \mathrm{o}(1)$ such that $\Delta^{m-k}w = \Delta^m x - b$. Choose $z \in \mathrm{SQ}$ such that $\Delta^k z = w$. Then $z \in \Delta^{-k}\mathrm{o}(1)$ and

$$\Delta^m z = \Delta^{m-k}\Delta^k z = \Delta^{m-k}w = \Delta^m x - b. \quad \text{So} \quad x = x - z + z \in \Delta^{-m}b + \Delta^{-k}\mathrm{o}(1).$$

(d) If $\sum_{n=1}^{\infty} n^{m-1-k}(|a_n| + |b_n|) < \infty$, then $\Delta^m x \in 0 + \mathrm{O}(|a| + |b|)$ and by (c) we obtain

$$x \in \Delta^{-m}0 + \Delta^{-k}\mathrm{o}(1) = \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1).$$

The proof is complete. □

4 The iterated remainder operator and fixed point lemmas

In the first three lemmas we present some basic properties of the iterated remainder operator. Next we obtain some fixed point lemmas (Lemma 4.7 and Lemma 4.9). These lemmas will be used in Sections 5 and 6.

Lemma 4.1. *Assume $x, y \in \mathrm{SQ}$, $m \in \mathbb{N}(1)$ and $p \in \mathbb{N}(0)$. Then*

(a) *if $|x| \in \mathrm{S}(m)$, then $x \in \mathrm{S}(m)$ and $|r^m x| \le r^m|x|$,*

(b) *$|x| \in \mathrm{S}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty$,*

(c) *if $|x| \in \mathrm{S}(m)$, then $r_k^m|x| \le \sum_{n=k}^{\infty} n^{m-1}|x_n|$ for any $k \in \mathbb{N}(1)$,*

(d) *if $x \in \mathrm{S}(m)$, then $\Delta^m r^m x = (-1)^m x$,*

(e) *if $x = \mathrm{o}(1)$, then $\Delta^m x \in \mathrm{S}(m)$ and $r^m \Delta^m x = (-1)^m x$,*

(f) *$\Delta^m(\mathrm{S}(0)) = \mathrm{S}(m)$, $\quad r^m(\mathrm{S}(m)) = \mathrm{S}(0)$,*

(g) *$\Delta^p(\mathrm{S}(m)) = \mathrm{S}(m+p)$, $\quad r^p(\mathrm{S}(m+p)) = \mathrm{S}(m)$,*

(h) *if $x, y \in \mathrm{S}(m)$ and $x_n \le y_n$ for $n \ge p$, then $r_n^m x \le r_n^m y$ for $n \ge p$,*

(i) *if $x \in \mathrm{S}(m)$ and $y_n = x_n$ for $n \ge p$, then $y \in \mathrm{S}(m)$ and $r_n^m y = r_n^m x$ for $n \ge p$.*

Proof. The assertion (a) is proved in Lemma 1 of [28] and (b) is proved in Lemma 3 of [28]. The assertion (c) follows from Lemma 2 of [28] and from the proof of Lemma 3 in [28], while (d) is proved in Lemma 5 of [28]. The assertion (e) follows from Lemma 6 of [28], while (f) is an easy consequence of (e) and (d). The assertion (g) is a consequence of (f) and (e). The assertion (h) is obvious for $m = 1$. For $m > 1$ it can be easily proved by induction. The assertion (i) is an easy consequence of (h). □

Lemma 4.2. *Assume $x \in \mathrm{SQ}$, $m \in \mathbb{N}(1)$, $s \in (-\infty, 0]$ and*

$$\sum_{n=1}^{\infty} n^{m-1-s}|x_n| < \infty. \tag{4.1}$$

Then $x \in \mathrm{S}(m)$, $r^m x = \mathrm{o}(n^s)$ and

$$r_n^m x = \sum_{i_1=n}^{\infty}\sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m} = \sum_{k=0}^{\infty}\binom{m+k-1}{m-1} x_{n+k}. \tag{4.2}$$

Moreover, if $k \in \mathbb{N}(0, m)$, then $\Delta^k r^m x = \mathrm{o}(n^{s-k})$.

Proof. Let $u_n = n^{-s}$. Then by Lemma 3.10 there exists a sequence $w = \mathrm{o}(n^s)$ such that $\Delta^m w = x$. By Lemma 4.1 (f) we obtain

$$x = \Delta^m w \in \Delta^m \mathrm{o}(n^s) \subset \Delta^m \mathrm{o}(1) = \mathrm{S}(m).$$

By Lemma 4.1 (e) we obtain $r^m x = r^m \Delta^m w = (-1)^m w \in \mathrm{o}(n^s)$. The assertion (4.2) follows from Lemma 2 of [28]. Let $k \in \mathbb{N}(0, m)$. Then $m - 1 - s = m - k - 1 - (s - k)$ and

$$\sum_{n=1}^{\infty} n^{(m-k)-1-(s-k)} |x_n| < \infty.$$

Hence, by the first part of the proof we obtain $r^{m-k} x = \mathrm{o}(n^{s-k})$. On the other hand

$$\Delta^k r^m x = \Delta^k r^k r^{m-k} x = (-1)^k r^{m-k} x.$$

Hence $\Delta^k r^m x = \mathrm{o}(n^{s-k})$. □

Remark 4.3. In general the condition $z \in \mathrm{o}(n^s)$ does not imply the condition $\Delta^m z \in \mathrm{o}(n^{s-m})$.

Example 4.4. Let $t, s \in \mathbb{R}$, $t < s$, $m \in \mathbb{N}$, $m > s - t$ and $z_n = (-1)^n n^t$. Then

$$\begin{aligned} |\Delta^m z_n| &= \left| \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} z_{n+k} \right| = \left| \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} (-1)^{n+k} (n+k)^t \right| \\ &= \left| (-1)^{m+n} \sum_{k=0}^{m} \binom{m}{k} (n+k)^t \right| \geq n^t \end{aligned}$$

and we obtain

$$\frac{|\Delta^m z_n|}{n^{s-m}} = \frac{|\Delta^m z_n|}{n^{t+s-t-m}} = n^{m-s+t} \frac{|\Delta^m z_n|}{n^t} \geq n^{m-s+t} \to \infty.$$

Hence

$$z \in \mathrm{o}(n^s) \quad \text{and} \quad \Delta^m z \notin \mathrm{o}(n^{s-m}).$$

Lemma 4.5. *Assume $m \in \mathbb{N}(1)$, $\rho \in \mathrm{SQ}$, $|\rho| \in \mathrm{S}(m)$ and*

$$S = \{x \in \mathrm{SQ} : \ |x| \leq |\rho|\}.$$

Then $S \subset \mathrm{S}(m)$ and the map $r^m|S$ is continuous.

Proof. By Lemma 4.1 (b) we have $\sum_{n=1}^{\infty} n^{m-1}|\rho_n| < \infty$ and $S \subset \mathrm{S}(m)$. Let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ and $\delta > 0$ such that

$$\sum_{n=p}^{\infty} n^{m-1}|\rho_n| < \varepsilon \quad \text{and} \quad \delta \sum_{n=1}^{p} n^{m-1} < \varepsilon.$$

Let $x, y \in S$, $\|x - y\| < \delta$. Then

$$\begin{aligned} \|r^m x - r^m y\| &= \|r^m(x - y)\| = \sup_n |r_n^m(x - y)| \leq \sup_n r_n^m |x - y| \\ &= r_1^m |x - y| \leq \sum_{n=1}^{\infty} n^{m-1}|x_n - y_n| \leq \sum_{n=1}^{p} n^{m-1}|x_n - y_n| + \sum_{n=p}^{\infty} n^{m-1}|x_n - y_n| \\ &\leq \delta \sum_{n=1}^{p} n^{m-1} + \sum_{n=p}^{\infty} n^{m-1}(|\rho_n| + |\rho_n|) < 3\varepsilon. \end{aligned}$$

□

Remark 4.6. The operator $r^m : \mathrm{S}(m) \to \mathrm{S}(0)$ is discontinuous for any $m \in \mathbb{N}(1)$. For example if

$$u_k \in \mathrm{SQ}, \quad u_k(n) = \begin{cases} 1 & \text{for } n \leq k \\ 0 & \text{for } n > k, \end{cases}$$

then $u_k \in \mathrm{S}(m)$, $\|u_k\| = 1$ and $\|r^m(u_k)\| \geq \|r(u_k)\| = k$. Hence r^m is a linear unbounded operator. Therefore it is discontinuous.

Lemma 4.7 (Schauder's fixed point lemma)**.** *Assume $y, \rho \in \mathrm{SQ}$, $\rho \geq 0$, and $\lim \rho_n = 0$. In the set*

$$S = \{x \in \mathrm{SQ} : \ |x - y| \leq \rho\}$$

we define the metric by the formula $d(x,z) = \sup_{n\in\mathbb{N}} |x_n - z_n|$. Then every continuous map

$$H : S \to S$$

has a fixed point.

Proof. Assume $H : S \to S$ is continuous and let $T = \{x \in \mathrm{BS} : \ |x| \leq \rho\}$. Obviously T is a convex and closed subset of BS. Choose an $\varepsilon > 0$. Then there exists $p \in \mathbb{N}(1)$ such that $\rho_n \leq \varepsilon$ for any $n \geq p$. For $n = 1, \dots, p$ let G_n denote a finite ε-net for the interval $[-\rho_n, \rho_n]$ and let

$$G = \{x \in T : \ x_n \in G_n \ \text{ for } \ n \leq p \ \text{ and } \ x_n = 0 \ \text{ for } \ n > p\}.$$

Then G is a finite ε-net for T. Hence T is a complete and totally bounded metric space and so, T is compact. Hence T is a convex and compact subset of the Banach space BS. Let $F : T \to S$ be a map given by $F(x)(n) = x_n + y_n$. Then F is an isometry of T onto S. Let $B : T \to T$, $B = F^{-1} \circ H \circ F$. Then B is continuous. By Schauder's fixed point theorem there exists $u \in T$ such that $B(u) = u$. Let $x = F(u)$. Then

$$x = F(u) = F(B(u)) = F(F^{-1}HF(u)) = HF(u) = H(x).$$

The proof is complete. □

The following lemma is a version of the Knaster–Tarski fixed point theorem. This theorem may be found in [1] or in [16] but we use a simpler version. For the convenience of the reader we cite the proof from [3].

Lemma 4.8 (Knaster–Tarski)**.** *If X is a complete partially ordered set and a map $T : X \to X$ is nondecreasing then there exists $x_0 \in X$ such that $T(x_0) = x_0$.*

Proof. Let $Z = \{z \in X : \ z \leq T(z)\}$, $x_0 = \sup Z$. If $z \in Z$ then

$$z \leq x_0 \Longrightarrow T(z) \leq T(x_0) \Longrightarrow z \leq T(z) \leq T(x_0).$$

Hence $z \leq T(x_0)$ for every $z \in Z$. Therefore $x_0 = \sup Z \leq T(x_0)$. Thus $x_0 \in Z$. Moreover $x_0 \leq T(x_0)$ implies $T(x_0) \leq T(T(x_0))$. Hence $T(x_0) \in Z$. Therefore $T(x_0) \leq \sup Z = x_0$. It follows that $T(x_0) = x_0$. □

Lemma 4.9 (Knaster–Tarski fixed point lemma)**.** *Let $y, \rho \in \mathrm{SQ}$ and let S denote the set $\{x \in \mathrm{SQ} : |x - y| \leq \rho\}$ with natural order defined by: $x \leq z$ if $x_n \leq z_n$ for any $n \in \mathbb{N}(1)$. Then every nondecreasing map $T : S \to S$ has a fixed point.*

Proof. By Lemma 4.8 it follows that it is sufficient to show that the set S is complete; i.e. for every $B \subset S$ there exists a $\sup B \in S$. Let $B \subset S$. For $n \in \mathbb{N}$ let $B_n = \{x_n : x \in B\}$. Then B_n is a subset of the complete partially ordered set

$$Y_n = [y_n - \rho_n, y_n + \rho_n] \subset \mathbb{R}.$$

Let $s_n = \sup B_n$ in Y_n. We obtain a sequence $s \in S$. Obviously $s = \sup B$. □

5 Approximative solutions of continuous type equations

In this section, in Theorem 5.1, assuming the function f is continuous, we establish conditions under which for any $y \in \Delta^{-m}b$ such that $(y \circ \sigma)w = \mathrm{O}(1)$ there exists a solution x of (E) such that $x = y + \mathrm{o}(n^s)$. In Theorem 5.2 we obtain an analogous result with a full solution x.

In this section we will use the following condition

(A) $\quad f$ is continuous, g is locally bounded,

$$s \in (-\infty, 0], \ \sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty \ \text{ and } \ w = \mathrm{O}(1).$$

Theorem 5.1. *Assume (A), $y \in \mathrm{SQ}$, $\Delta^m y = b$ and*

$$(y \circ \sigma)w = \mathrm{O}(1). \tag{5.1}$$

Then there exists a solution x of (E) such that $x = y + \mathrm{o}(n^s)$ and moreover,

$$\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$$

for any $k \in \mathbb{N}(0, m)$.

Proof. Recall that for $x \in \mathrm{SQ}$ we denote by $\bar{x}$ the sequence defined by

$$\bar{x}_n = f(n, x_{\sigma(n)}).$$

Let

$$T = \{x \in \mathrm{SQ} : |x - y| \leq 1\}.$$

By (A) and (5.1), there exists a constant K such that if $x \in T$ and $n \in \mathbb{N}$, then

$$\begin{aligned} |w_n x_{\sigma(n)}| &= |w_n x_{\sigma(n)} - w_n y_{\sigma(n)} + w_n y_{\sigma(n)}| \\ &\leq |w_n||x_{\sigma(n)} - y_{\sigma(n)}| + |w_n y_{\sigma(n)}| \leq K. \end{aligned}$$

By (A) there exists $M > 0$ such that $g([0, K]) \subset [0, M]$. Therefore, using (G) we have

$$g(|w_n x_{\sigma(n)}|) \leq M \ \text{ and } \ |\bar{x}_n| \leq g(|x_{\sigma(n)} w_n|) \leq M \tag{5.2}$$

for $x \in T$ and $n \in \mathbb{N}$. There exists $p \geq 1$ such that $M r_n^m |a| \leq 1$ for $n \geq p$. Let

$$\mu(n) = 0 \ \text{ for } \ n < p, \ \ \mu(n) = 1 \ \text{ for } \ n \geq p, \ \ \rho = \mu M r^m |a|.$$

Let $S \subset \mathrm{SQ}$ and $A\colon S \to \mathrm{SQ}$ be defined by

$$S = \{x \in \mathrm{SQ} : \ |x - y| \leq \rho\}, \qquad A(x) = y + (-1)^m \mu r^m (a\bar{x}).$$

Then $S \subset T$. If $x \in S$, then, using Lemma 4.1 (a) and (h), we get

$$|Ax - y| = |\mu r^m (a\bar{x})| \leq \mu r^m |a\bar{x}| \leq \rho.$$

Hence $A(S) \subset S$. Choose $\varepsilon > 0$. There exist $q \geq p$ and $\alpha > 0$ such that

$$2M \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon \quad \text{and} \quad \alpha q^{m-1} \sum_{n=1}^{q} |a_n| < \varepsilon. \tag{5.3}$$

Let

$$L = \max\{|y_{\sigma(n)} - y_n| : n \in \mathbb{N}(1,q)\}, \quad W = \{(n,t) \in \mathbb{R}^2 : n \in \mathbb{N}(1,q), |t - y_n| \le L+1\}.$$

By compactness of W, the function f is uniformly continuous on W. Hence, there exists $\delta > 0$ such that if $(n,s),(n,t) \in W$ and $|s-t| < \delta$, then $|f(n,s) - f(n,t)| < \alpha$. Assume $x,z \in S$, $|x-z| < \delta$, and $u = \bar{x} - \bar{z}$. Then $|Ax - Az| = |\mu r^m(au)|$. Using Lemma 4.1 (a) and (c) we get

$$d(Ax,Az) = \sup_{n\in\mathbb{N}} |Ax_n - Az_n| = \sup_{n\in\mathbb{N}} |r_n^m(au)| \le \sup_{n\in\mathbb{N}} r_n^m|au| \le \sum_{n=1}^{\infty} n^{m-1}|a_n u_n|.$$

Hence

$$d(Ax,Az) \le \sum_{n=1}^{q} n^{m-1}|a_n u_n| + \sum_{n=q}^{\infty} n^{m-1}|a_n u_n|. \tag{5.4}$$

By (5.2), $|u| \le 2M$. If $n \in \mathbb{N}(1,q)$, then

$$|x_{\sigma(n)} - y_n| \le |x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)} - y_n| \le \rho(n) + L \le L+1.$$

Hence $(n, x_{\sigma(n)}) \in W$. Analogously $(n, z_{\sigma(n)}) \in W$ and $|u_n| \le \alpha$ for $n \le q$. By (5.3) and (5.4) we get

$$d(Ax,Az) \le \alpha q^{m-1} \sum_{n=1}^{q} |a_n| + 2M \sum_{n=q}^{\infty} n^{m-1}|a_n| < \varepsilon + \varepsilon.$$

Thus the map A is continuous and, by Lemma 4.7, there exists a sequence $x \in S$ such that $Ax = x$. Then

$$x_n = y_n + (-1)^m r_n^m(a_n f(n, x_{\sigma(n)})) \ \text{ for } \ n \ge p$$

and, by Lemma 4.1 (d), $\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n$ for $n \ge p$. Now, the assertion follows from Lemma 4.2. □

Theorem 5.2. *Assume* (A)*, the function* g *is bounded,* $y \in \mathrm{SQ}$*,* $\Delta^m y = b$ *and*

$$w(y \circ \sigma) = \mathrm{O}(1).$$

Then there exists a full solution x *of* (E) *such that*

$$\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$$

for any $k \in \mathbb{N}(0,m)$*.*

Proof. By (2.1) and (G), there exists a constant M such that $|\bar{x}| \le M$ for every $x \in \mathrm{SQ}$. Let $\rho = r^m(|a|M + |b|)$,

$$S = \{x \in \mathrm{SQ} : |x - y| \le \rho\}, \quad A(x) = y + (-1)^m r^m(a\bar{x} + b).$$

Similarly as in the proof of Theorem 5.1 it can be shown that there exists a sequence $x \in S$ such that $Ax = x$. Then

$$x = y + (-1)^m r^m(a\bar{x}).$$

Hence

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \ \text{ for any } n.$$

This means that x is a full solution of (E). Moreover, by Lemma 4.2, we have $\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$ for any $k \in \mathbb{N}(0,m)$. □

Corollary 5.3. *Assume* (A), $\varphi \in \operatorname{Pol}(m-1)$, $w(\varphi \circ \sigma) = \mathrm{O}(1)$ *and*

$$\sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

Then there exists a solution x *of* (E) *such that*

$$\Delta^k x = \Delta^k \varphi + \mathrm{o}(n^{s-k}) \quad \text{for} \quad k \in \mathbb{N}(0, m).$$

Moreover, if g *is bounded, then we may assume* x *is full.*

Proof. By Lemma 4.2, $b \in \mathrm{S}(m)$ and $r^m b = \mathrm{o}(n^s)$. Let $u = (-1)^m r^m b$ and $y = \varphi + u$. Then

$$\Delta^m y = \Delta^m \varphi + \Delta^m u = \Delta^m u = b.$$

Let $k \in \mathbb{N}(0, m)$. By Lemma 4.2 we have $\Delta^k u = \mathrm{o}(n^{s-k})$ and by Theorem 5.1 there exists a solution x of (E) such that

$$\begin{aligned} \Delta^k x &= \Delta^k y + \mathrm{o}(n^{s-k}) = \Delta^k \varphi + \Delta^k u + \mathrm{o}(n^{s-k}) \\ &= \Delta^k \varphi + \mathrm{o}(n^{s-k}) + \mathrm{o}(n^{s-k}) = \Delta^k \varphi + \mathrm{o}(n^{s-k}). \end{aligned}$$

If g is bounded, then by Theorem 5.2, we can assume x is full. $\square$

Remark 5.4. Note that if $w_n = n^{-k}$ for certain $k \in \mathbb{N}(0)$, then the condition $w(y \circ \sigma) = \mathrm{O}(1)$ takes the form $y \circ \sigma = \mathrm{O}(n^k)$. Corollary 5.3, in the case $w_n = n^{-k}$ generalizes Theorem 3.1 of [29].

Example 5.5. Let $f(n,t) = t^2/n^2$, $g(t) = t^2$, $w_n = 1/n$, $a_n = 1/n^3$, $b_n = 0$, $\alpha > 0$, $\varphi(n) = \alpha n$ and $\sigma(n) = n$. Consider the equation

$$\Delta^2 x_n = a_n f(n, x_n) + b_n = \frac{x_n^2}{n^5} \tag{E1}$$

Then $|f(n,t)| = (t/n)^2 = g(|tw_n|)$ and by Corollary 5.3, the sequence φ is asymptotically equivalent to a certain solution x of the equation (E1) (i.e., $x - \varphi = \mathrm{o}(1)$).
Suppose now that $p \in \mathbb{N}$, $\Delta^2 x_n = x_n^2/n^5$ for $n \geq p$ and there exists a sequence $u = \mathrm{o}(1)$ such that $x = \varphi + u$. Then

$$\Delta^2 u = \Delta^2 u + \Delta^2 \varphi = \Delta^2(u + \varphi) = \Delta^2 x \geq 0 \;\text{ for }\; n \geq p.$$

Hence Δu_n is convergent to zero and nondecreasing for $n \geq p$. Therefore $\Delta u_n \leq 0$ for $n \geq p$. Hence the sequence u_n is convergent to zero and nonincreasing for $n \geq p$. Therefore $u_n \geq 0$ for $n \geq p$. Moreover,

$$u_{n+2} - 2u_{n+1} + u_n = \Delta^2 u_n = \Delta^2 x_n = \frac{x_n^2}{n^5} = \frac{\alpha^2 n^2 + 2\alpha n u_n + u_n^2}{n^5}$$

for $n \geq p$. Hence for $n \geq p$ we obtain

$$u_{n+2} - u_{n+1} = u_{n+1} - u_n + \frac{\alpha^2}{n^3} + \frac{2\alpha u_n}{n^4} + \frac{u_n^2}{n^5} \geq \left(\frac{1}{n^5}u_n^2 - u_n\right) + \frac{\alpha^2}{n^3}.$$

Since $\lambda t^2 - t \geq -1/4\lambda$ for every $t \in \mathbb{R}$, we obtain

$$0 \geq \Delta u_{n+1} = u_{n+2} - u_{n+1} \geq \frac{\alpha^2}{n^3} - \frac{n^5}{4}$$

for $n \geq p$. If $2\alpha > p^4$ then $4\alpha^2 > p^8$. Hence $\alpha^2/p^3 - p^5/4 > 0$ and we obtain $0 \geq \Delta u_{p+1} > 0$ which is impossible. Hence if $2\alpha > p^4$ then the sequence $\varphi(n) = \alpha n$ is not asymptotically equivalent to any sequence (x_n) fulfilling the equation (E1) for every $n \geq p$. In particular if $\alpha > 1/2$ then the sequence $\varphi(n) = \alpha n$ is not asymptotically equivalent to any full solution of (E1).

6 Approximative solutions of monotone type equations

In this section we obtain results analogous to that obtained in Section 5. We replace the continuity of f by monotonicity of f with respect to second variable.

We will use the following conditions

(B) g is locally bounded,

$$w = \mathrm{O}(1), \qquad s \in (-\infty, 0], \qquad \sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$$

and one of the following conditions is satisfied:

- (a) f is nondecreasing with respect to the second variable and $(-1)^m a_n \geq 0$ for all large n,
- (b) f is nonincreasing with respect to the second variable and $(-1)^m a_n \leq 0$ for all large n.

(C) one of the following conditions is satisfied:

- (c) f is nondecreasing with respect to the second variable and $(-1)^m a_n \geq 0$ for all n,
- (d) f is nonincreasing with respect to the second variable and $(-1)^m a_n \leq 0$ for all n.

Theorem 6.1. *Assume (B), $y \in \mathrm{SQ}$, $\Delta^m y = b$ and*

$$w(y \circ \sigma) = \mathrm{O}(1).$$

Then there exists a solution x of (E) such that

$$\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$$

for any $k \in \mathbb{N}(0, m)$.

Proof. Assume that the condition (a) is fulfilled. The proof in the case (b) is analogous. We define the sets T, S, the index p and the operator A as in the proof of Theorem 5.1. We may assume $(-1)^m a_n \geq 0$ for $n \geq p$. Similarly as in the proof of Theorem 5.1 it can be shown that $A(S) \subset S$. Assume $x, z \in S$ and $x \leq z$. Then

$$(-1)^m a_n f(n, x_{\sigma(n)}) + b_n \leq (-1)^m a_n f(n, z_{\sigma(n)}) + b_n$$

for $n \geq p$. Since the operator r^m is nondecreasing, $A(x) \leq A(z)$. By Lemma 4.9, there exists $x \in S$ such that $A(x) = x$. The rest of the proof is analogous to the proof of the Theorem 5.1. $\square$

Theorem 6.2. *Assume* (B), (C), *the function* g *is bounded,* $y \in \mathrm{SQ}$, $\Delta^m y = b$ *and*

$$w(y \circ \sigma) = \mathrm{O}(1).$$

Then there exists a full solution x *of* (E) *such that*

$$\Delta^k x = \Delta^k y + \mathrm{o}(n^{s-k})$$

for any $k \in \mathbb{N}(0, m)$.

Proof. There exists a constant M such that $|\bar{x}| \le M$ for any $x \in \mathrm{SQ}$. Let

$$\rho = r^m(|a|M + |b|), \qquad S = \{x \in \mathrm{SQ} : |x - y| \le \rho\}, \qquad A(x) = y + (-1)^m r^m(a\bar{x}).$$

Similarly as in the proof of Theorem 5.2 it can be shown that there exists a sequence $x \in S$ such that $A(x) = x$. Then

$$x = y + (-1)^m r^m(a\bar{x}).$$

Hence

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n \text{ for any } n.$$

Now, the assertion follows from Lemma 4.2. □

Corollary 6.3. *Assume* (B), $\varphi \in \mathrm{Pol}(m-1)$, $w(\varphi \circ \sigma) = \mathrm{O}(1)$ *and*

$$\sum_{n=1}^{\infty} n^{m-1-s}|b_n| < \infty.$$

Then there exists a solution x *of* (E) *such that*

$$\Delta^k x = \Delta^k \varphi + \mathrm{o}(n^{s-k}) \;\; \textit{for} \;\; k \in \mathbb{N}(0, m).$$

Moreover, if g *is bounded and condition* (C) *is satisfied, then we may assume* x *is full.*

Proof. See the proof of Corollary 5.3. □

7 Approximations of solutions

In this section we obtain the conditions under which all solutions of (E) are asymptotically polynomial or 'translated' asymptotically polynomial. In Theorem 7.4 we use a certain discrete version of the Bihari lemma and the 'small difference method' (Lemma 3.11). In Theorem 7.5 we use the 'small difference method' directly. Next, as a consequence of Theorem 7.5, we establish conditions under which all bounded solutions of (E) are convergent (or asymptotically periodic). We obtain also conditions under which all solutions of (E) are unbounded.

The first three lemmas are used in the proof of Theorem 7.4. In Lemma 7.1 we extend the discrete version of the Bihari lemma obtained by Demidovič in Theorem 1 of [7]. Note that we do not assume continuity of g. Lemma 7.2 is a consequence of Lemma 7.1.

Lemma 7.1. *Assume* a, u *are nonnegative sequences,* $b, c \in \mathbb{R}$, $p, m \in \mathbb{N}$, $0 \le b < c$, $p \le m$, $g\colon [0, \infty) \to [0, \infty)$ *is nondecreasing,* $g(b) > 0$,

$$\sum_{k=p}^{m-1} a_k \le \int_b^c \frac{dt}{g(t)} \;\; \textit{and} \;\; u_n \le b + \sum_{k=p}^{n-1} a_k g(u_k) \;\; \textit{for} \;\; n = p, \dots, m.$$

Then $u_n \le c$ *for* $n = p, \dots, m$.

Proof. For $n \in \mathbb{N}(p,m)$ let

$$s_n = b + \sum_{k=p}^{n-1} a_k g(u_k).$$

Then

$$\Delta s_n = s_{n+1} - s_n = a_n g(u_n) \le a_n g(s_n), \quad \frac{\Delta s_n}{g(s_n)} \le a_n \ \text{ for } \ n \in \mathbb{N}(p, m-1).$$

Since

$$g(s_k) \le g(s), \quad \frac{1}{g(t)} \le \frac{1}{g(s_k)} \ \text{ for } \ t \in [s_k, s_{k+1}],$$

we obtain

$$\int_{s_k}^{s_{k+1}} \frac{dt}{g(t)} \le \int_{s_k}^{s_{k+1}} \frac{dt}{g(s_k)} = \frac{\Delta s_k}{g(s_k)} \le a_k.$$

Hence

$$\int_{b}^{s_n} \frac{dt}{g(t)} = \sum_{k=p}^{n-1} \int_{s_k}^{s_{k+1}} \frac{dt}{g(t)} \le \sum_{k=p}^{m-1} a_k \le \int_{b}^{c} \frac{dt}{g(t)}.$$

Since the function g is strictly positive on $[b, \infty)$, we obtain $s_n \le c$. Hence $u_n \le s_n \le c$. □

Lemma 7.2. *Assume a, u are nonnegative sequences, $p \in \mathbb{N}$, $\lambda, \mu > 0$, and $b \ge 0$. Let $g\colon [0,\infty) \to [0,\infty)$ be nondecreasing, $g(b) > 0$,*

$$\sum_{k=0}^{\infty} a_k < \infty, \quad \int_{b}^{\infty} \frac{dt}{g(t)} = \infty, \ \text{ and } \ u_n \le b + \lambda \sum_{k=p}^{n-1} a_k g(\mu u_k) \ \text{ for } \ n \ge p.$$

Then the sequence u is bounded.

Proof. For $n \ge p$ we have

$$\mu u_n \le \mu b + \sum_{k=p}^{n-1} \lambda \mu a_k g(\mu u_k).$$

Obviously

$$g(\mu b) > 0, \quad \sum_{k=0}^{\infty} \lambda \mu a_k < \infty \ \text{ and } \ \int_{\mu b}^{\infty} dt/g(t) = \infty.$$

There exists $c > \mu b$ such that

$$\sum_{k=p}^{\infty} \lambda \mu a_k < \int_{\mu b}^{c} \frac{dt}{g(t)}.$$

Hence, by Lemma 7.1, we obtain $\mu u_n \le c$ for $n \ge p$. □

Lemma 7.3. *If x is a sequence of real numbers, $m \in \mathbb{N}(1)$ and $p \in \mathbb{N}(m)$ then there exists a positive constant $L = L(x, p, m)$ such that*

$$|x_n| \le n^{(m-1)} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right) \ \text{ for } \ n \ge p.$$

Proof. Induction over m. If $m = 1$, then

$$x_n = x_p + \sum_{i=p}^{n-1} \Delta x_i, \qquad |x_n| \leq |x_p| + \sum_{i=p}^{n-1} |\Delta x_i|.$$

Assume the assertion is true for certain $m \geq 1$. Then

$$|\Delta x_n| \leq n^{(m-1)} \left(L + \sum_{i=p}^{n-1} |\Delta^{m+1} x_i| \right) \quad \text{for} \quad n \geq p.$$

Hence for $n \geq p$ we obtain

$$\begin{aligned}
|x_n| &\leq |x_p| + \sum_{i=p}^{n-1} |\Delta x_i| \leq |x_p| + \sum_{i=p}^{n-1} i^{(m-1)} \left(L + \sum_{j=p}^{i-1} |\Delta^{m+1} x_j| \right) \\
&\leq |x_p| + (n-1)^{(m-1)} \sum_{i=p}^{n-1} \left(L + \sum_{j=p}^{n-1} |\Delta^{m+1} x_j| \right) \\
&\leq n^{(m)} |x_p| + (n-1)^{(m-1)} n \left(L + \sum_{j=p}^{n-1} |\Delta^{m+1} x_j| \right) = n^{(m)} \left(|x_p| + L + \sum_{j=p}^{n-1} |\Delta^{m+1} x_j| \right).
\end{aligned}$$

□

Theorem 7.4. *Assume $s \in (-\infty, m-1]$, $w = \mathrm{O}(n^{1-m})$, g is nondecreasing, $g(t) > 0$ for $t > 1$, $\sigma(n) \leq n$,*

$$\sum_{n=0}^{\infty} n^{m-1-s} |a_n| < \infty, \quad \sum_{n=0}^{\infty} n^{m-1-s} |b_n| < \infty, \quad \int_1^{\infty} \frac{dt}{g(t)} = \infty$$

and x is a solution of (E). Then

$$\Delta^m x = \mathrm{O}(|a| + |b|) \ \text{ and } \ x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s).$$

Moreover, if $s = k \in \mathbb{N}(0, m-1)$, then

$$x \in \mathrm{Pol}(m-1) + \Delta^{-k} \mathrm{o}(1).$$

Proof. Choose $M > 0$ such that $|w_n| n^{m-1} \leq M$. Then $|w_n| n^{(m-1)} \leq M$. By assumption

$$|\Delta^m x_n| = |a_n f(n, x_{\sigma(n)}) + b_n| \leq |a_n| |f(n, x_{\sigma(n)})| + |b_n| \leq |a_n| |g(|w_n x_{\sigma(n)}|)| + |b_n|.$$

By Lemma 7.3, there exists a positive constant L such that

$$|x_{\sigma(n)}| \leq \sigma(n)^{(m-1)} \left(L + \sum_{i=p}^{\sigma(n)-1} |\Delta^m x_i| \right) \leq n^{(m-1)} \left(L + \sum_{i=p}^{n-1} |\Delta^m x_i| \right).$$

Hence

$$|w_n x_{\sigma(n)}| \leq ML + M \sum_{j=1}^{n-1} |\Delta^m x_j|.$$

Then

$$|w_n x_{\sigma(n)}| \leq ML + M \sum_{j=1}^{n-1} |a_j| g(|w_j x_{\sigma(j)}|) + M \sum_{j=1}^{n-1} |b_j| \leq K + M \sum_{j=1}^{n-1} |a_j| g(|w_j x_{\sigma(j)}|).$$

Obviously $\int_K^\infty g(t)^{-1}\,dt = \infty$. By Lemma 7.2, the sequence $(w_n x_{\sigma(n)})$ is bounded. Choose $Q > 0$ such that $|w_n x_{\sigma(n)}| \le Q$ for every n. Choose $P \ge 1$ such that $g(Q) \le P$. Then $g(|w_n x_{\sigma(n)}|) \le P$ for every n. Hence

$$|\Delta^m x_n| \le |a_n| g(|w_n x_{\sigma(n)}|) + |b_n| \le P|a_n| + |b_n| \le P(|a_n| + |b_n|).$$

Therefore $\Delta^m x = \mathrm{O}(|a| + |b|)$. Now the conclusion follows from Lemma 3.11. □

Theorem 7.4 extends Theorem 4.1 of [29].

Theorem 7.5. *Assume x is a solution of* (E),

$$s \in (-\infty, m-1], \quad k \in \mathbb{N}(0, m-1), \quad c = |a| + |b|, \quad p \in \mathbb{N}(1), \quad X \subset \mathbb{R}$$

and one of the following conditions is satisfied:

(1) *the sequence $\bar{x}_n = f(n, x_{\sigma(n)})$ is bounded,*

(2) *f is bounded on $\mathbb{N}(p) \times X$ and $x_{\sigma(n)} \in X$ for large n,*

(3) *f is bounded on $\mathbb{N}(p) \times X$ and $x_n \in X$ for large n,*

(4) *f is bounded,*

(5) *g is locally bounded and the sequence $(x \circ \sigma)w$ is bounded.*

Then

(a) *if* $\sum_{n=1}^\infty n^{m-1-s}|a_n| < \infty$, *then* $x \in \Delta^{-m} b + \mathrm{o}(n^s)$,

(b) *if* $\sum_{n=1}^\infty n^{m-1-s}|c_n| < \infty$, *then* $x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s)$,

(c) *if* $\sum_{n=1}^\infty n^{m-1-k}|a_n| < \infty$, *then* $x \in \Delta^{-m} b + \Delta^{-k}\mathrm{o}(1)$,

(d) *if* $\sum_{n=1}^\infty n^{m-1-k}|c_n| < \infty$, *then* $x \in \mathrm{Pol}(m-1) + \Delta^{-k}\mathrm{o}(1)$.

Proof. Obviously $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. Assume (5). Then the sequence $z_n = g(|w_n x_{\sigma(n)}|)$ is bounded and

$$|\bar{x}_n| = |f(n, x_{\sigma(n)})| \le g(|w_n x_{\sigma(n)}|).$$

Hence $(5) \Rightarrow (1)$. If the sequence $\bar{x}$ is bounded, then by the equality $\Delta^m x_n = a_n \bar{x}_n + b_n$ for large n we obtain $\Delta^m x = \mathrm{O}(a) + b$. Hence the assertion follows from Lemma 3.11. □

Corollary 7.6. *Assume $w = \mathrm{O}(1)$, $s \in (-\infty, 0]$,*

$$\sum_{n=1}^\infty n^{m-1-s}|a_n| < \infty, \qquad \sum_{n=1}^\infty n^{m-1-s}|b_n| < \infty$$

and g is locally bounded. Then every bounded solution of (E) *is convergent. More precisely, for every bounded solution x of* (E) *there exists a constant λ such that $x = \lambda + \mathrm{o}(n^s)$.*

Proof. Let x be a bounded solution of (E). By Theorem 7.5, $x \in \mathrm{Pol}(m-1) + \mathrm{o}(n^s)$. Hence $x = \varphi + \mathrm{o}(n^s)$ for certain $\varphi \in \mathrm{Pol}(m-1)$. Moreover, $\varphi = x - \mathrm{o}(n^s)$ is bounded. Therefore $\varphi = \lambda$ for certain $\lambda \in \mathbb{R}$. □

Lemma 7.7. *Assume $m, q \in \mathbb{N}(1)$, $b \in \mathrm{SQ}$ is q-periodic and*

$$b_1 + b_2 + \cdots + b_q = 0.$$

Then there exists a q-periodic sequence $y \in \mathrm{SQ}$ such that $\Delta^m y = b$.

Proof. If $c \in \mathrm{SQ}$ is defined by

$$c_1 = 0 \text{ and } c_n = b_1 + \cdots + b_{n-1} \text{ for } n > 1,$$

then c is q-periodic and $\Delta c = b$. Let

$$\alpha = \frac{c_1 + \cdots + c_q}{q}$$

and let $d \in \mathrm{SQ}$ be defined by $d_n = c_n - \alpha$. Then d is q-periodic, $\Delta d = b$ and moreover

$$d_1 + d_2 + \cdots + d_q = c_1 - \alpha + c_2 - \alpha + \cdots + c_q - \alpha = c_1 + \cdots + c_q - q\alpha = 0.$$

Analogously there exists a q-periodic sequence h such that

$$\Delta h = d \text{ and } h_1 + \cdots + h_q = 0.$$

Then $\Delta^2 h = \Delta\Delta h = \Delta d = b$ and so on. After m steps we obtain the required sequence y. □

Corollary 7.8. *Assume $w = \mathrm{O}(1)$, $s \in (-\infty, 0]$, $q \in \mathbb{N}(1)$, b is q-periodic,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad b_1 + \cdots + b_q = 0$$

and g is locally bounded. Then every bounded solution of (E) is asymptotically q-periodic. More precisely, for every bounded solution x of (E) there exists a q-periodic sequence u such that

$$x = u + \mathrm{o}(n^s).$$

Proof. Assume x is a bounded solution of (E). By Theorem 7.5, $x \in \Delta^{-m}b + \mathrm{o}(n^s)$. By Lemma 7.7 there exists a q-periodic sequence y such that $\Delta^m y = b$. By Remark 2.1 there exists a polynomial $\varphi \in \mathrm{Pol}(m-1)$ such that $x = y + \varphi + \mathrm{o}(n^s)$. Since $\varphi = x - y - \mathrm{o}(n^s)$ is bounded, there exists a constant λ such that $\varphi = \lambda$. Let $u = y + \lambda$. Then u is q-periodic and $x = u + \mathrm{o}(n^s)$. □

Corollary 7.9. *Assume $w = \mathrm{O}(1)$, $s \in (-\infty, m-1]$, $q \in \mathbb{N}(1)$, b is q-periodic,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad b_1 + \cdots + b_q \neq 0$$

and g is locally bounded. Then every solution of (E) is unbounded. More precisely, if x is a solution of (E) and

$$\lambda = \frac{b_1 + \cdots + b_q}{qm!},$$

then $x_n = \lambda n^m + \mathrm{O}(n^{m-1})$.

Proof. Let $\mu = m!\lambda$ and $d = b - \mu$. Then d is q-periodic and $d_1 + \cdots + d_q = 0$. Hence, by Lemma 7.7, there exists a q-periodic sequence u such that $\Delta^m u = d$. Moreover, $\Delta^m(\lambda n^m) = \mu$. Let $y_n = \lambda n^m + u_n$. Then $\Delta^m y = b$. Hence, by Theorem 7.5 and Remark 2.1, we have

$$x \in \Delta^{-m} b + \mathrm{o}(n^s) = y + \mathrm{Pol}(m-1) + \mathrm{o}(n^s).$$

Therefore $x_n = \lambda n^m + \mathrm{O}(n^{m-1})$. □

Corollary 7.10. *Assume $w = \mathrm{O}(1)$, $s \in (-\infty, 0]$, $u, z \in \mathrm{SQ}$, $u = \mathrm{O}(1)$, $z = \mathrm{o}(n^s)$,*

$$b = \Delta^m(u + z), \qquad \sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty$$

and g is locally bounded. Then for every bounded solution x of (E) there exists a constant $\lambda \in \mathbb{R}$ such that $x = \lambda + u + \mathrm{o}(n^s)$.

Proof. Assume x is a bounded solution of (E). By Theorem 7.5, $x \in \Delta^{-m} b + \mathrm{o}(n^s)$. By Remark 2.1 there exists a polynomial $\varphi \in \mathrm{Pol}(m-1)$ such that $x = u + z + \varphi + \mathrm{o}(n^s)$. Hence

$$x = u + \varphi + \mathrm{o}(n^s) + \mathrm{o}(n^s) = u + \varphi + \mathrm{o}(n^s).$$

Since $\varphi = x - u - \mathrm{o}(n^s)$ is bounded, there exists a constant λ such that $\varphi = \lambda$. □

Corollary 7.11. *Assume $w = \mathrm{O}(1)$, $s \in (-\infty, m-1]$, $d \in \Delta^m \mathrm{o}(n^m)$, $\mu \in \mathbb{R}$, $\mu \neq 0$,*

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad b = d - \mu$$

and g is locally bounded. Then every solution of (E) is unbounded. More precisely, if x is a solution of (E) and $\lambda = \mu/m!$ then $x_n = \lambda n^m + \mathrm{o}(n^m)$.

Proof. Choose $u = \mathrm{o}(n^m)$ such that $\Delta^m u = d$. Let $y = \lambda n^m + u$. Then $\Delta^m y = \mu + b - \mu = b$. Hence by Theorem 7.5 and Remark 2.1 we have

$$x \in \Delta^{-m} b + \mathrm{o}(n^s) = y + \mathrm{Pol}(m-1) + \mathrm{o}(n^s) = \lambda n^m + \mathrm{o}(n^m).$$

□

References

[1] R. P. Agarwal, *Difference Equations and Inequalities: theory, methods, and applications*, Second edition, Marcel Dekker, Inc., New York, 2000. MR1740241

[2] R. P. Agarwal, S. Djebali, T. Moussaoui, O. G. Mustafa, On the asymptotic integration of nonlinear differential equations, *J. Comput. Appl. Math.* **202**(2007), 352–376. MR2319962

[3] G. Birkhoff, *Lattice theory*, American Mathematical Society, Colloquium Publications XXV, 1940; third edition 1967. MR0227053

[4] M. Bohner S. Stević, Asymptotic behavior of second-order dynamic equations, *Appl. Math. Comput.* **188**(2007), No. 2, 1503–1512. MR1202198

[5] M. Bohner, S. Stević, Trench's perturbation theorem for dynamics equations, *Discrete Dyn. Nat. Soc. Vol.* **2007**, Art. ID 75672, 11 pp. MR2375475; url

[6] S. S. Cheng, W. T. Patula, An existence theorem for a nonlinear difference equation, *Nonlinear Anal.* **20**(1993), No. 3, 193–203. MR1202198

[7] V. B. Demidovič, A certain criterion for the stability of difference equations, (Russian), *Differ. Uravn.* **5**(1969), 1247–1255. MR0261199

[8] J. Diblík, A criterion of asymptotic convergence for a class of nonlinear differential equations with delay, *Nonlinear Anal.* **47**(2001), No. 6, 4095–4106. MR1972350

[9] A. Drozdowicz, M. Migda, Asymptotic behavior of solutions of the m-th order nonhomogeneous difference equations, *J. Comput. Appl. Math.* **81**(1997), 75–81. MR1459357

[10] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of the second order difference equations, *Proc. Amer. Math. Soc.* **99**(1987), No. 1, 135–140. MR0866443

[11] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of an n-th order difference equations, *Annales Soc. Math. Pol., Com. Math.* XXIX (1990), 161–168. MR1059121

[12] J. Džurina, Asymptotic behavior of solutions of neutral nonlinear differential equations, *Arch. Math. (Brno)* **38**(2002), No. 4, 319–325. MR1942662

[13] M. Ehrnström, Linear asymptotic behaviour of second order ordinary differential equations, *Glasgow Math. J.* **49**(2007) 105–120. MR2337871

[14] A. Gleska, J. Werbowski, Comparison theorems for the asymptotic behavior of solutions of nonlinear difference equations, *J. Math. Anal. Appl.* **226**(1998), No. 2, 456–465. MR1650201

[15] Z. Guo, M. Liu, Existence of non-oscillatory solutions for a higher-order nonlinear neutral difference equation, *Electron. J. Differential Equations* **2010**, No. 146, 1–7. MR2729467

[16] I. Győri, G. Ladas, *Oscillation theory of delay differential equations with applications*, Clarendon Press, Oxford, 1991. MR1168471

[17] T. G. Hallam, Asymptotic behavior of the solutions of an nth order nonhomogeneous ordinary differential equation, *Trans. Amer. Math. Soc.* **122**(1966), 177–194. MR0188562

[18] M. Hasanbulli, Y. V. Rogovchenko, Asymptotic behavior nonoscillatory solutions to n-th order nonlinear neutral differential equations, *Nonlinear Anal.* **69**(2008), 1208–1218. MR2426686

[19] J. W. Hooker, W. T. Patula, A second-order nonlinear difference equation: oscillation and asymptotic behavior, *J. Math. Anal. Appl.* **91**(1983), No. 1, 9–29. MR0688528

[20] X. Huang, Z. Xu, Nonoscillatory Solutions of Certain Higher Order Neutral Difference Equations, *Southeast Asian Bull. Math.* **32**(2008), 445–458. MR2416160

[21] Q. Kong, Asymptotic behavior of a class of nonlinear differential equations of nth order, *Proc. Amer. Math. Soc.* **103**(1988), 831–838. MR0947667

[22] W. T. Li, S. S. Cheng, Asymptotically linear solutions of a discrete Emden-Fowler equation, *Far East J. Math. Sci.* **6**(1998), No. 4, 521–542. MR1657586

[23] O. Lipovan, On the asymptotic behavior of the solutions to a class of second order nonlinear differential equations, *Glasgow Math. J.* **45**(2003), 179–187. MR1973349

[24] M. Liu, Z. Guo, Solvability of a higher-order nonlinear neutral delay difference equation, *Adv. Difference Equ.* **2010**, Art. ID 767620, 14 pp. MR2727276; url

[25] M. Migda, J. Migda, On the asymptotic behavior of solutions of higher order nonlinear difference equations, *Nonlinear Anal.* **47**(2001), 4687–4695. MR1975862

[26] M. Migda, J. Migda, On a class of first order nonlinear difference equations of neutral type, *Math. Comput. Modelling* **40**(2004), 297–306. MR2091062

[27] M. Migda, J. Migda, Oscillatory and Asymptotic Properties of Solutions of Even Order Neutral Difference Equations, *J. Difference Equ. Appl.* **15**(2009), No. 11–12, 1077–1084. MR2569136

[28] J. Migda, Asymptotic Properties of Solutions of Nonautonomous Difference Equations, *Arch. Math. (Brno)* **46**(2010), 1–11. MR2644450

[29] J. Migda, Asymptotically polynomial solutions of difference equations, *Adv. Difference Equ.* **2013**, No. 92, 1–16. MR3053806; url

[30] M. Migda, Asymptotic properties of nonoscillatory solutions of higher order neutral difference equations, *Opuscula Math.* **26**(2006), No. 3, 497–504. MR2280277

[31] A. B. Mingarelli, K. Sadarangani, Asymptotic solutions of forced nonlinear second order differential equations and their extensions, *Electron. J. Differential Equations* **2007**, No. 40, 1–40. MR2299594

[32] O. G. Mustafa, Y. V. Rogovchenko, Asymptotic integration of a class of nonlinear differential equations, *Appl. Math. Lett.* **19**(2006), 849–853. MR2240473

[33] Ch. G. Philos, I. K. Purnaras, P. Ch. Tsamatos, Asymptotic to polynomials solutions for nonlinear differential equations, *Nonlinear Anal.* **59**(2004), 1157–1179. MR2098511

[34] J. Popenda, J. Werbowski, On the asymptotic behaviour of the solutions of difference equations of second order, *Comment. Math. Prace Mat.* **22**(1980), 135–142. MR0610341

[35] J. Popenda, Asymptotic properties of solutions of difference equations, *Proc. Indian Acad. Sci. Math. Sci.* **95**(1986), No. 2, 141–153. MR0913886

[36] J. Popenda, E. Schmeidel, On the asymptotic behaviour of nonhomogeneous linear difference equations, *Indian J. Pure Appl. Math.* **28**(1997), No. 3, 319–327. MR1442841

[37] Y. V. Rogovchenko, G. Villari, Asymptotic behaviour of solutions for second order nonlinear autonomous differential equation, *Nonlinear Differential Equations Appl.* **4**(1997), 271–282. MR1446220

[38] S. Stević, Asymptotic behaviour of second-order difference equation, *ANZIAM J.* **46**(2004), No. 1, 157–170. MR2075520

[39] E. Thandapani, R. Arul, J. R. Graef, P. W. Spikes, Asymptotic behavior of solutions of second order difference equations with summable coefficients, *Bull. Inst. Math. Acad. Sinica* **27**(1999), 1–22. MR1681601

[40] W. F. Trench, Asymptotic behavior of solutions of an n-th order differential equation, *Rocky Mountain J. Math.* **14**(1984), No. 2, 441–450. MR0747291

[41] Z. Wang, J. Sun, Asymptotic behavior of solutions of nonlinear higher-order neutral type difference equations, *J. Difference Equ. Appl.* **12**(2006), 419–432. MR2241385

[42] A. Zafer, Oscillatory and asymptotic behavior of higher order difference equations, *Math. Comput. Modelling* **21**(1995), 43–50. MR1317929

[43] Z. Q. Zhu, G. Q. Wang, S. S. Cheng, A classification scheme for nonoscillatory solutions of a higher order neutral difference equation, *Adv. Difference Equ.* **2006**, Art. ID 47654, 1–19. MR2209676; url