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Application of the omitted ray fixed point theorem

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Abstract. This paper presents a nontrivial application of the omitted ray fixed point theorem. Existence of solutions arguments to nonlinear boundary value problems utilizing the Krasnoselskii fixed point theorem, Leggett–Williams fixed point theorem and their functional generalizations are characterized by mapping portions of an inward boundary inward and portions of an outward boundary outward. In this application we demonstrate a technique that avoids requiring any portion of the inward boundary being mapped inward using the omitted ray fixed point theorem.

Keywords: fixed-point theorems, omitted ray, Altman, Leggett–Williams.

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1 Introduction

To prove the existence of solutions to nonlinear differential equations with specified boundary conditions, researchers often reformulate the problem using an integral operator, say T, the fixed points of which are solutions to the original boundary value problem. Using the Krasnoselskii [8] fixed point theorem to show the existence of at least one solution to a boundary value problem requires the inward boundary of a conical region to be mapped inward by some mapping β, that is,

$$\beta(Tx) < \beta(x) \text{ for all } \beta(x) = b.$$

Similarly, applications of the Leggett–Williams fixed point theorem [10] and their functional generalizations [2, 3, 5] require

$$\beta(Tx) < \beta(x) \quad \text{for all } \beta(x) = b \text{ with } \alpha(x) \geq a,$$

again requiring portions of the inward boundary to be mapped inward. The recent omitted ray fixed point theorem [4] provides flexibility by not requiring the inward boundary to be mapped inward, but instead requiring

$$\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x) \quad \text{for all } \beta(x) = b \text{ with } \alpha(x) \geq a$$

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for a certain kind of functional mapping γ.

Other work in this area include [6], which provides new criteria for the existence of nontrivial fixed points on cones assuming some monotonicity of the operator on a suitable conical shell, and [7], which provides new sufficient conditions for the existence of multiple fixed points for a map between ordered Banach spaces.

Our main result below illustrates one successful approach to applying the omitted ray fixed point theorem to guarantee the existence of at least one solution to a second-order nonlinear right-focal boundary value problem. We end the paper with an example where no portion of the inward boundary is mapped inward, thus highlighting the relaxed conditions and potent applicability of the omitted ray technique.

2 Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a *nonnegative continuous concave functional* on a cone P of a real Banach space E if $\alpha\colon P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a *nonnegative continuous convex functional* on a cone P of a real Banach space E if $\beta\colon P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. We say the map γ is a *continuous sub-homogeneous functional* on a real Banach space E if $\gamma\colon E \to \mathbb{R}$ is continuous and

$$\gamma(tx) \leq t\gamma(x) \;\text{ for all }\; x \in E, \;\; t \in [0,1] \;\text{ and }\; \gamma(0) = 0.$$

Similarly we say the map ρ is a *continuous super-homogeneous functional* on a real Banach space E if $\rho\colon E \to \mathbb{R}$ is continuous and

$$\rho(tx) \geq t\rho(x) \;\text{ for all }\; x \in E, \;\; t \in [0,1] \;\text{ and }\; \rho(0) = 0.$$

Let ψ and δ be nonnegative continuous functionals on P; then, for positive real numbers a and b, we define the following sets:

$$P(\psi, b) = \{x \in P : \psi(x) < b\}$$

and

$$P(\delta, \psi, b, a) = P(\delta, b) - \overline{P(\psi, a)} = \{x \in P : a < \psi(x) \text{ and } \delta(x) < b\}.$$

The following theorem is the omitted ray fixed point theorem [4], which utilizes a functional version of Altman's condition [1] applying the techniques found in the Leggett–Williams fixed point theorem [10] and generalizations of the Leggett–Williams fixed point theorem [2, 3, 5].

Theorem 2.4. *Suppose P is a cone in a real Banach space E, α and κ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ and ψ are continuous super-homogeneous functionals on E, and $T\colon P \to P$ is a completely continuous operator. Furthermore, suppose that there exist nonnegative numbers a, b, c and d and $x_0, x_1 \in P$ such that*

$(A1)$ $x_0 \in \{x \in P \,:\, a \leq \alpha(x) \text{ and } \beta(x) < b\}$;

$(A2)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;

$(A3)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;

$(A4)$ $x_1 \in \{x \in P \,:\, c < \kappa(x) \text{ and } \theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;

$(A5)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;

$(A6)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$.

If

$(H1)$ $\overline{P(\kappa, c)} \subsetneq P(\beta, b)$, *then* T *has a fixed point* $x \in P(\beta, \kappa, b, c)$,

whereas, if

$(H2)$ $\overline{P(\beta, b)} \subsetneq P(\kappa, c)$, *then* T *has a fixed point* $x \in P(\kappa, \beta, c, b)$.

3 Application

In this section we illustrate a nontrivial technique for verifying the existence of a positive solution for a right-focal boundary value problem using the omitted ray fixed point theorem that does not require any portion of the inward boundary to be mapped inward.

To proceed, consider the second-order nonlinear right-focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{3.1}$$

$$x(0) = 0 = x'(1), \tag{3.2}$$

where $f\colon \mathbb{R} \to [0, \infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s))\, ds,$$

where

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1]$$

is the Green's function for the operator L defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (3.1), (3.2). Throughout this section of the paper we will use the following facts, namely that $G(t,s)$ is nonnegative and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in t.

Define the cone $P \subset E = C[0,1]$, where E is equipped with the supremum norm, by

$$P := \{x \in E \ : \ x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\}.$$

Thus if $x \in P$ and $\nu \in (0,1)$, then by the concavity of x we have $x(\nu) \geq \nu x(1)$ since

$$\frac{x(\nu) - x(0)}{\nu - 0} \geq \frac{x(1) - x(0)}{1 - 0}.$$

In the following application we demonstrate how to use the maximum of the sum of functions principle and the minimum of the sum of functions principle to verify the inequalties that characterize the omitted ray fixed point theorem (Theorem 2.4). That is, one can show that

$$\max_{t \in I}(Tx - x_0) < \max_{t \in I}(x - x_0) + \max_{t \in I}(Tx - x)$$

by showing that two of the three maximums are achieved at different points – so for example, the inequality would be verified if

$$\max_{t \in I}(x - x_0) = (x - x_0)(t_0) \ \text{ and } \ \max_{t \in I}(Tx - x) = (Tx - x)(t_1)$$

with $t_0 \neq t_1$, $(x - x_0)(t_0) \neq (x - x_0)(t_1)$ and $(Tx - x)(t_1) \neq (Tx - x)(t_0)$. Note that in this application we verify the existence of a solution to the boundary value problem (3.1), (3.2) with the property that

$$b < x^* \left(\frac{1}{2}\right) < c. \tag{3.3}$$

It is important to note that in the proof of the omitted ray fixed point theorem one proves that the index of $P(\beta, b)$ is one and the index of $P(\kappa, c)$ is zero, so in the following application we can also say that there is a solution x^{**} with

$$\beta(x^{**}) = x^{**} \left(\frac{1}{2}\right) < b.$$

It is also worthy of note that there are other fixed point theorems that could be utilized to show the existence of a fixed point using conditions (a) and (b), in particular, by applying Theorem 2.13 of Lan [9] any function that satisfies conditions (a) and (b) below has a solution x^{***} with

$$\|x^{***}\| < \frac{13b}{8}.$$

However, Lan's results would not yield a solution with the property

$$b < x^{**} \left(\frac{1}{2}\right) < c$$

using conditions (c) and (d) below since to apply Theorem 2.10 from [9] one needs $f(z) \geq 4c$ for all $z \in [c, 2c]$. In short, the result below yields at least two positive solutions to (3.1), (3.2) however it is the techniques used to verify a solution x^* such that (3.3) holds, which is the focus of the application.

Theorem 3.1. *If b is a positive real number, $c > 2b$ and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function such that*

(a) $2b < f(x) < \frac{5b}{2}$ *for* $0 \le x \le b$,

(b) $\frac{7b}{3} \le f(x) < \frac{11b}{4}$ *for* $b \le x \le \frac{13b}{8}$,

(c) $\frac{7b}{3} \le f(x) < 5b$ *for* $\frac{13b}{8} \le x \le 2b$, *and*

(d) $\frac{15c}{4} < f(x)$ *for* $c \le x \le 2c$,

then the focal problem (3.1), (3.2) *has at least one positive solution x^* such that*

$$b < x^*\left(\frac{1}{2}\right) < c.$$

Proof. For $x \in P$, if $t \in (0,1)$, then by the properties of the Green's function and the non-negativity of f we have

$$(Tx)''(t) = -f(x(t)) \le 0,\ (Tx)'(t) = \int_0^1 f(x(s))\,ds \ge 0, \text{ and } Tx(0) = 0 = (Tx)'(1).$$

Therefore we have that $T\colon P \to P$. By the Arzelà–Ascoli theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f.

For $x \in P$ let

$$\beta(x) = \kappa(x) = x\left(\frac{1}{2}\right),\ \alpha(x) = x(1), \text{ and } \theta(x) = x\left(\frac{1}{4}\right)$$

and for $z \in E$ let

$$\gamma(z) = \max\{|z(1/2)|, |z(1)|\} \text{ and } \rho(z) = \min\{z(1/4), z(1/2)\}. \tag{3.4}$$

Furthermore, let $a = \frac{5b}{4}$ and $d = \frac{5c}{8}$.

Clearly $P(\kappa, c)$ is a bounded subset of the cone P, since if $x \in P(\kappa, c)$, then by the concavity of x,

$$c > x\left(\frac{1}{2}\right) \ge \frac{x(1)}{2} = \frac{\|x\|}{2}$$

hence $\|x\| < 2c$. Also, if $x \in \overline{P(\beta, b)}$, then

$$c > b \ge \beta(x) = \kappa(x)$$

and hence $\kappa(x) < c$, that is,

$$\overline{P(\beta, b)} \subset P(\kappa, c).$$

Let x_0 and x_1 be defined by

$$x_0(s) = \frac{5bs}{4} \text{ and } x_1(s) = \frac{5cs}{2}.$$

Consequently we have that

$$x_0 \in \left\{x \in P\ : a = \frac{5b}{4} \le \alpha(x) \text{ and } \beta(x) < b\right\},$$

thus verifying $(A1)$ of Theorem 2.4, and that

$$x_1 \in \{x \in P \ : c < \kappa(x) \ \text{ and } \ \theta(x) \le d\},$$

thus verifying $(A4)$ of Theorem 2.4, after noting that $x_2(s) = (b+c)s \in P(\kappa, c)$ so $P(\kappa, c) \neq \emptyset$. Also since $x_2 \in P(\kappa, c) - \overline{P(\beta, b)}$ we have that

$$\overline{P(\beta, b)} \subsetneq P(\kappa, c).$$

Claim 1: If $x \in P$ with $\beta(x) = b$, then

$$\gamma(Tx - x_0) = Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right), \tag{3.5}$$

where γ is given in (3.4).

To prove Claim 1, let $x \in P$ with $\beta(x) = b$. By the definition of β and the concavity of x,

$$x(1/2) = b \quad \text{and} \quad x(1) \le 2b. \tag{3.6}$$

Then

$$\begin{aligned} Tx\,(1) - Tx\left(\frac{1}{2}\right) &= \int_{\frac{1}{2}}^{1} (Tx)'(t)\,dt = \int_{\frac{1}{2}}^{1}\int_{t}^{1} f(x(s))\,ds\,dt \\ &< \int_{\frac{1}{2}}^{1} 5b(1-t)\,dt = \frac{5b}{8} = x_0(1) - x_0\left(\frac{1}{2}\right), \end{aligned}$$

using conditions (b) and (c) on f from the statement of the theorem, so

$$Tx\,(1) - x_0(1) < Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right). \tag{3.7}$$

Moreover,

$$Tx\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} s\, f(x(s))\,ds + \int_{\frac{1}{2}}^{1} \frac{f(x(s))}{2}\,ds > \int_0^{\frac{1}{2}} 2bs\,ds + \int_{\frac{1}{2}}^{1} \frac{7b}{6}\,ds = \frac{b}{4} + \frac{7b}{12} = \frac{5b}{6}$$

yields

$$Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) > \frac{5b}{24}.$$

Since

$$Tx\,(1) = \int_0^1 s\, f(x(s))\,ds > \int_0^{\frac{1}{2}} 2bs\,ds + \int_{\frac{1}{2}}^{1} \frac{7bs}{3}\,ds = \frac{b}{4} + \frac{7b}{8} = \frac{9b}{8}$$

implies

$$Tx\,(1) - x_0\,(1) > \frac{9b}{8} - \frac{5b}{4} = \frac{-b}{8}, \tag{3.8}$$

from (3.7) and (3.8) we have

$$|Tx(1/2) - x_0(1/2)| > |Tx(1) - x_0(1)|.$$

It follows that (3.5) holds and Claim 1 is established.

Claim 2: $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$ for all $x \in P$ with $\beta(x) = b$ and $\alpha(x) \ge a$.

Let $x \in P$ with $\beta(x) = b$ and $\alpha(x) \geq a = \frac{5b}{4}$. By Claim 1 we know that (3.5) holds, and we have (3.6) as well. Now, either $2b \geq x(1) > \frac{13b}{8}$ or $\frac{13b}{8} \geq x(1) \geq \frac{5b}{4} = a$.

Case 1: Suppose $2b \geq x(1) > \frac{13b}{8}$. Then

$$x(1) - x_0(1) > \frac{13b}{8} - \frac{5b}{4} = \frac{3b}{8} = x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right).$$

Hence $\gamma(x - x_0) = x(1) - x_0(1)$, so

$$\begin{aligned}\gamma(Tx - x_0) &= Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) = Tx\left(\frac{1}{2}\right) - x\left(\frac{1}{2}\right) + x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) \\ &< Tx\left(\frac{1}{2}\right) - x\left(\frac{1}{2}\right) + x(1) - x_0(1) \leq \gamma(Tx - x) + \gamma(x - x_0).\end{aligned}$$

Case 2: Suppose $\frac{13b}{8} \geq x(1) \geq \frac{5b}{4} = a$. Then

$$\begin{aligned}Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) &= \int_0^1 G(1/2, s)\, f(x(s))\, ds - \frac{5b}{8} \\ &= \int_0^{\frac{1}{2}} s\, f(x(s))\, ds + \int_{\frac{1}{2}}^1 \frac{f(x(s))}{2}\, ds - \frac{5b}{8} \\ &< \int_0^{\frac{1}{2}} \frac{5bs}{2}\, ds + \int_{\frac{1}{2}}^1 \frac{11b}{8}\, ds - \frac{5b}{8} \\ &= \frac{5b}{16} + \frac{11b}{16} - \frac{5b}{8} = \frac{3b}{8} = x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right),\end{aligned}$$

and thus

$$\begin{aligned}\gamma(Tx - x_0) &= Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) \\ &< x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) = \gamma(x - x_0) \\ &\leq \gamma(Tx - x) + \gamma(x - x_0).\end{aligned}$$

Therefore, in either case we have that

$$\gamma(Tx - x_0) < \gamma(Tx - x) + \gamma(x - x_0),$$

which verifies condition $(A2)$ of Theorem 2.4.

Claim 3: $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$ for all $x \in P$ with $\beta(x) = b$ and $\alpha(Tx) < a$. We have

$$Tx(1) - Tx\left(\frac{1}{2}\right) = \int_{\frac{1}{2}}^1 (Tx)'(t)\, dt = \int_{\frac{1}{2}}^1 \int_t^1 f(x(s))\, ds\, dt \geq \int_{\frac{1}{2}}^1 \left(\frac{7b}{3}\right)(1 - t)\, dt = \frac{7b}{24}.$$

Since $Tx(1) = \alpha(Tx) < a = \frac{5b}{4}$, we have

$$b > \frac{23b}{24} = \frac{5b}{4} - \frac{7b}{24} > Tx\left(\frac{1}{2}\right),$$

and hence

$$\frac{3b}{8} > \frac{23b}{24} - \frac{5b}{8} > Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right).$$

As a result, using (3.5) we have

$$\gamma(Tx - x_0) = Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) < x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right) = \gamma(x - x_0) \leq \gamma(Tx - x) + \gamma(x - x_0),$$

hence we have shown that

$$\gamma(Tx - x_0) < \gamma(Tx - x) + \gamma(x - x_0).$$

This verifies condition $(A3)$ of Theorem 2.4 with $\delta = \gamma$.

Claim 4: $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$ for $x \in P$ with $\kappa(x) = c$.

Let $x \in P$ with $\kappa(x) = c$. Then $x\left(\frac{1}{2}\right) = c$, hence $\frac{c}{2} \leq x\left(\frac{1}{4}\right) \leq c$ and we have that

$$x\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) = \frac{-c}{4} < x\left(\frac{1}{4}\right) - x_1\left(\frac{1}{4}\right)$$

therefore $\rho(x - x_1) = x\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right)$.

We also have

$$\begin{aligned} Tx\left(\frac{1}{2}\right) - Tx\left(\frac{1}{4}\right) &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(s - \frac{1}{4}\right) f(x(s))\, ds + \int_{\frac{1}{2}}^{1} \left(\frac{1}{4}\right) f(x(s))\, ds \\ &> 2b\left(\frac{1}{32}\right) + \frac{15c}{32} = \frac{c}{2} \geq x\left(\frac{1}{2}\right) - x\left(\frac{1}{4}\right), \end{aligned}$$

thus $Tx\left(\frac{1}{2}\right) - x\left(\frac{1}{2}\right) > Tx\left(\frac{1}{4}\right) - x\left(\frac{1}{4}\right)$ hence $\rho(Tx - x) = Tx\left(\frac{1}{4}\right) - x\left(\frac{1}{4}\right)$.

Therefore, if $\rho(Tx - x_1) = Tx\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right)$ then

$$\begin{aligned} \rho(Tx - x_1) &= Tx\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) \\ &= Tx\left(\frac{1}{2}\right) - x\left(\frac{1}{2}\right) + x\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) \\ &> Tx\left(\frac{1}{4}\right) - x\left(\frac{1}{4}\right) + x\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) \\ &= \rho(Tx - x) + \rho(x - x_1), \end{aligned}$$

and if $\rho(Tx - x_1) = Tx\left(\frac{1}{4}\right) - x_1\left(\frac{1}{4}\right)$ then

$$\begin{aligned} \rho(Tx - x_1) &= Tx\left(\frac{1}{4}\right) - x_1\left(\frac{1}{4}\right) \\ &= Tx\left(\frac{1}{4}\right) - x\left(\frac{1}{4}\right) + x\left(\frac{1}{4}\right) - x_1\left(\frac{1}{4}\right) \\ &> Tx\left(\frac{1}{4}\right) - x\left(\frac{1}{4}\right) + x\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) \\ &\geq \rho(Tx - x) + \rho(x - x_1). \end{aligned}$$

Hence, $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$.

Therefore, the conditions of Theorem 2.4 are satisfied and the operator T has at least one fixed point x^* with

$$x^* \in P(\kappa, \beta, c, b).$$

□

Remark 3.2. Note that there are $z \in \partial P(\beta, b)$ with $\alpha(z) \geq a$ such that $\beta(Tz) > \beta(z)$, which illustrates that this is an example that could not have been done using standard Krasnoselskii or Leggett–Williams techniques. This is indicative of the rich opportunities opening up now to find new techniques to verify the existence of solutions to boundary value problems by applying the omitted ray fixed point theorem.

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