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New existence and multiplicity of homoclinic solutions for second order non-autonomous systems

Huiwen Chen and **Zhimin He** ✉

School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P. R. China

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Abstract. In this paper, we study the second order non-autonomous system

$$\ddot{u}(t) + A\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R},$$

where A is an antisymmetric $N \times N$ constant matrix, $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ may not be uniformly positive definite for all $t \in \mathbb{R}$, and $W(t, u)$ is allowed to be sign-changing and local superquadratic. Under some simple assumptions on A, L and W, we establish some existence criteria to guarantee that the above system has at least one homoclinic solution or infinitely many homoclinic solutions by using the mountain pass theorem or the fountain theorem, respectively. Recent results in the literature are generalized and significantly improved.

Keywords: non-autonomous systems, homoclinic solutions, variational methods, mountain pass theorem, fountain theorem.

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1 Introduction

Consider the following second order non-autonomous system:

$$\ddot{u}(t) + A\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $u \in \mathbb{R}^N$, A is an antisymmetric $N \times N$ constant matrix, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix valued function. As usual, we say that a solution u of system (1.1) is homoclinic to zero if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

The motivation of our work stems from both theoretical and practical aspects. The importance of homoclinic orbits for dynamical systems has been recognized by Poincaré [14]. Thus, the existence and multiplicity of homoclinic solutions has become one of most important problems in the research of dynamical systems.

When $A = 0$, system (1.1) is just the following second order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}. \tag{1.2}$$

✉ Corresponding author. Email: hezhimin@csu.edu.cn

The existence and multiplicity of homoclinic solutions for system (1.2) has been intensively studied in many recent papers via variational methods under various hypotheses on L and W; see [1, 4–13, 15, 17–23, 26, 28, 31] and references therein. Most of them treated the case where $L(t)$ and $W(t,u)$ are either independent of t or T-periodic in t; see [4,6,9,11,13,15] and the references therein. In this case, the existence of homoclinic solutions can be obtained by going to the limit of $2kT$-periodic solutions of approximating problems. If $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, the problem of existence of homoclinic solutions for system (1.2) is quite different from the one just described, because of the lack of compactness of the Sobolev embedding; see for instance [1,5,7,10,12,17,19–23,26,28,31] and the references therein. In [17], Rabinowitz and Tanaka studied system (1.2) without a periodicity assumption for both L and W and obtained the existence of homoclinic solutions for system (1.2) under the Ambrosetti–Rabinowitz growth condition

$$0 < \mu_0 W(t,u) \leq (\nabla W(t,u), u), \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\},$$

where $\mu_0 > 2$.

Compared with the case where $A = 0$, the case where $A \neq 0$ is more complex. To the best of our knowledge, there are only a few papers that have studied this case; see [25, 27, 29, 30]. More precisely, in [27], Yuan and Zhang studied system (1.1) without a periodicity assumption, both for L and W. In detail, they obtained the following results.

Theorem 1.1 ([27]). *Assume that A, L and W satisfy the following conditions:*

(A_1) *$L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exist a function $l \in C(\mathbb{R},(0,\infty))$ and a constant $\beta > 0$ such that $(L(t)u,u) \geq l(t)|u|^2 \geq \beta|u|^2$ and $l(t) \to \infty$ as $|t| \to \infty$.*

(A_2) $\|A\| < \sqrt{\beta}$.

(A_3) *There exists $\mu > 2$ such that*

$$0 < \mu W(t,u) \leq (\nabla W(t,u), u), \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}.$$

(A_4) *$|\nabla W(t,u)| = o(|u|)$, as $|u| \to 0$ uniformly for all $t \in \mathbb{R}$.*

(A_5) *There exists $\hat{W} \in C(\mathbb{R}^N, \mathbb{R})$ such that $|\nabla W(t,u)| \leq |\hat{W}(u)|$ for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$.*

(A_6) *W is even in u.*

Then system (1.1) has infinitely many homoclinic solutions.

Theorem 1.2 ([27]). *Assume that (A_1)–(A_5) hold. Then system (1.1) possesses at least one nontrivial homoclinic solution.*

In the present paper, motivated by the above papers, we will study the existence and multiplicity of homoclinic solutions for system (1.1) under more relaxed assumptions on A, L and W.

We will use the following conditions:

(H_1) $l_1(t) = \inf_{|u|=1}(L(t)u,u) \to \infty$ as $|t| \to \infty$.

(H_2) There exists $\alpha_1 > 0$ such that $\|A\| < \sqrt{\alpha_1}$.

(H_3) $W(t,0)=0$ and there exist $c>0$, $\nu>2$ such that

$$|\nabla W(t,u)| \le c(|u|+|u|^{\nu-1}),\ \ \forall (t,u)\in \mathbb{R}\times\mathbb{R}^N.$$

(H_4) There exist $\lambda>2$, $h_0>0$, $0\le h_1<\frac{\lambda-2}{2}(1-\frac{\|A\|}{\sqrt{\alpha_1}})$ and $0<\gamma<2$ such that

$$(\nabla W(t,u),u)-\lambda W(t,u)\ge -h_0|u|^2-h_1(L(t)u,u)-h_2(t)|u|^\gamma-h_3(t),\ \ \forall (t,u)\in\mathbb{R}\times\mathbb{R}^N,$$

where $h_2,h_3\colon \mathbb{R}\to\mathbb{R}^+$ are positive continuous functions such that $h_2\in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R}^+)$ and $h_3\in L^1(\mathbb{R},\mathbb{R}^+)$.

(H_5) $\lim_{|u|\to+\infty}\frac{W(t,u)}{|u|^2}=+\infty$ uniformly for all $t\in\mathbb{R}$.

(H_6) There exists $\eta>0$ such that $W(t,u)\ge -\eta|u|^2$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^N$.

(H_7) There exist $\theta\ge\nu-1$, $d>0$ and $R>0$ such that

$$(\nabla W(t,u),u)-2W(t,u)\ge d|u|^\theta,\quad \forall t\in\mathbb{R},\forall |u|\ge R,$$

$$(\nabla W(t,u),u)\ge 2W(t,u),\quad \forall t\in\mathbb{R},\forall |u|\le R.$$

(H_8) There exist $-\infty<a<b<+\infty$ such that

$$\liminf_{|u|\to+\infty}\frac{W(t,u)}{|u|^2}>\left(1+\frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2}+\frac{l_2}{2}\right),\ \ \forall t\in[a,b],$$

where $l_2=\max_{|u|=1,t\in[a,b]}(L(t)u,u)$.

(H_8') There exist $-\infty<\bar a<\bar b<+\infty$ such that

$$\liminf_{|u|\to+\infty}\frac{W(t,u)}{|u|^2}=+\infty,\quad \text{a.e. } t\in[\bar a,\bar b].$$

(H_9) There exist $\mu_1>2$, $0\le l_3<\frac{\mu_1-2}{2}\left(1-\frac{\|A\|}{\sqrt{\beta}}\right)$, $k_0,k_1>1$ and $0<\vartheta<2$ such that

$$\begin{aligned}(\nabla W(t,u),u)-\mu_1 W(t,u)\ge\ & -l_3(L(t)u,u)-l_6(t)\frac{|u|^2}{\ln(k_0+|u|)}\\ & -l_7(t)|u|\ln(k_1+|u|)-l_4(t)|u|^\vartheta-l_5(t)\end{aligned}$$

for all $(t,u)\in\mathbb{R}\times\mathbb{R}^N$, where $l_4,l_5,l_6,l_7\colon\mathbb{R}\to\mathbb{R}^+$ are positive continuous functions such that $l_4\in L^{\frac{2}{2-\vartheta}}(\mathbb{R},\mathbb{R}^+)$, $l_6\in L^1(\mathbb{R},\mathbb{R}^+)\cap L^2(\mathbb{R},\mathbb{R}^+)$ and $l_5,l_7\in L^1(\mathbb{R},\mathbb{R}^+)$.

(H_{10}) $W(t,0)=0$ and there exist $0<k_2<m\left(1-\frac{\|A\|}{\sqrt{\beta}}\right)$ and $T_0>0$ such that

$$|\nabla W(t,u)|\le k_2|u|\quad \forall t\in\mathbb{R},\forall|u|\le T_0,$$

where $m=\min\{l(t):t\in\mathbb{R}\}$.

(H_{11}) There exist $D>0$ and $\gamma_0\ge 2$ such that

$$|\nabla W(t,u)|\le D(1+|u|^{\gamma_0-1}).$$

Now, we state our main results.

Theorem 1.3. *Assume that* (A_6) *and* (H_1)–(H_6) *hold. Then system* (1.1) *has infinitely many homoclinic solutions.*

Remark 1.4. Obviously, condition (H_1) is weaker than (A_1), condition (H_2) is weaker than (A_2), and conditions (H_4)–(H_6) are weaker than (A_3). Therefore, Theorem 1.3 generalizes Theorem 1.1 by relaxing conditions (A_1)–(A_3) and (A_5) and removing condition (A_4). Let

$$L(t) = (t^2 - 6)I_N, \quad W(t,u) = f(t)\left(10|u|^2 + |u|^6 - 15|u|^4\right), \quad \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^N,$$

where I_N is the unit matrix of order N and f is a continuous bounded function with positive lower bound, and A is an arbitrary antisymmetric $N \times N$ constant matrix. It is easy to check that A, L and W satisfying our Theorem 1.3 but not satisfying Theorem 1.1.

Remark 1.5. When $A = 0$, Theorem 1.3 generalizes the corresponding result in [12].

Theorem 1.6. *Assume that* (A_6), (H_1)–(H_3) *and* (H_5)–(H_7) *hold. Then system* (1.1) *has infinitely many homoclinic solutions.*

Remark 1.7. It is clear that Theorem 1.6 generalizes Theorem 1.1 by relaxing conditions (A_1)–(A_3) and (A_5) and removing condition (A_4). Let

$$L(t) = (t^2 - 3)I_N,\ W(t,u) = f_1(t)\left(-5|u|^2 + |u|^2\ln(|u|^4 - |u|^3 + 3|u|^2 + 2)\right), \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^N,$$

where I_N is the unit matrix of order N and f_1 is a continuous bounded function with positive lower bound, and A is an arbitrary antisymmetric $N \times N$ constant matrix. It is easy to check that A, L and W satisfying our Theorem 1.6 but not satisfying Theorem 1.1.

Remark 1.8. When $A = 0$, Theorem 1.6 generalizes Theorem 1.1 of [26] and Theorem 1.4 of [22].

Theorem 1.9. *Assume that* (A_1), (A_2) *and* (H_8)–(H_{11}) *hold. Then system* (1.1) *possesses at least one nontrivial homoclinic solution.*

Remark 1.10. Obviously, Theorem 1.9 treats the local superquadratic case and Theorem 1.2 just treats the global superquadratic case. Hence, Theorem 1.9 generalizes Theorem 1.2 by relaxing conditions (A_1)–(A_5).

Remark 1.11. When $A = 0$, Theorem 1.9 generalizes Theorem 5.4 in [17].

Theorem 1.12. *Assume that* (A_1), (A_2), (A_6), (H_3), (H_8') *and* (H_9) *hold. Then system* (1.1) *has infinitely many homoclinic solutions.*

Remark 1.13. Obviously, Theorem 1.12 treats the local superquadratic case and Theorem 1.1 just treats the global superquadratic case. Hence, Theorem 1.12 generalizes Theorem 1.1 by relaxing conditions (A_1)–(A_3) and (A_5) and removing condition (A_4). Furthermore, there are many functions W satisfying our Theorem 1.12 and not satisfying Theorem 1.1. For example, let $W(t,u) = f_2(t)\left(|u|^6 + \frac{6|u|^2}{\ln(4+|u|)}\right)$, where

$$f_2(t) = \begin{cases} 2 - 2\cos t, & t \in (0, 2\pi), \\ 0, & t \in \mathbb{R} \setminus (0, 2\pi). \end{cases}$$

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorems 1.3, 1.6, 1.9 and 1.12.

2 Preliminaries

In this section, the following theorems will be needed in our argument. Assume that E is a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, where X_j are finite dimensional subspace of E. For each $k \in \mathbb{N}$, let $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. The functional φ is said to satisfy the Palais–Smale condition if any sequence $\{u_n\}$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 2.1 ([3, 24])**.** *Suppose that the functional $\varphi \in C^1(E, \mathbb{R})$ is even. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(G_1) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0.$

(G_2) $b_k := \inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ *as* $k \to \infty$.

(G_3) φ *satisfies the Palais–Smale condition.*

Then φ possesses an unbounded sequence of critical values.

We will get a critical point of φ by using a standard version of the mountain pass theorem. Now we state this theorem precisely.

Theorem 2.2 ([2, 16])**.** *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfy the Palais–Smale condition. If φ satisfies the following conditions:*

(i) $\varphi(0) = 0$;

(ii) there exist constants $\rho, \alpha > 0$ such that $\varphi_{\partial B_\rho(0)} \ge \alpha$;

(iii) there exists $e \in E \setminus \bar{B}_\rho(0)$ such that $\varphi(e) \le 0$;

then φ possesses a critical value $\bar{d} \ge \alpha$ given by

$$\bar{d} = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)),$$

where $B_\rho(0)$ is an open ball in E of radius ρ around 0, and

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0,\ g(1) = e\}.$$

Before establishing the variational setting for system (1.1), we have the following.

Remark 2.3. It follows from (H_1) that there exists $\alpha_2 > \alpha_1 > 0$ such that $(\hat{L}(t)u, u) = ((L(t) + \alpha_2 I_N)u, u) \ge \alpha_1|u|^2$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$, where α_1 is defined in condition (H_2). Let $\nabla\hat{W}(t, u) = \nabla W(t, u) + \alpha_2 u$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$ and consider the following new second order non-autonomous system:

$$\ddot{u}(t) + A\dot{u}(t) - \hat{L}(t)u(t) + \nabla\hat{W}(t, u(t)) = 0, \quad \forall t \in \mathbb{R}. \tag{2.1}$$

Then system (2.1) is equivalent to system (1.1). It is easy to see that all conditions of Theorem 1.3 (or Theorem 1.6) still hold for A, $\hat{L}$ and $\hat{W}$ provided that those hold for A, L and W. Hence we can assume without loss of generality that $(L(t)u, u) \ge \alpha_1|u|^2$ in (H_1).

We will present some definitions and lemmas that will be used in the proof of our results. In view of Remark 2.3 (or (A_1)), we consider the function space

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \right] dt < +\infty \right\}$$

equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}} \big[(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) \big] dt, \quad \forall u, v \in X, \tag{2.2}$$

and the norm

$$\|u\| := \langle u, v \rangle^{\frac{1}{2}} = \left\{ \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \right] dt \right\}^{\frac{1}{2}}, \quad \forall u \in X. \tag{2.3}$$

Then E is a Hilbert space with this inner product, and it is easy to verify that E is continuously embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$. Let $\| \cdot \|_p$ denote the usual norm on $L^p(\mathbb{R}, \mathbb{R}^N)$ $(p \in [1, \infty])$. Note that E is continuously embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, +\infty]$. Therefore, there exists a constant $C_p > 0$ such that

$$\|u\|_p \leq C_p \|u\|, \quad \forall u \in E, \tag{2.4}$$

for all $p \in [2, +\infty]$.

Define the functional φ on E by

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}} \left[\frac{1}{2} |\dot{u}(t)|^2 + \frac{1}{2}(Au(t), \dot{u}(t)) + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)) \right] dt \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot{u}(t))\, dt - \int_{\mathbb{R}} W(t, u(t))\, dt. \end{aligned} \tag{2.5}$$

From the assumptions it follows that φ is defined on E and belongs to $C^1(E, \mathbb{R})$, and one can easily check that

$$\langle \varphi'(u), v \rangle = \int_{\mathbb{R}} \big[(\dot{u}(t), \dot{v}(t)) + (Au(t), \dot{v}(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t)) \big] dt \tag{2.6}$$

for any $u, v \in E$. Furthermore, it is routine to verify that any critical point of φ in E is a classical solution of system (1.1) with $u(\pm\infty) = 0 = \dot{u}(\pm\infty)$ (see [27]).

Lemma 2.4. *Assume that L satisfies (A_1) (or (H_1)). Then E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for any $2 \leq p \leq \infty$.*

Proof. The proof is similar to the proof of [9, Lemma 2.1], and we omit it here. □

Lemma 2.5 ([20])**.** *Under assumption (A_1), for $u \in H^1(\mathbb{R}, \mathbb{R}^N)$,*

$$\|u\|_\infty \leq \frac{\sqrt{2}}{2} \|u\|_{H^1(\mathbb{R}, \mathbb{R}^N)} = \frac{\sqrt{2}}{2} \left[\int_{\mathbb{R}} \left(|\dot{u}(s)|^2 + |u(s)|^2 \right) ds \right]^{\frac{1}{2}}; \tag{2.7}$$

and for $u \in E$,

$$\|u\|_\infty \leq \frac{1}{\sqrt{2\sqrt{m}}} \|u\| = \frac{1}{\sqrt{2\sqrt{m}}} \left\{ \int_{\mathbb{R}} \left[|\dot{u}(s)|^2 + (L(s)u(s), u(s)) \right] ds \right\}^{\frac{1}{2}}, \tag{2.8}$$

$$|u(t)| \leq \left\{ \int_t^{\infty} \frac{1}{\sqrt{l(s)}} \left[|\dot{u}(s)|^2 + (L(s)u(s), u(s)) \right] \right\}^{\frac{1}{2}}, \quad t \in \mathbb{R}, \tag{2.9}$$

and

$$|u(t)| \leq \left\{ \int_{-\infty}^{t} \frac{1}{\sqrt{l(s)}} \left[|\dot{u}(s)|^2 + (L(s)u(s), u(s)) \right] \right\}^{\frac{1}{2}}, \quad t \in \mathbb{R}, \tag{2.10}$$

where $m = \min\{l(t) : t \in \mathbb{R}\}$.

3 Proof of Theorems 1.3, 1.6, 1.9 and 1.12

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We choose a completely orthonormal basis $\{e_j\}$ of E and define $E_j := \mathbb{R}e_j$, then Z_k and Y_k can be defined as that in Section 2. By (A_6) and (2.6), we obtain that $\varphi \in C^1(E, \mathbb{R})$ is even. Next we will check that all conditions in Theorem 2.1 are satisfied.

Step 1. We verify condition (G_2) in Theorem 2.1. Set $\beta_k = \sup_{u \in Z_k, \|u\|=1} \|u\|_2$, $\lambda_k = \sup_{u \in Z_k, \|u\|=1} \|u\|_\nu$, then $\beta_k \to 0$ and $\lambda_k \to 0$ as $k \to \infty$ since E is compactly embedded into both $L^2(\mathbb{R}, \mathbb{R}^N)$ and $L^\nu(\mathbb{R}, \mathbb{R}^N)$ (see [24]). By (2.5), (H_1)–(H_3) and Remark 2.3, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot{u}(t))dt - \int_{\mathbb{R}} W(t, u(t))dt \\
&\geq \left(\frac{1}{2} - \frac{\|A\|}{2\sqrt{\alpha_1}} \right) \|u\|^2 - c\left(\|u\|_2^2 + \|u\|_\nu^\nu \right) \\
&\geq \left(\frac{1}{2} - \frac{\|A\|}{2\sqrt{\alpha_1}} \right) \|u\|^2 - c\beta_k^2 \|u\|^2 - c\lambda_k^\nu \|u\|^\nu.
\end{aligned} \tag{3.1}$$

Let $\zeta = \frac{1}{2} - \frac{\|A\|}{2\sqrt{\alpha_1}}$, it follows from (H_2) and Remark 2.3 that $\zeta > 0$. Since $\beta_k \to 0$ as $k \to \infty$, there exists a positive constant N_0 such that

$$c\beta_k^2 \leq \frac{1}{2}\zeta, \quad \forall k \geq N_0. \tag{3.2}$$

By (3.1) and (3.2), we get

$$\varphi(u) \geq \frac{1}{2}\zeta\|u\|^2 - c\lambda_k^\nu \|u\|^\nu, \quad \forall k \geq N_0. \tag{3.3}$$

We choose $r_k = \left(\frac{4c\lambda_k^\nu}{\zeta} \right)^{\frac{1}{2-\nu}}$, then

$$b_k = \inf_{u \in Z_k, \|u\|=r_k} \varphi(u) \geq \frac{1}{4}\zeta r_k^2, \quad \forall k \geq N_0. \tag{3.4}$$

Since $\lambda_k \to 0$ as $k \to \infty$ and $\nu > 2$, we have

$$b_k \to +\infty \quad \text{as } k \to \infty.$$

Step 2. We verify condition (G_1) in Theorem 2.1. We follow the idea of the proof of Theorem 1.1 in [26]. Firstly, we claim that there exists $\sigma > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : |u(t)| \geq \sigma\|u\|\} \geq \sigma, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.5}$$

If not, there exists a sequence $\{v_n\} \subset Y_k$ with $\|v_n\| = 1$ such that

$$\text{meas}\left\{t \in \mathbb{R} : |v_n(t)| \geq \frac{1}{n}\right\} \leq \frac{1}{n}. \tag{3.6}$$

Since $\dim Y_k < \infty$, it follows from the compactness of the unit sphere of Y_k that there exists a subsequence, say $\{v_n\}$, such that v_n converges to some v_0 in Y_k. Hence, we have $\|v_0\| = 1$. Since all norms are equivalent in the finite-dimensional space, we have $v_n \to v_0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$. Then one has

$$\int_{\mathbb{R}} |v_n - v_0|^2 \, dt \to 0, \quad \text{as } n \to \infty. \tag{3.7}$$

Thus there exist $\sigma_1, \sigma_2 > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : |v_0(t)| \geq \sigma_1\} \geq \sigma_2. \tag{3.8}$$

In fact, if not, we have

$$\text{meas}\left\{t \in \mathbb{R} : |v_0(t)| \geq \frac{1}{n}\right\} = 0, \tag{3.9}$$

for all positive integers n, which implies that

$$\int_{\mathbb{R}} |v_0(t)|^4 \, dt \leq \frac{1}{n^2} |v_0|_2^2 \to 0$$

as $n \to \infty$. Hence $v_0 = 0$ which contradicts that $\|v_0\| = 1$. Therefore, (3.8) holds.

Now let

$$\Omega_0 = \{t \in \mathbb{R} : |v_0(t)| \geq \sigma_1\}, \quad \Omega_n = \left\{t \in \mathbb{R} : |v_n(t)| < \frac{1}{n}\right\}$$

and $\Omega_n^c = \mathbb{R} \setminus \Omega_n = \{t \in \mathbb{R} : |v_n(t)| \geq \frac{1}{n}\}$. Combining (3.6) and (3.8), we have

$$\begin{aligned} \text{meas}(\Omega_n \cap \Omega_0) &= \text{meas}(\Omega_0 \setminus \Omega_n^c \cap \Omega_0) \\ &\geq \text{meas}(\Omega_0) - \text{meas}(\Omega_n^c \cap \Omega_0) \\ &\geq \sigma_2 - \frac{1}{n} \end{aligned}$$

for all positive integers n. Let n be large enough such that $\sigma_2 - \frac{1}{n} \geq \frac{1}{2}\sigma_2$ and $\sigma_1 - \frac{1}{n} \geq \frac{1}{2}\sigma_1$. Then we have

$$|v_n(t) - v_0(t)|^2 \geq \left(\sigma_1 - \frac{1}{n}\right)^2 \geq \frac{\sigma_1^2}{4}, \quad \forall t \in \Omega_n \cap \Omega_0.$$

This implies that

$$\begin{aligned} \int_{\mathbb{R}} |v_n - v_0|^2 dt &\geq \int_{\Omega_n \cap \Omega_0} |v_n - v_0|^2 \, dt \\ &\geq \frac{\sigma_1^2}{4} \text{meas}(\Omega_n \cap \Omega_0) \\ &\geq \frac{\sigma_1^2}{4} \left(\sigma_2 - \frac{1}{n}\right) \\ &\geq \frac{\sigma_1^2 \sigma_2}{8} > 0 \end{aligned}$$

for all large n, which is a contradiction to (3.7). Therefore, (3.5) holds. For the σ given in (3.5), let

$$\Omega_u = \{t \in \mathbb{R} : |u(t)| \geq \sigma \|u\|\}, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.10}$$

By (3.5), we obtain

$$\text{meas}(\Omega_u) \geq \sigma, \quad \forall u \in Y_k \setminus \{0\}. \tag{3.11}$$

It follows from (H_5) that for any $M_1 > 0$ there exists $\varrho = \varrho(M_1) > 0$ such that

$$W(t,u) \geq M_1|u|^2, \quad \forall |u| \geq \varrho, \ \forall t \in \mathbb{R}. \tag{3.12}$$

Hence we have

$$W(t,u) \geq M_1|u|^2 \geq M_1\sigma^2\|u\|^2, \quad \forall t \in \Omega_u, \tag{3.13}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{\varrho}{\sigma}$. It follows from (H_2), (H_6), (2.4), (3.11), (3.13) and Remark 2.3 that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}}(Au(t),\dot{u}(t))dt - \int_{\mathbb{R}} W(t,u(t))dt \\
&\leq \left(\frac{1}{2} + \frac{\|A\|}{2\sqrt{\alpha_1}}\right)\|u\|^2 - \int_{\Omega_u} W(t,u)dx + \eta\int_{\mathbb{R}\setminus\Omega_u}|u|^2dx \\
&\leq \left(\frac{1}{2} + \frac{\|A\|}{2\sqrt{\alpha_1}} + \eta C_2^2\right)\|u\|^2 - M_1\sigma^2\|u\|^2\text{meas}(\Omega_u) \\
&\leq \left(\frac{1}{2} + \frac{\|A\|}{2\sqrt{\alpha_1}} + \eta C_2^2\right)\|u\|^2 - M_1\sigma^3\|u\|^2
\end{aligned}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{\varrho}{\sigma}$. Choose M_1 sufficiently large such that

$$\frac{1}{2} + \frac{\|A\|}{2\sqrt{\alpha_1}} + \eta C_2^2 - M_1\sigma^3 < 0.$$

Thus, we can choose $\|u\| = \rho_k$ large enough $(\rho_k > r_k)$ such that

$$a_k = \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \leq 0.$$

Step 3. We prove that φ satisfies the Palais–Smale condition. Let $\{u_n\}$ be a Palais–Smale sequence, that is, $\{\varphi(u_n)\}$ is bounded, and $\varphi'(u_n) \to 0$ as $n \to \infty$. We now prove that $\{u_n\}$ is bounded in E. In fact, if not, we may assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Let $w_n := \frac{u_n}{\|u_n\|}$. Clearly, $\|w_n\| = 1$ and there is $w_0 \in E$ such that, up to a subsequence,

$$w_n \rightharpoonup w_0 \ \text{ in } E, \quad w_n \to w_0 \ \text{ a.e. in } \mathbb{R}, \tag{3.14}$$

$$w_n \to w_0 \ \text{ in } L^p(\mathbb{R},\mathbb{R}^N),\ 2 \leq p \leq +\infty \ \text{ as } n \to \infty. \tag{3.15}$$

Case 1. $w_0 = 0$. In view of (2.4), (H_2), (H_4), Remark 2.3 and the Hölder's inequality, one has

$$\begin{aligned}
\lambda M_2 + M_2\|u_n\| &\geq \lambda\varphi(u_n) - \langle\varphi'(u_n),u_n\rangle \\
&= \left(\frac{\lambda}{2}-1\right)\|u_n\|^2 + \left(\frac{\lambda}{2}-1\right)\int_{\mathbb{R}}(Au_n(t),\dot{u}_n(t))dt \\
&\quad + \int_{\mathbb{R}}\left[(\nabla W(t,u_n),u_n) - \lambda W(t,u_n)\right]dt \\
&\geq \xi\|u_n\|^2 - \int_{\mathbb{R}}\left[h_0|u_n|^2 + h_1(L(t)u_n,u_n) + h_2(t)|u_n|^\gamma + h_3(t)\right]dt \\
&\geq (\xi - h_1)\|u_n\|^2 - h_0\|u_n\|_2^2 - \|h_2\|_{\frac{2}{2-\gamma}}\|u_n\|_2^\gamma - \|h_3\|_1 \\
&\geq (\xi - h_1)\|u_n\|^2 - h_0\|u_n\|_2^2 - C_2^\gamma\|h_2\|_{\frac{2}{2-\gamma}}\|u_n\|^\gamma - \|h_3\|_1.
\end{aligned} \tag{3.16}$$

for some $M_2 > 0$, where $\xi = (\frac{\lambda}{2} - 1)\left(1 - \frac{\|A\|}{\sqrt{\alpha_1}}\right) > 0$. Divided by $\|u_n\|^2$ on both sides of (3.16), noting that $0 \le h_1 < \frac{\lambda-2}{2}\left(1 - \frac{\|A\|}{\sqrt{\alpha_1}}\right)$ and $0 < \gamma < 2$, we obtain

$$\|w_n\|_2^2 \ge \frac{\xi - h_1}{h_0} > 0 \quad \text{as } n \to \infty. \tag{3.17}$$

It follows from (3.15) and (3.17) that $w_0 \neq 0$. That is a contradiction.

Case 2. $w_0 \neq 0$. Since $\{\varphi(u_n)\}$ is bounded, there exists $M_3 > 0$ such that

$$\varphi(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}}(Au_n(t), \dot{u}_n(t))\,dt - \int_{\mathbb{R}} W(t, u_n(t))\,dt \ge -M_3. \tag{3.18}$$

Divided by $\|u_n\|^2$ on both sides of (3.18), noting that Remark 2.3, we have

$$\int_{\mathbb{R}} \frac{W(t,u_n)}{\|u_n\|^2}\,dt \le \frac{1}{2} + \frac{\|A\|}{\sqrt{\alpha_1}} + \frac{M_3}{\|u_n\|^2} < \infty. \tag{3.19}$$

Let $\Lambda := \{t \in \mathbb{R} : w_0(t) \neq 0\}$, then $\operatorname{meas}(\Lambda) > 0$. It follows from (3.14) that

$$u_n(t) = w_n(t)\|u_n\| \to \infty, \quad \text{for } t \in \Lambda.$$

Combining (H_5) and (H_6), we obtain

$$\lim_{n\to\infty}\left(\frac{W(t,u_n)}{|u_n|^2} + \eta\right)|w_n|^2 \to \infty, \quad \text{for } t \in \Lambda.$$

Therefore, by Fatou's lemma, (H_6) and (3.15), we get

$$\begin{aligned}
\int_{\mathbb{R}} \frac{W(t,u_n)}{\|u_n\|^2}dt &= \int_{\Lambda} \frac{W(t,u_n)}{|u_n|^2}|w_n|^2 dt + \int_{\mathbb{R}\setminus\Lambda} \frac{W(t,u_n)}{|u_n|^2}|w_n|^2 dt \\
&\ge \int_{\Lambda} \frac{W(t,u_n)}{|u_n|^2}|w_n|^2 dt - \eta\int_{\mathbb{R}\setminus\Lambda}|w_n|^2 dx \\
&= \int_{\Lambda} \frac{W(t,u_n) + \eta|u_n|^2}{|u_n|^2}|w_n|^2 dt - \eta\int_{\mathbb{R}}|w_n|^2 dt \to \infty.
\end{aligned}$$

This contradicts (3.19). Therefore, $\{u_n\}$ is bounded in E, that is, there exists $M_3 > 0$ such that

$$\|u_n\| \le M_3. \tag{3.20}$$

In view of the boundedness of $\{u_n\}_{n=1}^{\infty}$, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_n\}$, $u_n \rightharpoonup u$ in E. Next we will verify that $\{u_n\}$ strongly converges to u in E. By virtue of (H_3), (2.4), (3.20) and Lemma 2.4, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}}(\nabla W(t,u_n) - \nabla W(t,u), u_n - u)dt\right| &\le \int_{\mathbb{R}}(|\nabla W(t,u_n)| + |\nabla W(t,u)|)\,|u_n - u|\,dt \\
&\le c\int_{\mathbb{R}}\left(|u_n| + |u_n|^{\nu-1}\right)|u_n - u|\,dt \\
&\quad + c\int_{\mathbb{R}}\left(|u| + |u|^{\nu-1}\right)|u_n - u|\,dt \\
&\le c\left(\|u_n\|_2 + \|u_n\|_{2\nu-2}^{\nu-1}\right)\|u_n - u\|_2 \\
&\quad + c\left(\|u\|_2 + \|u\|_{2\nu-2}^{\nu-1}\right)\|u_n - u\|_2 \\
&\le c(C_2\|u_n\| + C_{2\nu-2}^{\nu-1}\|u_n\|^{\nu-1})\|u_n - u\|_2 \\
&\quad + c(\|u\|_2 + \|u\|_{2\nu-2}^{\nu-1})\|u_n - u\|_2 \\
&\le M_4\|u_n - u\|_2 \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.21}$$

where $M_4 = c\big(C_2M_3 + C_{2\nu-2}^{\nu-1}M_3^{\nu-1} + \|u\|_2 + \|u\|_{2\nu-2}^{\nu-1}\big)$. By Lemma 2.4 and Hölder's inequality, one has

$$\int_{\mathbb{R}} (Au_n - Au, \dot{u}_n - \dot{u})\, dt \le \|A\|\|\dot{u}_n - \dot{u}\|\|u_n - u\|_2 \to 0 \quad \text{as } n \to \infty. \tag{3.22}$$

It follows from $u_n \rightharpoonup u$, (3.21) and (3.22) that

$$\begin{aligned}\|u_n - u\|^2 &= \langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle - \int_{\mathbb{R}} (Au_n - Au, \dot{u}_n - \dot{u})\, dt \\ &\quad + \int_{\mathbb{R}} \big(\nabla W(t,u_n) - \nabla W(t,u), u_n - u\big)\, dt \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus, φ satisfies the Palais–Smale condition.

It follows from Theorem 2.1 that φ has a sequence of critical points $\{u_k\} \subset E$ such that $\varphi(u_k) \to \infty$ as $k \to \infty$. Hence system (1.1) has infinitely many homoclinic solutions. □

Now we give the proof of Theorem 1.6.

Proof of Theorem 1.6. The proof of Theorem 1.6 is similar to that of Theorem 1.3. In fact, we only need to prove that φ satisfies the Palais–Smale condition. Let $\{u_n\}$ be a Palais–Smale sequence, that is, $\{\varphi(u_n)\}$ is bounded, and $\varphi'(u_n) \to 0$ as $n \to \infty$. We now prove that $\{u_n\}$ is bounded in E. In fact, if not, we may assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. We take w_n as in the proof of Theorem 1.3.

Case 1. $w_0 = 0$. By (H_7), we have

$$\begin{aligned}2\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle &= \int_{\mathbb{R}} \big[(\nabla W(t,u_n), u_n) - 2W(t,u_n)\big] dt \\ &\ge \int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} \big[(\nabla W(t,u_n), u_n) - 2W(t,u_n)\big] dt \\ &\ge d \int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} |u_n|^\theta\, dt,\end{aligned} \tag{3.23}$$

which implies that

$$\frac{\int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} |u_n|^\theta\, dt}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{3.24}$$

In view of (2.4), (H_2), (H_3) and Remark 2.3, we obtain

$$\begin{aligned}M_5 &\ge \varphi(u_n) \\ &= \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\mathbb{R}} (Au_n(t), \dot{u}_n(t))dt - \int_{\mathbb{R}} W(t, u_n(t))\, dt \\ &\ge \frac{\xi_1}{2}\|u_n\|^2 - c\int_{\mathbb{R}} \left(|u_n|^2 + |u_n|^\nu\right)\, dt \\ &\ge \frac{\xi_1}{2}\|u_n\|^2 - c\|u_n\|_2^2 - c\int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} |u_n|^\nu\, dt - c\int_{\{t\in\mathbb{R}:|u_n(t)|\le R\}} |u_n|^\nu\, dt \\ &\ge \frac{\xi_1}{2}\|u_n\|^2 - c\|u_n\|_2^2 - c\|u_n\|_\infty \int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} |u_n|^{\nu-1}\, dt \\ &\quad - cR^{\nu-2}\int_{\{t\in\mathbb{R}:|u_n(t)|\le R\}} |u_n|^2\, dt \\ &\ge \frac{\xi_1}{2}\|u_n\|^2 - c\left(1 + R^{\nu-2}\right)\|u_n\|_2^2 - cC_\infty\|u_n\|R^{\nu-1-\theta}\int_{\{t\in\mathbb{R}:|u_n(t)|\ge R\}} |u_n|^\theta\, dt\end{aligned} \tag{3.25}$$

for some $M_5 > 0$, where $\xi_1 = \left(1 - \frac{\|A\|}{\sqrt{\alpha_1}}\right) > 0$. Divided by $\|u_n\|^2$ on both sides of (3.25), noting that (3.24) and $\theta \geq \nu - 1$, we have

$$\|w_n\|_2^2 \geq \frac{\xi_1}{2c(1+R^{\nu-2})} > 0 \quad \text{as } n \to \infty. \tag{3.26}$$

It follows from (3.15) and (3.26) that $w_0 \neq 0$. That is a contradiction.

Case 2. $w_0 \neq 0$. The proof is the same as that in Theorem 1.3, and we omit it here. Therefore, $\{u_n\}$ is bounded in E. Similar to the proof of Theorem 1.3, we can prove that $\{u_n\}$ has a convergent subsequence in E. Hence, φ satisfies the Palais-Smale condition. The proof is completed. □

Now we give the proof of Theorem 1.9.

Proof of Theorem 1.9. Obviously, $\varphi \in C^1(E, \mathbb{R})$ and $\varphi(0) = 0$. Next we divide our proof into third parts in order to show Theorem 1.9.

Firstly, we prove that φ satisfies the Palais–Smale condition. Suppose that $\{u_n\} \subset E$ such that $\{\varphi(u_n)\}$ be a bounded sequence and $\varphi'(u_n) \to 0$ as $n \to \infty$. By (2.4), (A_2), (H_9) and the Hölder's inequality, we have

$$\begin{aligned}
\mu_1 M_6 + M_6\|u_n\| &\geq \mu_1\varphi(u_n) - \langle\varphi'(u_n), u_n\rangle \\
&= \left(\frac{\mu_1}{2} - 1\right)\|u_n\|^2 + \left(\frac{\mu_1}{2} - 1\right)\int_{\mathbb{R}}(Au_n(t), \dot{u}_n(t))dt \\
&\quad + \int_{\mathbb{R}}\left[(\nabla W(t,u_n), u_n) - \mu_1 W(t,u_n)\right]dt \\
&\geq \xi_2\|u_n\|^2 - \int_{\mathbb{R}}\left[l_3(L(t)u_n, u_n) + l_4(t)|u_n|^\vartheta + l_5(t)\right]dt \\
&\quad - \int_{\mathbb{R}} l_6(t)\frac{|u_n|^2}{\ln(k_0 + |u_n|)}dt - \int_{\mathbb{R}} l_7(t)|u_n|\ln(k_1 + |u_n|)dt \\
&\geq (\xi_2 - l_3)\|u_n\|^2 - \|l_4\|_{\frac{2}{2-\vartheta}}\|u_n\|_2^\vartheta - \|l_5\|_1 - \|l_7\|_1\|u_n\|_\infty\ln(k_1 + \|u_n\|_\infty) \\
&\quad - \int_{\{t\in\mathbb{R}:|u_n(t)|\geq\sqrt{\|u_n\|}\}} l_6(t)\frac{|u_n|^2}{\ln(k_0+|u_n|)}dt \\
&\quad - \int_{\{t\in\mathbb{R}:|u_n(t)|\leq\sqrt{\|u_n\|}\}} l_6(t)\frac{|u_n|^2}{\ln(k_0+|u_n|)}dt \\
&\geq (\xi_2 - l_3)\|u_n\|^2 - C_2^\vartheta\|l_4\|_{\frac{2}{2-\vartheta}}\|u_n\|^\vartheta - \|l_5\|_1 \\
&\quad - C_\infty\|l_7\|_1\|u_n\|\ln(k_1 + C_\infty\|u_n\|) - \frac{\|l_6\|_2\|u_n\|_4^2}{\ln(k_0 + \sqrt{\|u_n\|})} - \frac{\|l_6\|_1}{\ln k_0}\|u_n\| \\
&\geq (\xi_2 - l_3)\|u_n\|^2 - C_2^\vartheta\|l_4\|_{\frac{2}{2-\vartheta}}\|u_n\|^\vartheta - \|l_5\|_1 \\
&\quad - C_\infty\|l_7\|_1\|u_n\|\ln(k_1 + C_\infty\|u_n\|) - \frac{C_4^2\|l_6\|_2\|u_n\|^2}{\ln(k_0 + \sqrt{\|u_n\|})} - \frac{\|l_6\|_1}{\ln k_0}\|u_n\|
\end{aligned}$$

for some $M_6 > 0$, where $\xi_2 = \left(\frac{\mu_1}{2} - 1\right)\left(1 - \frac{\|A\|}{\sqrt{\beta}}\right) > 0$. Since $0 < \vartheta < 2$ and $0 \leq l_3 < \frac{\mu_1 - 2}{2}\left(1 - \frac{\|A\|}{\sqrt{\beta}}\right)$, we get that $\{u_n\}$ is bounded in E. Similar to the proof of Theorem 1.3, we can prove that $\{u_n\}$ has a convergent subsequence in E. Hence, φ satisfies the Palais–Smale condition.

Secondly, we verify condition (ii) in Theorem 2.2. If $\|u\| \le \sqrt{2\sqrt{m}}T_0 = \rho$, then it follows from (2.8) that $|u(t)| \le T_0$ for any $t \in \mathbb{R}$. Let $\alpha := \sqrt{m}T_0^2\Big(1 - \frac{\|A\|}{\sqrt{\beta}} - \frac{k_2}{m}\Big)$, by ($H_{10}$), one has $\alpha > 0$. By virtue of (H_{10}), we have

$$|W(t,u)| \le \frac{k_2}{2}|u|^2 \quad \forall t \in \mathbb{R}, \forall |u| \le T_0. \tag{3.27}$$

Hence, for any $u \in E$ with $\|u\| \le \rho$, by (3.27) and (A_2), we get

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}}(Au(t), \dot{u}(t))dt - \int_{\mathbb{R}} W(t,u(t))\,dt \\
&\ge \frac{1}{2}\left(1 - \frac{\|A\|}{\sqrt{\beta}}\right)\|u\|^2 - \frac{k_2}{2}\int_{\mathbb{R}}|u|^2\,dt \\
&\ge \frac{1}{2}\left(1 - \frac{\|A\|}{\sqrt{\beta}}\right)\|u\|^2 - \frac{k_2}{2m}\int_{\mathbb{R}}(L(t)u,u)\,dt \\
&\ge \frac{1}{2}\left(1 - \frac{\|A\|}{\sqrt{\beta}} - \frac{k_2}{m}\right)\|u\|^2.
\end{aligned} \tag{3.28}$$

(3.28) shows that $\|u\| = \rho$ implies that $\varphi(u) \ge \alpha$.

Finally, we verify condition (iii) in Theorem 2.2. By (H_8), there exist $\varepsilon > 0$ and $R_2 > 0$ such that

$$W(t,u) \ge \left[\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2} + \frac{l_2}{2}\right) + \varepsilon\right]|u|^2, \quad \forall |u| \ge R_2,\ \forall t \in [a,b].$$

Let $R_3 = \max_{t\in[a,b],|u|\le R_2}|W(t,u)|$, hence we obtain

$$W(t,u) \ge \left[\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2} + \frac{l_2}{2}\right) + \varepsilon\right]\left(|u|^2 - R_2^2\right) - R_3 \tag{3.29}$$

for all $u \in \mathbb{R}^N$ and $t \in [a,b]$. Let

$$e(t) = \begin{cases} \eta_1|\sin(\omega(t-a))|e_1, & t \in [a,b], \\ 0, & t \in \mathbb{R} \setminus [a,b], \end{cases}$$

where $\omega = \frac{2\pi}{b-a}$ and $e_1 = (1,0,\dots,0)^\top$. By (H_{10}) and (3.29), we obtain

$$\begin{aligned}
\varphi(e) &= \frac{1}{2}\|e\|^2 + \frac{1}{2}\int_{\mathbb{R}}(Ae(t), \dot{e}(t))dt - \int_{\mathbb{R}} W(t,e(t))dt \\
&\le \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\|e\|^2 - \int_{\mathbb{R}} W(t,e(t))dt \\
&= \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\int_a^b\left[|\dot{e}(t)|^2 + (L(t)e(t),e(t))\right]dt - \int_a^b W(t,e(t))dt \\
&\le \frac{1}{2}\eta_1^2\omega^2\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\int_a^b|\cos(\omega(t-a))|^2dt \\
&\quad + \frac{1}{2}l_2\eta_1^2\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\int_a^b|\sin(\omega(t-a))|^2dt
\end{aligned} \tag{3.30}$$

$$
\begin{aligned}
&-\eta_1^2\left[\left(1+\frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2}+\frac{l_2}{2}\right)+\varepsilon\right]\int_a^b|\sin(\omega(t-a))|^2dt\\
&+(b-a)\left[\left(\left(1+\frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2}+\frac{l_2}{2}\right)+\varepsilon\right)R_2^2+R_3\right]\\
=&-\frac{\varepsilon(b-a)}{2}\eta_1^2+(b-a)\left[\left(\left(1+\frac{\|A\|}{\sqrt{\beta}}\right)\left(\frac{2\pi^2}{(b-a)^2}+\frac{l_2}{2}\right)+\varepsilon\right)R_2^2+R_3\right]\to-\infty
\end{aligned}
$$

as $\eta_1 \to \infty$. Thus, we can choose a large enough η_1 such that $\|e\| > \rho$ and $\varphi(e) \leq 0$. By Theorem 2.2, φ possesses a critical value $\bar{d_1} \geq \alpha$ given by

$$\bar{d_1} = \inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0,\ g(1) = e\}.$$

Hence, there exists $u^* \in E$ such that

$$\varphi(u^*) = \bar{d_1} \text{ and } \varphi'(u^*) = 0.$$

Then u^* is a desired classical solution of system (1.1). Since $\bar{d_1} > 0$, u^* is a nontrivial homoclinic solution. □

Now we give the proof of Theorem 1.12.

Proof of Theorem 1.12. By (A_6) and (2.6), we obtain that $\varphi \in C^1(E,\mathbb{R})$ is even. Next we will check that all conditions in Theorem 2.1 are satisfied.

For any $k \in \mathbb{N}$, we can choose $k+1$ disjoint open sets $\{\Upsilon_i | i = 0,1,\ldots,k\}$ such that

$$\bigcup_{i=0}^{k}\Upsilon_i \subset [\bar{a},\bar{b}].$$

For $i = 0,1,\ldots,k$, let $v_i \in (H_0^1(\Upsilon_i)\cap E)\setminus\{0\}$ and $\|v_i\| = 1$, then $v_0, v_1,\ldots,v_k$ can extended to be an orthonormal basis $\{v_n\}$ of E. Define $X_j := \mathbb{R}v_j$, then Z_k and Y_k can be defined as that in Section 2.

Step 1. We verify condition (G_2) in Theorem 2.1. The proof is similar to the proof of *Step 1* in Theorem 1.3.

Step 2. We prove that φ satisfies the Palais–Smale condition. The proof is the same as that the proof of Theorem 1.9.

Step 3. We verify condition (G_1) in Theorem 2.1. For any $u \in Y_k$, there exist ζ_i $(i = 0,1,\ldots,k)$ such that

$$u = \sum_{i=0}^{k}\zeta_i v_i.$$

Then we have

$$\begin{aligned}\|u\| &= \left(\int_{\mathbb{R}} \left(|\nabla u|^2 + (L(t)u, u)\right) dt\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=0}^{k} \zeta_i^2 \int_{Y_i} \left(|\nabla v_i|^2 + (L(t)v_i, v_i)\right) dt\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=0}^{k} \zeta_i^2 \|v_i\|^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=0}^{k} \zeta_i^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.31}$$

In view of (H_8'), for any $M_8 > 0$ there exists $T_1 = T_1(M_8) > 0$ such that

$$W(t,u) \geq M_8|u|^2, \ \text{ a.e. } t \in [\bar{a}, \bar{b}], \ |u| \geq T_1. \tag{3.32}$$

Since all norms are equivalent in the finite-dimensional space, there exist constants $M_{10} > 0$, $M_{11} > 0$ such that

$$M_{10}\|v_i\| \leq \|v_i\|_2 \leq M_{11}\|v_i\|, \ \ M_{10}\|v_i\| \leq \|v_i\|_\nu \leq M_{11}\|v_i\|, \ i = 0, 1, \dots, k. \tag{3.33}$$

By virtue of (A_1), (H_3), (3.32) and (3.33), one has

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot{u}(t))dt - \int_{\mathbb{R}} W(t, u(t))dt\\ &\leq \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\|u\|^2 - \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \int_{\mathbb{R}} W(t, \zeta_i v_i)dt - \sum_{i\in\{j:|\zeta_j|< T_1\}} \int_{\mathbb{R}} W(t, \zeta_i v_i)dt\\ &\leq \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)\|u\|^2 - M_8 \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \zeta_i^2 \int_{Y_i} |v_i|^2 dt\\ &\quad + c \sum_{i\in\{j:|\zeta_j|< T_1\}} \int_{Y_i} \left(|\zeta_i v_i|^2 + |\zeta_i v_i|^\nu\right) dt\\ &\leq \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right) \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \zeta_i^2 + \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)(k+1)T_1^2\\ &\quad - M_8 \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \zeta_i^2 \int_{Y_i} |v_i|^2 dt + c\sum_{i=0}^{k}\left(T_1^2 \int_{Y_i} |v_i|^2 dt + T_1^\nu \int_{Y_i} |v_i|^\nu dt\right)\\ &\leq \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right) \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \zeta_i^2 + \frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right)(k+1)T_1^2 - M_8 M_{10}^2 \sum_{i\in\{j:|\zeta_j|\geq T_1\}} \zeta_i^2\\ &\quad + c(k+1)\left(T_1^2 M_{11}^2 + T_1^\nu M_{11}^\nu\right)\end{aligned} \tag{3.34}$$

for any $u \in Y_k$ with $\|u\| = \rho_k$. Choosing M_8 sufficiently large such that

$$\frac{1}{2}\left(1 + \frac{\|A\|}{\sqrt{\beta}}\right) - M_8 M_{10}^2 < 0.$$

When ρ_k large enough, $\{j : |\zeta_j| \geq T_1\} \neq \emptyset$ and there exists $i \in \{j : |\zeta_j| \geq T_1\}$ such that $|\zeta_i|^2 \geq \frac{\rho_k^2}{k+1}$. Thus, we can choose $\|u\| = \rho_k$ large enough $(\rho_k > r_k)$ such that

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Psi(u) \leq 0.$$

It follows from Theorem 2.1 that φ has a sequence of critical points $\{u_k\} \subset E$ such that $\varphi(u_k) \to \infty$ as $k \to \infty$. Hence system (1.1) has infinitely many homoclinic solutions. □

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