Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **33**, 1–13; http://www.math.u-szeged.hu/ejqtde/

Weighted L^p estimates for the elliptic Schrödinger operator

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Received 25 March 2014, appeared 24 July 2014
Communicated by Patrizia Pucci

Abstract. In this paper we study weighted L^p estimates for the elliptic Schrödinger operator $P = -\Delta + V(x)$ with non-negative potentials $V(x)$ on $\mathbb{R}^n$ $(n \geq 3)$ which belongs to certain reverse Hölder class.

Keywords: weighted, regularity, L^p estimates, elliptic, Schrödinger operator.

2010 Mathematics Subject Classification: 35J10, 35J15.

1 Introduction

Shen [19] proved the L^p boundedness with $1 < p \leq 2$ of the nontangential maximal function of ∇u for the L^p-Neumann problem of the elliptic Schrödinger operator

$$P = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3 \tag{1.1}$$

with $V \in V_\infty$ (see Definition 1.1) in a domain $\Omega \subset \mathbb{R}^n$. Moreover, Shen [20] has obtained the following L^p estimates for (1.1)

$$\int_{\mathbb{R}^n} \left| D^2 \left(-\triangle + V(x) \right)^{-1} f \right|^p dx \leq C \int_{\mathbb{R}^n} |f|^p \, dx$$

for $1 < p \leq q$, assuming that $V \in V_q$ for some $q \geq n/2$. In this paper we consider weighted L^p estimates for the elliptic Schrödinger operator (1.1)

$$P = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3$$

with $V \in V_\infty$, where $x = (x^1, \dots, x^n)$ and $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

Definition 1.1. *The function $V(x)$ is said to belong to the reverse Hölder class V_q for some $1 < q \leq \infty$ if $V \in L^q_{loc}(\mathbb{R}^n)$, $V \geq 0$ almost everywhere and there exists a constant C such that for all balls B_r of $\mathbb{R}^n$,*

$$\left(\fint_{B_r} V^q(x) \, dx \right)^{1/q} \leq C \fint_{B_r} V(x) \, dx,$$

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with

$$\fint_{B_r} V(x)\,dx = \frac{1}{|B_r|}\int_{B_r} V(x)\,dx.$$

If $q = \infty$, then the left-hand side is the essential supremum on B_r, i.e.,

$$\sup_{B_r} |V(x)| \le C \fint_{B_r} V(x)\,dx.$$

In fact, if $V \in V_\infty$, it clearly implies $V \in V_q$ for every $q > 1$.

We can refer to [2, 20, 21] regarding the reverse Hölder class. In particular, $V(x) = |x|^\alpha \in V_q$ if $\alpha q > -n$.

We use the Hardy–Littlewood maximal function which controls the local behavior of a function.

Definition 1.2. *Let v be a locally integrable function. The Hardy–Littlewood maximal function $\mathcal{M}v(x)$ is defined as*

$$\mathcal{M}v(x) = \sup \fint_Q |v(y)|dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x.

It is well known that the maximal functions satisfy strong p-p estimate for any $1 < p < \infty$ and weak $(1,1)$ estimate (see [21]).

We now introduce the weighted Lebesgue spaces (see [11, 12, 15, 16, 18, 21, 22]).

Definition 1.3. *A_q for some $q > 1$ is the class of the Muckenhoupt weights: $w \in A_q$ if $w \in L^1_{loc}(\mathbb{R}^n)$, $w > 0$ almost everywhere and there exists a constant C such that for all balls B_r in $\mathbb{R}^n$,*

$$\left(\fint_{B_r} w(x)\,dx\right)\left(\fint_{B_r} w(x)^{\frac{-1}{q-1}}\,dx\right)^{q-1} \le C.$$

Moreover, we denote

$$A_\infty = \bigcup_{1<q<\infty} A_q$$

and

$$w(\Omega) = \int_\Omega w(x)\,dx,$$

where $\Omega \subset \mathbb{R}^n$. Furthermore, the corresponding weighted Lebesgue space $L^q_w(\Omega)$ consists of all functions h which satisfy

$$\|h\|_{L^q_w(\Omega)} := \left(\int_\Omega |h|^q\,w(x)\,dx\right)^{1/q} < \infty.$$

Remark 1.4. We remark that $A_{q_1} \subset A_{q_2}$ for any $1 < q_1 \le q_2 < \infty$ (see [21, p. 195]).

Lemma 1.5. *If $w \in A_q$ with $q > q_1 > 1$, then we have*

$$L^q_w(B_r) \subset L^{q_1}(B_r).$$

Proof. From Hölder's inequality we have

$$\int_{B_r} |f|^{q_1}\,dx = \int_{B_r} |f|^{q_1}\,w(x)^{\frac{q_1}{q}}\,w(x)^{-\frac{q_1}{q}}\,dx$$
$$\leq \left(\int_{B_r} |f|^q\,w(x)\,dx\right)^{\frac{q_1}{q}} \left(\int_{B_r} w(x)^{-\frac{q_1}{q-q_1}}\,dx\right)^{1-\frac{q_1}{q}}.$$

Since $w \in A_q$ with $q > q_1 > 1$, from Remark 1.4 we find that $w \in A_{q/q_1}$. Furthermore, we conclude that

$$\fint_{B_r} w(x)^{-\frac{q_1}{q-q_1}}\,dx = \fint_{B_r} w(x)^{-\frac{1}{q/q_1-1}}\,dx \leq C\left(\frac{|B_r|}{w(B_r)}\right)^{\frac{q_1}{q-q_1}}.$$

Thus, if $f \in L^q_w(B_r)$, then we have

$$\int_{B_r} |f|^{q_1}\,dx \leq C\left(\int_{B_r} |f|^q\,w(x)\,dx\right)^{\frac{q_1}{q}} |B_r|^{1-\frac{q_1}{q}} \left(\frac{|B_r|}{w(B_r)}\right)^{\frac{q_1}{q}} \leq C,$$

since $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ almost everywhere. This finishes our proof. □

Lemma 1.6 (see [5, 6, 12, 15, 16, 21, 22])**.** *Assume that $w(x) \in A_q$ for some $q > 1$ and $g \in L^q_w(\mathbb{R}^n)$. Then we have*

(1)

$$\|\mathcal{M}g\|_{L^q_w(\mathbb{R}^n)} \leq C\|g\|_{L^q_w(\mathbb{R}^n)}.$$

(2)

$$w\left(\{x \in \mathbb{R}^n : \mathcal{M}g(x) > \mu\}\right) \leq \frac{C}{\mu^q}\int_{\mathbb{R}^n} |g|^q\,w(x)\,dx.$$

(3)

$$\int_{\mathbb{R}^n} |g|^q w(x)\,dx = q\int_0^\infty \mu^{q-1} w\left(\{x \in \mathbb{R}^n : |g| > \mu\}\right)d\mu.$$

Next, we shall give some lemmas on the properties of A_q weight.

Lemma 1.7 (see [5, 6, 15, 16, 22])**.** *If $w \in A_q$ for some $q > 1$ and $B_r \subset B_R \subset \mathbb{R}^n$, then there exists a constant $C_1 > 0$ such that*

$$\frac{w(B_R)}{w(B_r)} \leq C_1\left(\frac{|B_R|}{|B_r|}\right)^q.$$

Furthermore, we have the following reverse Hölder inequality.

Lemma 1.8 (see [22, Theorem 3.5 in Chapter 9])**.** *If $w \in A_q$ for some $q > 1$, then there exists a small positive constant $\epsilon_0 < 1$ and a constant $C_2 > 1$ such that*

$$\left(\fint_{B_r} w(x)^{1+\epsilon_0}\,dx\right)^{\frac{1}{1+\epsilon_0}} \leq C_2\fint_{B_r} w(x)\,dx$$

for any ball $B_r \subset \mathbb{R}^n$.

Lemma 1.9. *If $w \in A_q$ for some $q > 1$ and $B_r \subset B_R \subset \mathbb{R}^n$, then there exists $\sigma > 0$ such that*

$$\frac{w(B_r)}{w(B_R)} \leq C_2 \left(\frac{|B_r|}{|B_R|} \right)^{\sigma}.$$

Proof. We first conclude that

$$\begin{aligned} w(B_r) &= \int_{B_r} w(x)\, dx \\ &\leq \left(\int_{B_r} w(x)^{1+\epsilon_0}\, dx \right)^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} \\ &\leq \left(\fint_{B_R} w(x)^{1+\epsilon_0}\, dx \right)^{\frac{1}{1+\epsilon_0}} \cdot |B_R|^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} \end{aligned}$$

by using Hölder's inequality. Thus, it follows from the lemma above that

$$w(B_r) \leq C_2 \fint_{B_R} w(x)\, dx \cdot |B_R|^{\frac{1}{1+\epsilon_0}} \cdot |B_r|^{\frac{\epsilon_0}{1+\epsilon_0}} = C_2 w\left(B_R\right) \left(\frac{|B_r|}{|B_R|} \right)^{\frac{\epsilon_0}{1+\epsilon_0}},$$

which finishes our proof by selecting $\sigma = \epsilon_0/(1+\epsilon_0)$. □

Now let us state the main results of this work: Theorem 1.10 and Theorem 1.11. We shall give the direct proofs of the main results via the maximal function approach which was employed by [1, 5, 7, 13, 15, 16, 17].

Theorem 1.10. *Assume that $w(x) \in A_p$ for some $p > 1$ and $f \in L^p_w(\mathbb{R}^n)$. If u is the solution of the Poission equation*

$$-\Delta u = f(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3, \tag{1.2}$$

then we have

$$\int_{\mathbb{R}^n} \left|D^2 u\right|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx.$$

Theorem 1.11. *Assume that $w(x) \in A_p$ for some $p > 1$, $V \in V_\infty$ and $f \in L^p_w(\mathbb{R}^n)$. If $u \in C_0^\infty(\mathbb{R}^n)$ is the solution of the following elliptic Schrödinger equation*

$$-\Delta u(x) + V(x)u(x) = f(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3, \tag{1.3}$$

then we have

$$\int_{\mathbb{R}^n} |Vu|^p w(x)\, dx + \int_{\mathbb{R}^n} \left|D^2 u\right|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx.$$

Remark 1.12. Assume that $u \in C_0^\infty(\mathbb{R}^n)$ and $V \in V_q$ with $1 < p \leq q$ and $q \geq n/2$. The authors of [4] proved that

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \leq C \left(\|f\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right)$$

for (1.3) and the general case.

2 Proofs of the main results

In this section we shall finish the proofs of the main results: Theorem 1.10 and Theorem 1.11.

2.1 Proof of Theorem 1.10

We first give the following Calderón–Zygmund decomposition, which is much influenced by [14].

Lemma 2.1. *Let D be a cube in $\mathbb{R}^n$ and $A, B \subset D$ be measurable sets. Assume that $0 < w(A) < \mu w(D)$ for $0 < \mu < 1$. Then there exists a sequence of disjoint cubes $\{Q_k\}_{k\in\mathbb{N}}$ satisfying*

(1) $w\left(A \setminus \bigcup_{k\in\mathbb{N}} Q_k\right) = 0$,

(2) $w\left(A \cap Q_k\right) > \mu w\left(Q_k\right)$,

(3) $w\left(A \cap \widetilde{Q_k}\right) \leq \mu w\left(\widetilde{Q_k}\right)$ if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.

Furthermore, if for any Q_k, its predecessor $\widetilde{Q_k}$ satisfies

$$w\left(B \cap \widetilde{Q_k}\right) > \alpha w\left(\widetilde{Q_k}\right) \quad \text{for } 0 < \alpha < 1, \tag{2.1}$$

then we have

$$w(A) \leq \frac{\mu}{\alpha} w(B).$$

Proof. **1.** We first divide D into 2^n $\left(\text{denote by } \{Q_1^{j_1}\}_{j_1=1}^{2^n}\right)$ disjoint cubes (daughters) with the same size. Choose those cubes satisfying $w(A \cap Q_1^{j_1}) > \mu w(Q_1^{j_1})$ and continue to divide every remaining cube $Q_1^{j_1}$ into 2^n $\left(\text{denote by } \{Q_2^{j_1,j_2}\}_{j_2=1}^{2^n}\right)$ disjoint cubes with the same size. Therefore, we obtain a sequence of disjoint cubes $\{Q_k\}_{k\in\mathbb{N}}$ which satisfies (2)–(3) by repeating the process above. If $x \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, then there is a sequence of cubes P_i containing x with the diameters of P_i converging to 0 and

$$w\left(A \cap P_i\right) \leq \mu w\left(P_i\right).$$

From elementary measure theory and the fact that $w(x) > 0$ almost everywhere we can conclude that for almost every $x \in D \setminus \{Q_k\}_{k\in\mathbb{N}}$, $x \in D \setminus A$. That is say, (1) is true.

2. Let $\widetilde{Q_k}$ be the predecessor (father) of Q_k. Now we choose a disjoint predecessor subsequence $\left\{\widetilde{Q_{k_j}}\right\}$ $\left(\text{still denoted by } \left\{\widetilde{Q_k}\right\}\right)$ such that $\bigcup_{k\in\mathbb{N}} Q_k \subset \bigcup_{k\in\mathbb{N}} \widetilde{Q_k}$. Thus, from (1), (3) and the hypothesis (2.1) we deduce that

$$w(A) = \sum_k w\left(A \cap \widetilde{Q_k}\right) \leq \mu \sum_k w\left(\widetilde{Q_k}\right) < \frac{\mu}{\alpha} \sum_k w\left(B \cap \widetilde{Q_k}\right) \leq \frac{\mu}{\alpha} w\left(B\right),$$

which finishes our proof. □

Next, we shall prove the following important result.

Lemma 2.2. *Assume that $1 < q < p$. For any $\mu, \alpha \in (0, 1)$ there exist two constants $M_2 = M_2(n) > 1$ and $\delta = \delta(n, \mu) \in (0, 1)$, such that if*

$$\left|\left\{x \in \widetilde{Q} : \mathcal{M}\left(\left|D^2 u\right|^q\right)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|^q\right)(x) \leq \delta^q\right\}\right| > \alpha \left|\widetilde{Q}\right|, \tag{2.2}$$

then we have

$$\left|\left\{x \in Q : \mathcal{M}\left(\left|D^2 u\right|^q\right)(x) \geq M_2^q\right\}\right| \leq \mu \left|Q\right|.$$

Proof. **1.** From the hypothesis (2.2) there exists $x_0 \in \widetilde{Q}$ such that

$$\mathcal{M}\left(\left|D^2u\right|^q\right)(x_0) \le 1 \quad \text{and} \quad \mathcal{M}\left(|f|^q\right)(x_0) \le \delta^q. \tag{2.3}$$

Since $x_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$⨍_{4Q} \left|D^2u\right|^q dx \le 1 \quad \text{and} \quad ⨍_{4Q} |f|^q dx \le \delta^q. \tag{2.4}$$

Let v_1 be the solution of

$$-\Delta v_1 = \bar{f} \qquad \text{on } \mathbb{R}^n,$$

where $\bar{f}$ is the zero extention of f from $4Q$ to $\mathbb{R}^n$. Then from the elementary L^p-type estimates we have

$$\int_{\mathbb{R}^n} \left|D^2v_1\right|^q dx \le \int_{\mathbb{R}^n} |\bar{f}|^q dx,$$

which implies that

$$\int_{4Q} \left|D^2v_1\right|^q dx \le \int_{\mathbb{R}^n} \left|D^2v_1\right|^q dx \le \int_{\mathbb{R}^n} |\bar{f}|^q dx = \int_{4Q} |f|^q dx. \tag{2.5}$$

Therefore, from (2.4) we conclude that

$$⨍_{4Q} \left|D^2v_1\right|^q dx \le ⨍_{4Q} |f|^q dx \le \delta^q. \tag{2.6}$$

Set $h_1 = u - v_1$. From the definition of $\bar{f}$, we find that h_1 satisfies

$$-\Delta h_1 = 0 \qquad \text{in } 4Q. \tag{2.7}$$

Moreover, it follows from $W^{2,\infty}_{loc}$ regularity that

$$\sup_{3Q} \left|D^2h_1\right| \le M_1,$$

where $M_1 > 1$ only depends on n.

2. The proof is totally similar to the proof of Lemma 2.8. Here we omit the details. □

Corollary 2.3 (cf. Corollary 2.9)**.** *Assume that $1 < q < p$ and $w \in A_p$. For any $\mu \in (0,1)$ there exist two constants $M_3 = M_3(n) > 1$ and $\delta = \delta(n,\sigma,\mu) \in (0,1)$ such that if*

$$w\left(\left\{x \in \widetilde{Q} : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \le 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|^q\right)(x) \le \delta^q\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.8}$$

then we have

$$w\left(\left\{x \in Q : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \ge M_3^q\right\}\right) \le \mu w(Q).$$

Corollary 2.4 (cf. Corollary 2.10)**.** *Let D be a cube in $\mathbb{R}^n$. Assume that q, w, μ, δ, M_3 satisfy the same conditions as those in Corollary 2.3. If*

$$w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \ge M_3^q\right\}\right) \le \mu w(D),$$

then we have

$$\begin{aligned} & w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \ge M_3^q\right\}\right) \\ & \quad \le 2\mu\Big[w\left(\left\{x \in D : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) > 1\right\}\right) \\ & \qquad + w\left(\left\{x \in D : \mathcal{M}\left(|f|^q\right)(x) > \delta^q\right\}\right)\Big]. \end{aligned} \tag{2.9}$$

Corollary 2.5 (cf. Corollary 2.11)**.** *Assume that $\mu \in (0,1)$ with $C_2\mu^\sigma < 1$ and q, w, δ, M_3 satisfy the same conditions as those in Corollary 2.3. For any $\lambda > 0$ we have*

$$\begin{aligned} & w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) \geq \lambda^q M_3^q\right\}\right) \\ & \quad \leq 2C_2\mu^\sigma\Big[w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|D^2u\right|^q\right)(x) > \lambda^q\right\}\right) \\ & \qquad + w\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(|f|^q\right)(x) > \lambda^q\delta^q\right\}\right)\Big]. \end{aligned} \tag{2.10}$$

The rest of the proof of Theorem 1.10 is totally similar to that of Theorem 1.11 in §2.2.

2.2 Proof of Theorem 1.11

We first recall the following result (see [21, p. 195]).

Lemma 2.6. *If $V \in V_\infty$, then there exist $t \in [1,\infty)$ and $C > 0$ such that*

$$⨍_Q g\, dx \leq \left(\frac{C}{V(Q)}\int_Q V g^t\, dx\right)^{\frac{1}{t}}$$

holds for any nonnegative function g and all cubes Q, where

$$V(Q) = \int_Q V\, dx.$$

Furthermore, we have the following local boundedness property.

Lemma 2.7. *Assume that $V \in V_\infty$. If $h(x)$ satisfies $-\Delta h(x) + V(x)h(x) = 0$ in $2Q$, then*

$$\sup_Q |h| \leq \frac{C}{V(2Q)}\int_{2Q} V|h|\, dx,$$

where C depends on n.

Proof. Since $V \in V_\infty$ and $u \in C_0^\infty(\mathbb{R}^n)$ satisfies $-\Delta u(x) + V(x)u(x) = f(x)$, we may as well assume that

$$\operatorname{supp} u \subset B_{r_0}, \quad V(x) \equiv 0 \quad \text{in } \mathbb{R}^n \setminus B_{r_0} \quad \text{and} \quad |V(x)| \leq C \quad \text{in } \mathbb{R}^n$$

for some $r_0 > 0$. Recalling the elementary local boundedness property of the second-order elliptic equation (see [9, Theorem 9.20], or [10, Theorem 4.1]), we have

$$\sup_Q |h| \leq C\left(⨍_{2Q} |h|^r\, dx\right)^{\frac{1}{r}}$$

for any $r > 0$. Then using the above inequality and Lemma 2.6 with $r = \frac{1}{t}$, we find that

$$\sup_Q |h| \leq C\left(⨍_{2Q} |h|^{\frac{1}{t}}\, dx\right)^t \leq \frac{C}{V(2Q)}\int_{2Q} V|h|\, dx.$$

This completes our proof. □

Next, we shall prove the following important result.

Lemma 2.8. *For any $\mu, \alpha \in (0,1)$ there exist two constants $N_2 = N_2(n) > 1$ and $\delta = \delta(n,\mu) \in (0,1)$, such that if*

$$\left|\left\{x \in \widetilde{Q} : \mathcal{M}\left(V|u|\right)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}\left(|f|\right)(x) \leq \delta\right\}\right| > \alpha\left|\widetilde{Q}\right|, \tag{2.11}$$

then we have

$$\left|\left\{x \in Q : \mathcal{M}\left(V|u|\right)(x) \geq N_2\right\}\right| \leq \mu\left|Q\right|.$$

Proof. **1.** From the hypothesis (2.11) there exists $x_0 \in \widetilde{Q}$ such that

$$\mathcal{M}\left(V|u|\right)(x_0) \leq 1 \quad \text{and} \quad \mathcal{M}\left(|f|\right)(x_0) \leq \delta. \tag{2.12}$$

Since $x_0 \in \widetilde{Q} \subset 3Q$, we conclude that

$$⨍_{4Q} |Vu|\, dx \leq 1 \quad \text{and} \quad ⨍_{4Q} |f|\, dx \leq \delta. \tag{2.13}$$

Let v be the solution of

$$-\Delta v(x) + V(x)v(x) = \bar{f} \quad \text{on } \mathbb{R}^n,$$

where $\bar{f}$ is the zero extention of f from $4Q$ to $\mathbb{R}^n$. Then recalling the well-known L^1 estimate (see [3, 8]), we have

$$\int_{\mathbb{R}^n} V|v|\, dx \leq \int_{\mathbb{R}^n} |\bar{f}|\, dx,$$

which implies that

$$\int_{4Q} V|v|\, dx \leq \int_{\mathbb{R}^n} V|v|\, dx \leq \int_{\mathbb{R}^n} |\bar{f}|\, dx = \int_{4Q} |f|\, dx. \tag{2.14}$$

Therefore, from (2.13) we conclude that

$$⨍_{4Q} V|v|\, dx \leq ⨍_{4Q} |f|\, dx \leq \delta. \tag{2.15}$$

Set $h = u - v$. From the definition of $\bar{f}$, we find that h satisfies

$$-\Delta h(x) + V(x)h(x) = 0 \quad \text{in } 4Q. \tag{2.16}$$

Moreover, it follows from (2.13) and (2.15) that

$$⨍_{4Q} V|h|\, dx \leq ⨍_{4Q} V|v|\, dx + ⨍_{4Q} V|u|\, dx < 2.$$

Then from the above inequality and Lemma 2.7 we find that

$$\sup_{3Q} V|h| \leq C \sup_{4Q} V \ \left[V(4Q)\right]^{-1} \int_{4Q} V|h|\, dx \leq C \sup_{4Q} V \left(⨍_{4Q} V\, dx\right)^{-1},$$

which implies that

$$\sup_{3Q} V|h| \leq N_1, \tag{2.17}$$

since $V \in V_\infty$, where $N_1 > 1$ depends on n.

2. Next, we shall prove that

$$\{x \in Q : \mathcal{M}(V|u|)(x) > N_2\} \subset \{x \in Q : \mathcal{M}(|Vv|)(x) > N_1\}, \tag{2.18}$$

where $N_2 := \max\{2N_1, 3^n\}$. Actually, from (2.17) we find that

$$V|u| \leq V|v| + V|h| \leq V|v| + N_1 \quad \text{for any} \quad x \in 3Q.$$

Let x be a point in $\{x \in Q : \mathcal{M}(V|v|)(x) \leq N_1\}$. If $x \in Q_1 \subset 3Q$, then we have

$$⨍_{Q_1} V|u|dx \leq ⨍_{Q_1} V|v|dx + N_1 \leq 2N_1. \tag{2.19}$$

Moreover, if $x \in Q_1 \not\subset 3Q$, then we have $x \in Q \subset Q_1$ and $3Q \subset 3Q_1$. Therefore, from (2.12) we find that

$$⨍_{Q_1} V|u|dy \leq 3^n ⨍_{3Q_1} V|u|dy \leq 3^n, \tag{2.20}$$

since $x_0 \in \widetilde{Q} \subset 3Q \subset 3Q_1$ and $\mathcal{M}(V|u|)(x_0) \leq 1$. Thus, it follows from (2.19) and (2.20) that $\mathcal{M}(V|u|)(x) \leq N_2$, which implies that (2.18) is true. Finally, from (2.15), (2.18) and the weak $(1,1)$ estimate of the maximal functions we have

$$\begin{aligned}
&|\{x \in Q : \mathcal{M}(V|u|)(x) > N_2\}| \\
&\quad \leq |\{x \in Q : \mathcal{M}(|Vv|)(x) > N_1\}| \\
&\quad \leq C \int_Q |Vv|\, dx \leq C \int_{4Q} |Vv|\, dx \leq C\delta\, |4Q| \leq C\delta\, |Q| \leq \mu\, |Q|,
\end{aligned}$$

by choosing δ small enough satisfying the last inequality. Thus we complete the proof. □

Furthermore, we can directly obtain the following result from the lemma above.

Corollary 2.9. *Assume that $w \in A_p$ for $p > 1$. For any $\mu \in (0,1)$ there exist two constants $N_3 = N_3(n) > 1$ and $\delta = \delta(n, \sigma, \mu) \in (0,1)$ such that if*

$$w\left(\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \leq \delta\right\}\right) > \frac{1}{2} w\left(\widetilde{Q}\right), \tag{2.21}$$

then we have

$$w(\{x \in Q : \mathcal{M}(V|u|)(x) \geq N_3\}) \leq \mu w(Q).$$

Proof. From Lemma 1.9 and (2.21) we have

$$\begin{aligned}
&\frac{\left|\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \leq \delta\right\}\right|}{\left|\widetilde{Q}\right|} \\
&\quad \geq \left[\frac{w\left(\left\{x \in \widetilde{Q} : \mathcal{M}(V|u|)(x) \leq 1\right\} \cap \left\{x \in \widetilde{Q} : \mathcal{M}(|f|)(x) \leq \delta\right\}\right)}{C_2 w\left(\widetilde{Q}\right)}\right]^{\frac{1}{\sigma}} \\
&\quad \geq (2C_2)^{-\frac{1}{\sigma}} \in (0,1),
\end{aligned}$$

since $C_2 > 1$ and $\sigma > 0$. So, for any $\mu_1 \in (0,1)$ with $C_2 \mu_1^{\sigma} < 1$, it follows from Lemma 2.8 that there exist two positive constants $N_3 = N_3(n) > 1$ and $\delta = \delta(n, \sigma, C_2, \mu_1) \in (0,1)$ such that

$$|\{x \in Q : \mathcal{M}(V|u|)(x) \geq N_3\}| \leq \mu_1 |Q|.$$

Then Lemma 1.9 implies that

$$w\left(\{x \in Q : \mathcal{M}\left(V|u|\right)(x) \geq N_3\}\right) \leq C_2 \mu_1^{\sigma} w\left(Q\right),$$

which completes our proof by selecting $\mu = C_2 \mu_1^{\sigma}$. □

Furthermore, we can obtain the following result.

Corollary 2.10. *Let D be a cube in $\mathbb{R}^n$. Assume that w, μ, δ, N_3 satisfy the same conditions as those in Corollary 2.9. If*

$$w\left(\{x \in D : \mathcal{M}\left(V|u|\right)(x) \geq N_3\}\right) \leq \mu w\left(D\right),$$

then we have

$$\begin{aligned} & w\left(\{x \in D : \mathcal{M}\left(V|u|\right)(x) \geq N_3\}\right) \\ & \quad \leq 2\mu \Big[w\left(\{x \in D : \mathcal{M}\left(V|u|\right)(x) > 1\}\right) + w\left(\{x \in D : \mathcal{M}\left(|f|\right)(x) > \delta\}\right) \Big]. \end{aligned} \tag{2.22}$$

Proof. We denote

$$A = w\left(\{x \in D : \mathcal{M}\left(V|u|\right)(x) \geq N_3\}\right)$$

and

$$B = w\left(\{x \in D : \mathcal{M}\left(V|u|\right)(x) > 1\} \cup \{x \in D : \mathcal{M}\left(|f|\right)(x) > \delta\} \right).$$

Then $A, B \subset D$ and $w\left(A\right) \leq \mu w(D)$. Therefore, it follows from Lemma 2.1 that there exists a sequence of disjoint cubes $\{Q_k\}$ satisfying

(1) $w\left(A \setminus \bigcup_{k \in \mathbb{N}} Q_k\right) = 0$,

(2) $w\left(A \cap Q_k\right) > \mu w\left(Q_k\right)$,

(3) $w\left(A \cap \widetilde{Q_k}\right) \leq \mu w\left(\widetilde{Q_k}\right)$ if $\widetilde{Q_k}$ is the predecessor (father) of Q_k.

If $w\big(\widetilde{Q_k} \cap B\big) \leq \frac{1}{2} w\big(\widetilde{Q_k}\big)$, where $\widetilde{Q_k}$ is the predecessor of Q_k, then we obtain (2.21) with $\widetilde{Q}$ repacing by $\widetilde{Q_k}$. Furthermore, it follows from Corollary 2.9 that

$$w\left(A \cap Q_k\right) \leq w\left(\{x \in Q_k : \mathcal{M}\left(V|u|\right)(x) \geq N_3\}\right) \leq \mu w\left(Q_k\right).$$

So, we get a contradiction with (2) and then know that $w\big(\widetilde{Q_k} \cap B\big) > \frac{1}{2} w\big(\widetilde{Q_k}\big)$. Finally, we can use Lemma 2.1 again to get that

$$w\left(A\right) \leq 2\mu w\left(B\right),$$

which implies (2.22) is true. Thus, we finish the proof. □

Corollary 2.11. *Assume that $\mu \in (0,1)$ with $C_2 \mu^{\sigma} < 1$ and w, δ, N_3 satisfy the same conditions as those in Corollary 2.9. For any $\lambda > 0$ we have*

$$\begin{aligned} & w\left(\{x \in \mathbb{R}^n : \mathcal{M}\left(V|u|\right)(x) \geq \lambda N_3\}\right) \\ & \quad \leq 2C_2 \mu^{\sigma} \big[w\left(\{x \in \mathbb{R}^n : \mathcal{M}\left(V|u|\right)(x) > \lambda\}\right) \\ & \qquad + w\left(\{x \in \mathbb{R}^n : \mathcal{M}\left(|f|\right)(x) > \lambda\delta\}\right) \big]. \end{aligned} \tag{2.23}$$

Proof. Without loss of generality, we may as well assume that $\lambda = 1$. Let

$$\mathbb{R}^n = \bigcup_{i=1}^{\infty} \overline{Q_i},$$

where $\{Q_i\}$ is a sequence of disjoint same side-length cubes. Moreover, from the weak 1-1 estimate and L^1 estimate (see [3, 8]) we conclude that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) \geq N_3\}| \leq \frac{C}{N_3} \|Vu\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{N_3} \|f\|_{L^1(\mathbb{R}^n)}.$$

We may as well assume that $f \in C_0^\infty(\mathbb{R}^n)$ via an elementary approximation argument. So, we can obtain

$$|\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}| \leq \mu |Q_i|$$

by selecting $|Q_i|$ large enough for $i \in \mathbb{N}$. Furthermore, from Lemma 1.9 we have

$$w(\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}) \leq C_2 \mu^\sigma w(Q_i).$$

Thus, by Corollary 2.10 we obtain

$$\begin{aligned} & w(\{x \in Q_i : \mathcal{M}(V|u|)(x) \geq N_3\}) \\ & \quad \leq 2C_2\mu^\sigma \left[w(\{x \in Q_i : \mathcal{M}(V|u|)(x) > 1\}) + w(\{x \in Q_i : \mathcal{M}(|f|)(x) > \delta\}) \right], \end{aligned}$$

which implies that the desired estimate (2.23) is true. This finishes our proof. □

Now we are ready to prove Theorem 1.11.

Proof. From Lemma 1.6 (3) and Corollary 2.11 we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\, dx \\ & \quad = p \int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) > N_3\lambda\})\, d[N_3\lambda] \\ & \quad \leq 2C_2 p \mu^\sigma \int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(V|u|)(x) > \lambda\})\, d[N_3\lambda] \\ & \qquad + 2C_2 p \mu^\sigma \int_0^\infty (N_3\lambda)^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}(|f|)(x) > \lambda\delta\})\, d[N_3\lambda] \\ & \quad \leq C_3 \mu^\sigma \int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\, dx + C_4 \int_{\mathbb{R}^n} |\mathcal{M}(|f|)|^p w(x)\, dx \end{aligned}$$

for any $\mu \in (0,1)$ with $C_2\mu^\sigma < 1$, where $C_3 = C_3(p,n)$ and $C_4 = C_4(p,n,\mu,\sigma)$. Without loss of generality we may as well assume that $f \in C_0^\infty(\mathbb{R}^n)$. Then choosing a suitable μ such that $C_3\mu^\sigma < 1$, we obtain

$$\int_{\mathbb{R}^n} |\mathcal{M}(V|u|)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |\mathcal{M}(|f|)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p\, w(x)\, dx$$

in view of Lemma 1.6 (1). Thus, we can obtain

$$\int_{\mathbb{R}^n} |Vu|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx$$

by using the fact that $V|u|(x) \leq \mathcal{M}(V|u|)(x)$. Thus from Theorem 1.10 we observe that

$$\int_{\mathbb{R}^n} |D^2u|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f|^p w(x)\, dx,$$

which completes the proof.

□

Acknowledgments

The author wishes to thank the editor and the anonymous referee for offering valuable suggestions to improve the expressions. This work is supported in part by the Innovation Program of Shanghai Municipal Education Commission (14YZ027).

References

[1] E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.* **584**(2005), 117–148. MR2155087; url

[2] R. A. Adams, J. J. F. Fournier, *Sobolev spaces (2nd edition)*, Academic Press, New York, 2003. MR2424078

[3] P. Auscher, B. Ben Ali, Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials, *Ann. Inst. Fourier (Grenoble)* **57**(2007), 1975–2013. MR2377893

[4] M. Bramanti, L. Brandolini, E. Harboure, B. Viviani, Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition, *Ann. Mat. Pura Appl. (4)* **191**(2012), 339–362. MR2909802; url

[5] S. Byun, Dian K. Palagachev, S. Ryu, Weighted $W^{1,p}$ estimates for solutions of nonlinear parabolic equations over non-smooth domains, *Bull. Lond. Math. Soc.* **45**(2013), 765–778. MR3081545; url

[6] S. Byun, S. Ryu, Global weighted estimates for the gradient of solutions to nonlinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30**(2013), 291–313. MR3035978; url

[7] L. A. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, *Comm. Pure Appl. Math.* **51**(1998), 1–21. MR1486629; url

[8] T. Gallouët, J. M. Morel, Resolution of a semilinear equation in L^1, *Proc. Roy. Soc. Edinburgh Sect. A*, **96**(1984), 275–288. Corrigenda: *Proc. Roy. Soc. Edinburgh Sect. A* **99**(1985), 399. MR760776; url

[9] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order (3rd edition)*, Springer-Verlag, Berlin, 1998. MR1814364

[10] Q. Han, F. Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, Vol. 1., New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. MR2777537

[11] J. Jiménez Urrea, The Cauchy problem associated to the Benjamin equation in weighted Sobolev spaces, *J. Differential Equations* **254**(2013), 1863–1892. MR3003295; url

[12] A. Kufner, *Weighted Sobolev spaces*, John Wiley & Sons, Inc., New York, 1985. MR802206

[13] T. Kuusi, G. Mingione, Universal potential estimates, *J. Funct. Anal.* **262**(2012), 4205–4269. MR2900466; url

[14] D. Li, L. Wang, A new proof for the estimates of Calderón–Zygmund type singular integrals, *Arch. Math. (Basel)* **87**(2006), 458–467. MR2269929; url

[15] T. Mengesha, N. Phuc, Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains, *J. Differential Equations* **250**(2011), 2485–2507. MR2756073; url

[16] T. Mengesha, N. Phuc, Global estimates for quasilinear elliptic equations on Reifenberg flat domains, *Arch. Ration. Mech. Anal.* **203**(2012), 189–216. MR2864410; url

[17] G. Mingione, Gradient estimates below the duality exponent, *Math. Ann.* **346**(2010), 571–627. MR2578563; url

[18] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165**(1972), 207–226. MR0293384

[19] Z. Shen, On the Neumann problem for Schrödinger operators in Lipschitz domains, *Indiana Univ. Math. J.* **43**(1994), 143–176. MR1275456; url

[20] Z. Shen, L^p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)* **45**(1995), No. 2, 513–546. MR1343560

[21] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, 1993. MR1232192

[22] A. Torchinsky, *Real-variable methods in harmonic analysis*, Pure Appl. Math., Vol. 123, Academic Press, Inc., Orlando, FL, 1986. MR869816